NUMERICAL PEAK POINTS AND NUMERICAL ŠILOV BOUNDARY FOR HOLOMORPHIC FUNCTIONS

SUNG GUEN KIM

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ABSTRACT. In this paper, we characterize the numerical and numerical strong-peak points for $\mathcal{A}_\infty(B_E : E)$ when E is the complex space l_1 or $C(K)$. We also prove that $\{(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1 \text{ for all } n \in \mathbb{N}\}$ is the numerical Šilov boundary for $\mathcal{A}_\infty(B_{l_1} : l_1)$.

1. INTRODUCTION

Throughout this paper we will just consider *complex* Banach spaces. For a Banach space E , S_E and B_E will be the unit sphere and the closed unit ball of E , respectively. If E and F are Banach spaces, $\mathcal{C}_b(B_E : F)$ denotes the Banach space of the bounded continuous functions $f : B_E \rightarrow F$, endowed with the supremum norm. In the case that $F = \mathbb{C}$, we write $\mathcal{C}_b(B_E)$. An N -homogeneous polynomial P from E to F is a mapping such that there is an N -linear (and bounded) mapping L from E to F satisfying

$$P(x) = L(x, \dots, x), \quad \forall x \in E.$$

The set of all N -homogenous polynomials from E to F is denoted by $\mathcal{P}^N E : F$. We denote by $\mathcal{A}_\infty(B_E : F)$ the Banach space of the bounded continuous function $f : B_E \rightarrow F$ such that f is holomorphic on the open unit ball, endowed with the supremum norm. If $F = \mathbb{C}$, we write $\mathcal{A}_\infty(B_E)$.

A result of Šilov asserts that if $\mathcal{A}$ is a unital separating subalgebra of $C(K)$ (K is a compact Hausdorff topological space), there is a smallest closed subset $S \subset K$ such that every function of $\mathcal{A}$ attains its norm at some point of S ([7], Theorem I.4.2). Bishop [5] proved that if K is metrizable, in fact, there is a minimal subset of K satisfying the above condition for every separating subalgebra of $C(K)$. That subset is the set of peak points for $\mathcal{A}$. Globevnik [8] introduced the corresponding concepts of the boundary of a subalgebra $\mathcal{A}$ of $\mathcal{C}_b(\Omega)$, the set of bounded and continuous functions on a topological space Ω not necessarily compact, and studied

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the boundaries for $\Omega = B_{c_0}$ and $\mathcal{A}$ a certain space of holomorphic functions. A subset $B \subset B_E$ is a boundary for a subspace $\mathcal{A} \subset \mathcal{C}_b(B_E)$ if

$$\|f\| = \sup_{z \in B} |f(z)|, \quad \forall f \in \mathcal{A}.$$

An element $x \in S_E$ is called a *peak point* for $\mathcal{A}$ if there exists $h \in \mathcal{A}$ such that $|h(x)| = \|h\|_{B_E}$ and $|h(z)| < \|h\|_{B_E}$ for every $z \in B_E \setminus \{x\}$. In this case we say that h peaks at x . A peak point x is called a *strong peak point* for $\mathcal{A}$ if there exists $h \in \mathcal{A}$ such that $|h(x)| = \|h\|_{B_E}$ and, given $\epsilon > 0$, there exists a $\delta > 0$ such that for every $z \in B_E$ with $\|z - x\| > \epsilon$, we have $|h(z)| < \|h\|_{B_E} - \delta$. In this case we say that h peaks strongly at x .

In 1971, Harris [9] introduced the definition of a spatial numerical range for a bounded and holomorphic function defined on a Banach space and the corresponding concept of numerical radius. The spatial numerical range of a bounded function f from B_E to E is given by

$$W(f) := \{x^*(f(x)) : (x, x^*) \in \Pi(E)\},$$

where we denoted by $\Pi(E)$ the following subset:

$$\Pi(E) := \{(x, x^*) \in S_E \times S_{E^*} : x^*(x) = 1\}.$$

The numerical radius $v(f)$ is just the number

$$v(f) := \sup\{|\lambda| : \lambda \in W(f)\}.$$

For more background and information about numerical ranges and radii, we refer the reader to [6]. For a linear space $\mathcal{A} \subset \mathcal{C}_b(B_E : E)$, M. Acosta and the author [3] gave the corresponding definition of numerical boundary for $\mathcal{A}$. We say that $B \subset \Pi(E)$ is called a *numerical boundary* for $\mathcal{A}$ if

$$\sup_{(x, x^*) \in B} |x^*(f(x))| = v(f), \quad \forall f \in \mathcal{A}.$$

In the case that B is a $(\|\cdot\| \times w^*)$ -closed numerical boundary for $\mathcal{A}$ that is minimal under the previous conditions, B is said to be the *numerical Šilov boundary*. In [3], the authors studied numerical boundaries for holomorphic functions on some classical Banach spaces. Parallel to the concepts of peak and strong peak-points, we introduce the corresponding definition of numerical and numerical strong-peak points. An element $(x, x^*) \in \Pi(E)$ is called a *numerical peak point* for $\mathcal{A}$ if there exists $h \in \mathcal{A}$ such that $|x^*(h(x))| = v(h)$ and $|z^*(h(z))| < v(h)$ for every $(z, z^*) \in \Pi(E) \setminus \{(x, x^*)\}$. In this case we say that h peaks numerically at (x, x^*) . Also, $(x, x^*) \in \Pi(E)$ is called a *numerical strong-peak point* for $\mathcal{A}$ if there exists a function $h \in \mathcal{A}$ such that $|x^*(h(x))| = v(h)$, and for any $(x_n, x_n^*) \in \Pi(E)$ with $\lim_{n \rightarrow \infty} |x_n^*(h(x_n))| = v(h)$ we have that $x_n \rightarrow x$ in norm and $x_n^* \rightarrow x^*$ in w^* -topology. In this case we say that h peaks strong-numerically at (x, x^*) . It is clear that every numerical strong-peak point is a numerical peak point. Note that if E is a finite dimensional space, then every numerical peak point is also a numerical strong-peak point. It is immediate from definition that every $(\|\cdot\| \times w^*)$ -closed numerical boundary for $\mathcal{A}_\infty(B_E : E)$ contains all numerical strong-peak points.

In Section 2, we characterize the numerical peak points for $\mathcal{A}_\infty(B_{l_1} : l_1)$ and prove that every numerical peak point for $\mathcal{A}_\infty(B_{l_1} : l_1)$ is also a numerical strong-peak point. We prove that $\{(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1 \text{ for all } n \in \mathbb{N}\}$ is the numerical Šilov boundary for $\mathcal{A}_\infty(B_{l_1} : l_1)$.

In Section 3, we characterize the numerical peak points for $\mathcal{A}_\infty(B_{C(K)} : C(K))$ when K is a compact metrizable space. If K is an infinite compact Hausdorff topological space, we prove that there are no numerical strong-peak points for $\mathcal{A}_\infty(B_{C(K)} : C(K))$.

2. NUMERICAL PEAK POINTS AND NUMERICAL ŠILOV BOUNDARY FOR HOLOMORPHIC FUNCTIONS ON l_1

Lemma 2.1. *If $(x, x^*) \in \Pi(l_1)$ is a numerical peak point for $\mathcal{A}_\infty(B_{l_1} : l_1)$, then $|x^*(e_n)| = 1$ for all $n \in \mathbb{N}$.*

Proof. Let $x := (v_n)_n \in S_{l_1}$, $x^* := (w_n)_n \in S_{l_\infty}$. Assume that there exists a positive integer n_0 such that $|w_{n_0}| < 1$. Since the subset of peak points in S_E for $\mathcal{A}_\infty(B_{l_1})$ is invariant under surjective linear isometries on l_1 , we may assume that $n_0 = 1$, so $|w_1| < 1$. Let $h \in \mathcal{A}_\infty(B_{l_1} : l_1)$ peak numerically at (x, x^*) with $h := (h_n)_n$ for some $h_n \in \mathcal{A}_\infty(B_{l_1})$. Let $x_\lambda^* := \lambda e_1 + \sum_{n>1} w_n e_n$ for $|\lambda| \leq 1$. We claim that $v_1 \neq 0$. If $v_1 = 0$, then $(x, x_\lambda^*) \in \Pi(l_1)$ for $|\lambda| \leq 1$. We define the polynomial $\psi : \overline{D}(0, 1) \rightarrow \mathbb{C}$ by

$$\psi(\lambda) := \lambda h_1(x) + \sum_{n>1} w_n h_n(x) \quad (|\lambda| \leq 1).$$

Since $\max_{|\lambda| \leq 1} |\psi(\lambda)| = v(h) = |x^*(h(x))| = |\psi(w_1)|$, by the Maximum Modulus Theorem, $\psi(\lambda) = v(h)$ for all $|\lambda| \leq 1$. Choose any complex number β with $|\beta| = 1$. Then $(x, x_\beta^*) \neq (x, x^*) \in \Pi(l_1)$ and $|\psi(\beta)| = |x_\beta^*(h(x))| = v(h)$. Since h peaks numerically at (x, x^*) , we have a contradiction. Thus $v_1 \neq 0$. We define the 1-degree polynomial $u : \overline{D}(0, 1) \rightarrow \mathbb{C}$ by

$$u(\lambda) := \lambda v_1 + \sum_{n>1} w_n v_n \quad (|\lambda| \leq 1).$$

Since $1 = x^*(x) = |u(w_1)|$, by the Maximum Modulus Theorem, $u(\lambda) = 1$ for all $|\lambda| \leq 1$. By the same reason as in the above argument, $(x, x_\beta^*) \in \Pi(l_1)$ and $|\psi(\beta)| = |x_\beta^*(h(x))| = v(h)$ for any $\beta \in \mathbb{C}$ with $|\beta| = 1$. Thus we have a contradiction. Therefore $|w_1| = 1$. $\square$

Theorem 2.2. *$S := \{(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1 \text{ for all } n \in \mathbb{N}\}$ is the set of all numerical strong-peak points for the space of 2-degree polynomials in $\mathcal{A}_\infty(B_{l_1} : l_1)$ and every numerical peak point for the space of 2-degree polynomials in $\mathcal{A}_\infty(B_{l_1} : l_1)$ is a numerical strong-peak point.*

Proof. By Lemma 2.1, it is enough to show the first statement of the theorem. Let $(y_0, y_0^*) \in S$. We will prove that (y_0, y_0^*) is a numerical strong-peak point for the space of 2-degree polynomials in $\mathcal{A}_\infty(B_{l_1} : l_1)$. Let $J = \text{supp}(y_0)$. Since the subset of peak points in S_E for $\mathcal{A}_\infty(B_{l_1})$ is invariant under surjective linear isometries on l_1 , we can assume that $y_0(k) > 0$ for all $k \in J$. Let $a_n := y_0^*(e_n)$ for all $n \in \mathbb{N}$. Then $a_k = 1$ for all $k \in J$. By Theorem 2.6 in [4] there exists a 2-degree polynomial $f \in \mathcal{A}_\infty(B_{l_1})$ which peaks strongly at y_0 with $\|f\| = 1$. Let $\lambda_n := \text{sign}(a_n)$ and $z_0 := \sum_{n=1}^{\infty} \lambda_n b_n e_n \in S_{l_1}$ with $b_n > 0$ for all $n \in \mathbb{N}$. We define a 2-degree polynomial $h \in \mathcal{A}_\infty(B_{l_1} : l_1)$ by $h(x) := f(x)z_0$ ($x \in l_1$). Clearly $y_0^*(h(y_0)) = 1 = \|h\| = v(h)$. We claim that h peaks strong-numerically at (y_0, y_0^*) . Consider a sequence $\{(x_n, x_n^*)\}$ in $\Pi(l_1)$ such that $\lim_{n \rightarrow \infty} |x_n^*(h(x_n))| = v(h)$. We

will prove that $x_n \rightarrow y_0$ in norm and $x_n^* \rightarrow y_0^*$ in w^* -topology. Since

$$1 = \lim_{n \rightarrow \infty} |x_n^*(h(x_n))| \leq \lim_{n \rightarrow \infty} |f(x_n)| \leq 1$$

and f peaks strongly at y_0 , $x_n \rightarrow y_0$ in norm. Write $x_n^* := (a_j^{(n)})_j \in S_{l_\infty}$ and $x_n := (c_j^{(n)})_j \in S_{l_1}$ for all $n, j \in \mathbb{N}$. Let $x := (d_j) \in l_1$. First we claim that $\lim_{n \rightarrow \infty} a_k^{(n)} = 1$ for all $k \in J$. Let $i_0 \in J$ be fixed. Let $\{a_{i_0}^{(n_i)}\}$ be any subsequence of $\{a_{i_0}^{(n)}\}$. As the set $\{a_{i_0}^{(n_i)}\}$ is a bounded subset of $\mathbb{C}$, there exist a subsequence $\{a_{i_0}^{(n_{i_l})}\}$ and a complex number r with $|r| \leq 1$ such that $\lim_{l \rightarrow \infty} a_{i_0}^{(n_{i_l})} = r$. Since $1 = x_n^*(x_n)$ for all $n \in \mathbb{N}$, we have

$$1 = \sum_{j=1}^{\infty} a_j^{(n_{i_l})} c_j^{(n_{i_l})} \text{ for all } l \in \mathbb{N}.$$

Since $x_n \rightarrow y_0$ in norm, we have $\lim_{n \rightarrow \infty} c_{i_0}^{(n)} = y_0(i_0) > 0$. For a sufficiently large l , $c_{i_0}^{(n_{i_l})} > 0$. It follows that, for a sufficiently large l ,

$$\begin{aligned} 1 &= a_{i_0}^{(n_{i_l})} c_{i_0}^{(n_{i_l})} + \sum_{j \neq i_0}^{\infty} a_j^{(n_{i_l})} c_j^{(n_{i_l})} \\ &\leq |a_{i_0}^{(n_{i_l})}| c_{i_0}^{(n_{i_l})} + \sum_{j \neq i_0}^{\infty} |a_j^{(n_{i_l})}| |c_j^{(n_{i_l})}| \\ &\leq \sum_{j=1}^{\infty} |c_j^{(n_{i_l})}| = 1. \end{aligned}$$

Thus $1 = a_{i_0}^{(n_{i_l})} = |a_{i_0}^{(n_{i_l})}|$ for a sufficiently large l , so $r = 1$. Therefore $\lim_{n \rightarrow \infty} a_{i_0}^{(n)} = 1$. We claim that $\lim_{n \rightarrow \infty} a_j^{(n)} = a_j$ for all $j \in \mathbb{N} \setminus J$. Let $j_0 \in \mathbb{N} \setminus J$ be fixed. Let $\{a_{j_0}^{(n_i)}\}$ be any subsequence of $\{a_{j_0}^{(n)}\}$. As the set $\{a_{j_0}^{(n_i)}\}$ is a bounded subset of $\mathbb{C}$, there exist a subsequence $\{a_{j_0}^{(n_{i_l})}\}$ and a complex number ρ with $|\rho| \leq 1$ such that $\lim_{l \rightarrow \infty} a_{j_0}^{(n_{i_l})} = \rho$. Since

$$1 = \lim_{n \rightarrow \infty} |x_n^*(h(x_n))| \leq \lim_{n \rightarrow \infty} |f(x_n)| |x_n^*(z_0)| \leq 1,$$

we have

$$1 = \lim_{n \rightarrow \infty} |x_n^*(z_0)| = \lim_{n \rightarrow \infty} \left| \sum_{j=1}^{\infty} a_j^{(n)} \lambda_j b_j \right|.$$

It follows that for fixed $i_0 \in J$,

$$\begin{aligned} 1 &= \lim_{l \rightarrow \infty} \left| \sum_{j=1}^{\infty} a_j^{(n_{i_l})} \lambda_j b_j \right| \\ &\leq \lim_{l \rightarrow \infty} |a_{j_0}^{(n_{i_l})} \lambda_{j_0} b_{j_0} + a_{i_0}^{(n_{i_l})} b_{i_0}| + \sum_{j \in \mathbb{N} \setminus \{i_0, j_0\}} b_j \\ &\leq b_{j_0} + b_{i_0} + \sum_{j \in \mathbb{N} \setminus \{i_0, j_0\}} b_j = 1. \end{aligned}$$

Thus

$$\lim_{l \rightarrow \infty} |a_{j_0}^{(n_{i_l})} \lambda_{j_0} b_{j_0} + a_{i_0}^{(n_{i_l})} b_{i_0}| = |\rho \lambda_{j_0} b_{j_0} + b_{i_0}| = b_{j_0} + b_{i_0},$$

showing $\rho \lambda_{j_0} = 1$. We have $\rho = a_{j_0}$. Therefore $\lim_{n \rightarrow \infty} a_{j_0}^{(n)} = a_{j_0}$. Since the w^* -topology in l_∞ is the convergence in each coordinate, we complete the proof. $\square$

By Theorem 2.3 in [3], $\Pi(l_p)$ is the numerical Šilov boundary for $\mathcal{A}_\infty(B_{l_p} : l_p)$ for each $1 < p < \infty$. The following is an application of Theorem 2.2 to the numerical Šilov boundary.

Theorem 2.3. $S := \{(x, x^*) \in \Pi(l_1) : |x^*(e_n)| = 1 \text{ for all } n \in \mathbb{N}\}$ is the numerical Šilov boundary for $\mathcal{A}_\infty(B_{l_1} : l_1)$.

Proof. It is easy to show that S is $(\|\cdot\| \times w^*)$ -closed. By Theorem 2.2 and the fact that every $(\|\cdot\| \times w^*)$ -closed numerical boundary for $\mathcal{A}_\infty(B_{l_1} : l_1)$ contains all numerical strong-peak points, it suffices to prove that S is a numerical boundary for $\mathcal{A}_\infty(B_{l_1} : l_1)$. Let $h \in \mathcal{A}_\infty(B_{l_1} : l_1)$ and let $\epsilon > 0$. Choose $(x_0, x_0^*) \in \Pi(l_1)$ such that $v(h) - \epsilon < |x_0^*(h(x_0))|$. Let $x_0 := (v_n)_n$, $x_0^* := (w_n)_n$ and $h := (h_n)_n$ for some $h_n \in \mathcal{A}_\infty(B_{l_1})$. We claim that there exists a complex sequence $\{\lambda_n\}$ in the unit disk such that if

$$z_n^* := \sum_{1 \leq j \leq n} \lambda_j e_j + \sum_{j > n} w_j e_j$$

for all $n \in \mathbb{N}$, then $(x_0, z_n^*) \in \Pi(l_1)$ and $|z_{n+1}^*(h(x_0))| \geq |z_n^*(h(x_0))| > v(h) - \epsilon$ for all $n \in \mathbb{N}$. If $|w_1| = 1$, let $\lambda_1 := w_1$. Otherwise $w_1 = 0$; hence

$$(x_0, \lambda e_1 + \sum_{n > 1} w_n e_n) \in \Pi(l_1) \text{ for all } |\lambda| \leq 1.$$

If $h_1(x_0) = 0$, let $\lambda_1 := 1$. Assume that $h_1(x_0) \neq 0$. We define a nonconstant 1-degree polynomial $\psi_1 : \overline{D}(0, 1) \rightarrow \mathbb{C}$ by

$$\psi_1(\lambda) := \lambda h_1(x_0) + \sum_{n > 1} w_n h_n(x_0) \quad (|\lambda| \leq 1).$$

Since $\max_{|\lambda|=1} |\psi_1(\lambda)| \geq |\psi_1(w_1)| = |x_0^*(h(x_0))|$, by the Maximum Modulus Theorem, there exists a complex number λ_1 with $|\lambda_1| = 1$ such that

$$|\psi_1(\lambda_1)| \geq |x_0^*(h(x_0))| > v(h) - \epsilon.$$

Let $z_1^* := \lambda_1 e_1 + \sum_{n > 1} w_n e_n$. Then $(x_0, z_1^*) \in \Pi(l_1)$ and $|z_1^*(h(x_0))| \geq |x_0^*(h(x_0))| > v(h) - \epsilon$. If $|w_2| = 1$, let $\lambda_2 := w_2$. Otherwise $w_2 = 0$; hence

$$(x_0, \lambda_1 e_1 + \lambda e_2 + \sum_{n > 2} w_n e_n) \in \Pi(l_1) \text{ for all } |\lambda| \leq 1.$$

If $h_2(x_0) = 0$, let $\lambda_2 := 1$. Assume that $h_2(x_0) \neq 0$. We define a nonconstant 1-degree polynomial $\psi_2 : \overline{D}(0, 1) \rightarrow \mathbb{C}$ by

$$\psi_2(\lambda) := \lambda_1 h_1(x_0) + \lambda h_2(x_0) + \sum_{n > 2} w_n h_n(x_0) \quad (|\lambda| \leq 1).$$

Since $\max_{|\lambda|=1} |\psi_2(\lambda)| \geq |\psi_2(w_2)| = |\psi_1(\lambda_1)| > v(h) - \epsilon$, by the Maximum Modulus Theorem, there exists a complex number λ_2 with $|\lambda_2| = 1$ such that

$$|\psi_2(\lambda_2)| \geq |\psi_1(\lambda_1)| > v(h) - \epsilon.$$

Let $z_2^* := \sum_{1 \leq j \leq 2} \lambda_j e_j + \sum_{n > 2} w_n e_n$. Then $(x_0, z_2^*) \in \Pi(l_1)$ and $|z_2^*(h(x_0))| \geq |z_1^*(h(x_0))| > v(h) - \epsilon$. Continuing this process, we can get a complex sequence

$\{\lambda_n\}$ in the unit disk satisfying the claim. Let $z^* := (\lambda_n)_{n=1}^\infty \in l_\infty$. We will show that $(x_0, z^*) \in S$. Indeed, it follows that for each $n \in \mathbb{N}$,

$$|z^*(x_0) - 1| = |z^*(x_0) - z_n^*(x_0)| \leq \sum_{j>n} |\lambda_j - w_j| |v_j| \leq 2 \sum_{j>n} |v_j| \rightarrow 0,$$

as $n \rightarrow \infty$. Thus $z^*(x_0) = 1$. We will show that

$$\lim_{n \rightarrow \infty} |z_n^*(h(x_0))| = |z^*(h(x_0))|.$$

Let $h(x_0) := (\beta_n)_{n=1}^\infty \in l_1$. It follows that

$$|z^*(h(x_0)) - z_n^*(h(x_0))| \leq \sum_{j>n} |\lambda_j - w_j| |\beta_j| \leq 2 \sum_{j>n} |\beta_j| \rightarrow 0,$$

as $n \rightarrow \infty$. Thus we have

$$v(h) - \epsilon < \limsup_{n \rightarrow \infty} |z_n^*(h(x_0))| = |z^*(h(x_0))| \leq \sup_{(x, x^*) \in S} |x^*(h(x))| \leq v(h),$$

which shows that $\sup_{(x, x^*) \in S} |x^*(h(x))| = v(h)$. Since $h \in \mathcal{A}_\infty(B_{l_1} : l_1)$ is arbitrary, we complete the proof. $\square$

3. NUMERICAL PEAK POINTS AND NUMERICAL STRONG-PEAK POINTS ON $C(K)$

Theorem 3.1. *Let K be a compact Hausdorff topological space with at least two points. If $(x_0, x_0^*) \in \Pi(C(K))$ is a numerical peak point for $\mathcal{A}_\infty(B_{C(K)} : C(K))$, then:*

- (a) *There exists a unique $t_0 \in K$ such that*

$$|x_0(t)| = 1, \quad \forall t \in K \text{ and } x_0^* = \text{sign}(x_0(t_0))\delta_{t_0}.$$

Hence x_0 is an extreme point of $B_{C(K)}$.

- (b) *x_0 is a peak point for $\mathcal{A}_\infty(B_{C(K)})$.*

Proof. Let $h \in S_{\mathcal{A}_\infty(B_{C(K)} : C(K))}$ peak numerically at (x_0, x_0^*) . Then $v(h) = |x_0^*(h(x_0))|$. By Theorem 2.7 in [2], $1 = v(h) = \|h\|$. Choose an element $t_0 \in K$ such that $1 = \|h(x_0)\| = |\delta_{t_0} \circ h(x_0)| = \|\delta_{t_0} \circ h\|$. We claim that $|x_0(t)| = 1, \forall t \in K$. For every complex number λ in the unit disk, the function $x_0 + \lambda(1 - |x_0|) \in C(K)$, and for every $t \in K$, it is satisfied that $|x_0(t) + \lambda(1 - |x_0(t)|)| \leq 1$. Define the continuous function $\phi : \overline{D}(0, 1) \rightarrow \mathbb{C}$ by

$$\phi(\lambda) := \delta_{t_0}(h(x_0 + \lambda(1 - |x_0|))) \quad (|\lambda| \leq 1).$$

Note that ϕ is holomorphic on $D(0, 1)$ and $|\phi(\lambda)| \leq 1$ for every λ in the unit disk. Also $|\phi(0)| = |\delta_{t_0} \circ h(x_0)| = 1$. Since ϕ attains its maximum modulus at 0, ϕ is constant. We choose a complex number λ_0 satisfying the facts that $|\lambda_0| = 1$ and $|x_0(t_0) + \lambda_0(1 - |x_0(t_0)|)| = 1$. So $\phi(0) = \phi(\lambda_0)$. The element $z_0 := x_0 + \lambda_0(1 - |x_0|)$ is in the unit ball of $C(K)$ and $|z_0(t_0)| = 1$. Then $(z_0, \overline{z_0(t_0)}\delta_{t_0}) \in \Pi(C(K))$. Since $|\phi(\lambda_0)| = |\overline{z_0(t_0)}\delta_{t_0}(h(z_0))| = v(h) = 1$, we have $z_0 = x_0$ and $x_0^* = \overline{z_0(t_0)}\delta_{t_0}$. Thus $\lambda_0(1 - |x_0(t)|) = 0$ for all $t \in K$, so $|x_0(t)| = 1$ for all $t \in K$. So $x_0^* = \text{sign}(x_0(t_0))\delta_{t_0}$. The uniqueness of t_0 follows from Urysohn's Lemma. Therefore we have proved assertion (a).

We will show that $\delta_{t_0} \circ h$ peaks at x_0 . Let $y \in S_{C(K)}$ such that $1 = |\delta_{t_0} \circ h(y)| = \|\delta_{t_0} \circ h\|$. By the same argument as in the proof of assertion (a), we have $|y(t_0)| = 1$. Note that $(y, \text{sign}(y(t_0))\delta_{t_0}) \in \Pi(C(K))$. Since

$$1 = |\text{sign}(y(t_0))\delta_{t_0}(h(y))| = \|\delta_{t_0} \circ h\| = \|h\| = v(h),$$

we have $y = x_0$ and $x_0^* = \text{sign}(y(t_0))\delta_{t_0}$, which show assertion (b). $\square$

Theorem 3.2. (1) *Let K be a compact metrizable space.*

Then $M := \{(x, \text{sign}(x(t))\delta_t) : x \in \text{ext}B_{C(K)}, t \in K\}$ is the set of all numerical peak points for the space of 1-degree polynomials in $\mathcal{A}_\infty(B_{C(K)} : C(K))$.

(2) *If K is any infinite compact topological space, then there are no numerical strong-peak points for $\mathcal{A}_\infty(B_{C(K)} : C(K))$.*

Proof. (1): By (a) of Theorem 3.1, it suffices to show that if $(x_0, \text{sign}(x_0(t_0))\delta_{t_0}) \in M$, then it is a numerical peak point for the space of 1-degree polynomials in $\mathcal{A}_\infty(B_{C(K)} : C(K))$. Since the subset of peak points in $S_{C(K)}$ for $\mathcal{A}_\infty(B_{C(K)})$ is invariant under surjective linear isometries on $C(K)$, we can assume that $x_0(t) = 1$ for all $t \in K$. Since K is a metrizable space, there exists a dense subset $\{l_n\}$ in K . Choose $(\alpha_n) \in S_{l_1}$ with $\alpha_n > 0$ for all $n \in \mathbb{N}$. We claim that there exists a function $y_0 \in S_{C(K)}$ such that $y_0(t_0) = 1$ and $0 \leq y_0(t) < 1$ for all $t \in K \setminus \{t_0\}$. Indeed, let d be a metric in K . Let

$$A_n := \{t \in K : d(t, t_0) \geq \frac{1}{n}\} \quad (n \in \mathbb{N}).$$

Clearly A_n is a closed subset of K with $t_0 \notin A_n$ for all $n \in \mathbb{N}$. By the Tietze Extension Theorem, for each $n \in \mathbb{N}$, there exists a sequence $\{z_n\}$ in $C(K)$ such that $0 \leq z_n \leq 1$, $z_n(t_0) = 1$ and $z_n(A_n) = \{0\}$. We define the function

$$y_0(t) := \sum_{n=1}^{\infty} \frac{1}{2^n} z_n(t) \quad (t \in K).$$

We define a 1-degree polynomial $h \in \mathcal{A}_\infty(B_{C(K)} : C(K))$ by

$$h(x) := \sum_{n=1}^{\infty} \alpha_n (1 + x(l_n)) y_0 \quad (x \in C(K)).$$

We claim that h peaks numerically at (x_0, δ_{t_0}) . Let $(z_0, z_0^*) \in \Pi(C(K))$ be such that $v(h) = |z_0^*(h(z_0))|$. Since $2 \geq \|h\| \geq v(h) \geq \delta_{t_0}(h(x_0)) = 2$, we have $2 = v(h) = \|h\|$. Since $2 = |z_0^*(h(z_0))| = \|h(z_0)\|$, we have $z_0(l_n) = 1$ for all $n \in \mathbb{N}$. Since $\{l_n\}$ is a dense subset of K , $z_0(t) = 1 = x_0(t)$ for all $t \in K$. Thus $z_0 = x_0$. By the Riesz Representation Theorem on $C(K)^*$, there exists a unique regular complex Baire measure $\mu = v + iw$ on K (v and w are positive measures) satisfying

$$z_0^*(x) = \int_K x(t) d\mu = \int_K x(t) dv + i \int_K x(t) dw \quad (x \in C(K))$$

with $\|z_0^*\| = |\mu| = 1$. Since

$$1 = z_0^*(x_0) = \int_K x_0 dv + i \int_K x_0 dw = v(K) + iw(K),$$

$v(K) = 1$, $w(K) = 0$. Thus $w = 0$ and

$$z_0^*(x) = \int_K x(t) dv \quad (x \in C(K)).$$

It follows that

$$2 = |z_0^*(h(x_0))| = \left| \int_K h(x_0) dv \right| = 2 \left| \int_K y_0(t) dv \right| \leq 2 \int_K |y_0(t)| dv \leq 2.$$

Thus $1 = \int_K |y_0(t)| dv$. We claim that $v(\{t_0\}) = 1$. Otherwise, by the regularity of v and the choice of y_0 , there exists an open subset θ_0 of K containing t_0 such that $v(K \setminus \theta_0) > 0$. Let $\delta_0 := \max_{t \in K \setminus \theta_0} |y_0(t)| < 1$. It follows that

$$1 = \int_K |y_0(t)| dv \leq v(\theta_0) + \delta_0 v(K \setminus \theta_0) < v(K) = 1,$$

which is impossible. Thus $v(\{t_0\}) = 1$ and $v(K \setminus \{t_0\}) = 0$. Therefore, we have

$$z_0^*(x) = \int_K x(t) dv = x(t_0) v(\{t_0\}) = x(t_0) = \delta_{t_0}(x)$$

for all $x \in C(K)$, showing $z_0^* = \delta_{t_0}$.

(2): By (a) of Theorem 3.1, it suffices to show that if $(x_0, x_0^*) \in \Pi(C(K))$ with $x_0 \in \text{ext}B_{C(K)}$, then (x_0, x_0^*) is not a numerical strong-peak point. Let $\{t_n\}$ in K be the sequence such that there is a sequence $\{x_n\}$ in $B_{C(K)}$ such that $0 \leq x_n \leq 1$, $x_n(t_n) = 1$, for all n and $\text{supp}(x_n) \cap \text{supp}(x_m) = \emptyset$ ($n \neq m$). Let $h \in \mathcal{A}_\infty(B_{C(K)} : C(K))$ such that $1 = v(h) = |x_0^*(h(x_0))|$. We will show that h cannot peak strong-numerically at (x_0, x_0^*) . By Theorem 2.7 in [2], we have $v(h) = \|h\| = 1$. There exists a $t_0 \in K$ such that $\text{sign}(x_0(t_0))\delta_{t_0} \circ h \in \mathcal{A}_\infty(B_{C(K)})$ and

$$1 = \|h(x_0)\| = |\delta_{t_0}(h(x_0))| = |\text{sign}(x_0(t_0))\delta_{t_0}(h(x_0))| = \|\delta_{t_0} \circ h\|.$$

Let $z_n := x_0(1 - x_n)$ for all $n \in \mathbb{N}$. Since the support of x_n are pairwise disjoint, there is a positive integer N such that $x_n(t_0) = 0$ for all $n > N$. Let $\lambda_n := \text{sign}(z_n(t_0))$ for all $n > N$. Thus $(z_n, \lambda_n \delta_{t_0}) \in \Pi(C(K))$ for all $n > N$. Since $\{x_n\}$ is equivalent to a c_0 -basis, then it converges weakly to 0. By the Rainwater theorem, the sequence $\{z_n\}$ is in the unit ball of $C(K)$ and converges weakly to x_0 . Since $C(K)$ has the Dunford-Pettis property, then it has also the polynomial Dunford-Pettis property [10], and so, if we follow the argument in the proof Proposition 4.1 in [1], then

$$|\lambda_n \delta_{t_0} \circ h(z_n)| \rightarrow 1.$$

It follows that

$$\|z_n - x_0\| = \|x_0 x_n\| \geq |x_0(t_n) x_n(t_n)| = 1$$

for all $n \in \mathbb{N}$. Therefore, we have proved that (x_0, x_0^*) is not a numerical strong-peak point. $\square$

It is known in [3], Theorem 5.2, that there is no numerical Šilov boundary for $\mathcal{A}_\infty(B_{C(K)} : C(K))$ if K is an infinite compact Hausdorff topological space.

Corollary 3.3. Let $n \in \mathbb{N}$. Then $M := \{(x, \text{sign}(x(t))\delta_t) \in \Pi(l_\infty^n) : |x(k)| = 1 \text{ for all } k = 1, 2, \dots, n, \text{ for some } t = 1, 2, \dots, n\}$ is the numerical Šilov boundary for $\mathcal{A}_\infty(B_{l_\infty^n} : l_\infty^n)$.

Proof. ($\subseteq$): By Proposition 5.1 in [3], it follows.

($\supseteq$): Clearly M is $(\|\cdot\| \times w^*)$ -closed. By Theorem 3.2 (1), M is the set of all numerical peak points for $\mathcal{A}_\infty(B_{l_\infty^n} : l_\infty^n)$. Since l_∞^n is finite dimensional, M is the set of all numerical strong-peak points. $\square$

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DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, DAEGU 702-701, SOUTH KOREA

E-mail address: `sgk317@knu.ac.kr`