A COUNTEREXAMPLE TO THE MAXIMALITY OF TORIC VARIETIES

VALERIE HOWER

(Communicated by Ted Chinburg)

ABSTRACT. We present a counterexample to the conjecture of Bihan, Franz, McCrory, and van Hamel concerning the maximality of toric varieties. There exists a six dimensional projective toric variety X with the sum of the $\mathbb{Z}_2$ Betti numbers of $X(\mathbb{R})$ strictly less than the sum of the $\mathbb{Z}_2$ Betti numbers of $X(\mathbb{C})$.

Let $X = X(\mathbb{C})$ be a complex algebraic variety defined by equations with real coefficients. Complex conjugation $\sigma : X \rightarrow X$ acts on X , and the fixed points X^σ are the real points of X , denoted $X(\mathbb{R})$. It is of interest to try to understand the relationship between the topology of $X(\mathbb{C})$ and that of $X(\mathbb{R})$. If one looks at the $\mathbb{Z}_2$ Borel-Moore Betti numbers, the Smith-Thom inequality [Deg], §1.1, states

$$\sum_i b_i(X(\mathbb{R})) \leq \sum_j b_j(X(\mathbb{C})).$$

We say X is *maximal* when equality is achieved.

Toric varieties are defined by equations with integral coefficients. See Fulton [Ful] for background on toric varieties. Thus, if X is a toric variety, we may consider $X(\mathbb{R})$ the real points of X . In [Bih], Bihan et al. define a spectral sequence $G_{p,q}^r$ that converges to the $\mathbb{Z}_2$ Borel-Moore homology of $X(\mathbb{R})$ when X is a toric variety. We restrict ourselves to projective toric varieties, where the $\mathbb{Z}_2$ Borel-Moore homology is isomorphic to the ordinary $\mathbb{Z}_2$ homology. We also use different notation and indexing than in [Bih]. We write this spectral sequence as $\overline{E}_{p,q}^r$, where $G_{p,q}^r = \overline{E}_{p+q,-p}^r$. For each p and q we have

$$(1) \quad \overline{E}_{p,q}^1 \cong E_{p,q}^2,$$

where $E_{p,q}^r$ is the $\mathbb{Z}_2$ Leray spectral sequence for the moment map of $X(\mathbb{C})$. The spectral sequence $E_{p,q}^r$ converges to the ordinary $\mathbb{Z}_2$ homology of $X(\mathbb{C})$. Bihan et al. conjecture that every toric variety is maximal. Their conjecture is based on (1), numerous examples, and the fact that every nonsingular complete toric variety is maximal. In this paper, we present a counterexample to this conjecture. The author would like to thank Matthias Franz and Clint McCrory for ongoing discussions and emails.

Theorem. *There exists a six dimensional projective toric variety which is not maximal.*

Received by the editors May 4, 2007, and, in revised form, November 1, 2007.

2000 *Mathematics Subject Classification.* Primary 14M25, 14F45; Secondary 05B35.

©2008 American Mathematical Society
 Reverts to public domain 28 years from publication

To construct this toric variety, we will use a specific matroid. Matroids abstract the theory of dependence studied in linear algebra or graph theory. There are many equivalent definitions of a matroid; most are discussed in [Oxl]. A definition useful for our purposes involves a convex polytope P_M . Let $E \subset \binom{[n]}{k}$ be a collection of k element subsets of the ground set $[n] := \{1, 2, 3, \dots, n\}$. We define a convex polytope P via

$$P := \text{conv}\{e_I = \sum_{i \in I} e_i \mid I \in E\},$$

where e_i is the i th standard basis vector for $\mathbb{R}^n$. If each edge of P is parallel to $e_i - e_j$ for some i, j , then the set E is the set of *bases* of a *matroid* M . In this case $P = P_M$ is the *matroid polytope* for the matroid M . Subsets of bases are *independent sets*, and the *rank* of a matroid is the cardinality of one of its bases.

We define a toric variety from a matroid as in [Gel]. Let M be the rank 3 matroid F_7 , the Fano plane. We remind the reader that M is the projective plane over $\mathbb{F}_2$. The affine dependencies of M are depicted below, and the bases of M are the triangles in the diagram.

Next, consider the projective space $\mathbb{CP}^{27}$ whose coordinates are given by the bases of M

$$\{y_{ijk} \mid ijk \in \binom{[7]}{3} - \{123, 147, 156, 246, 257, 345, 367\}\}.$$

We define a linear action of the complex algebraic torus

$$T := (\mathbb{C}^*)^7 = \{(t_1, t_2, t_3, t_4, t_5, t_6, t_7) \mid t_i \in \mathbb{C}^*\}$$

on $\mathbb{CP}^{27}$ where the action on the coordinates is

$$(t_1, t_2, t_3, t_4, t_5, t_6, t_7) \cdot y_{ijk} = t_i t_j t_k y_{ijk}.$$

Note that the one-dimensional subtorus $\{(t, t, t, t, t, t, t) \mid t \in \mathbb{C}^*\}$ acts trivially. By restricting this action to the subtorus

$$T' := \{(t_1, t_2, t_3, t_4, t_5, t_6, 1) \mid t_i \in \mathbb{C}^*\},$$

we obtain an effective action of a six-dimensional algebraic torus on $\mathbb{CP}^{27}$. Let $X(\mathbb{C})$ be the closure of the torus orbit $T' \cdot (1, 1, 1, \dots, 1)$ in $\mathbb{CP}^{27}$. Then, X is a six-dimensional projective toric variety. The moment polytope for X is the projection of the matroid polytope P_M onto the first six coordinates. We use `torhom` [Fra] to compute the ranks of the entries in $E_{p,q}^2 = \overline{E}_{p,q}^1$ for X , which are listed in (2), and

the $\mathbb{Z}_2$ Betti numbers for $X(\mathbb{R})$. The p th entry of the vector (3) is the rank of the $\mathbb{Z}_2$ vector space $H_p(X(\mathbb{R}), \mathbb{Z}_2)$.

$$(2) \quad \begin{array}{c|ccccccc} & & & & & & 1 \\ & & & & & 15 & 0 \\ & & & & 22 & 0 & 0 \\ & & 6 & 26 & 0 & 0 & \\ q & 2 & 3 & 9 & 0 & 0 & \\ & 1 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & p & & & & & \end{array}$$

$$(3) \quad [1 \ 1 \ 1 \ 8 \ 57 \ 15 \ 1]$$

As the total grading of the spectral sequence $\overline{E}_{p,q}^r$ is p , we see that the spectral sequence $\overline{E}_{p,q}^r$ does not collapse at $\overline{E}^1$. Moreover,

$$\sum_{i=0}^6 b_i(X(\mathbb{R})) = 84.$$

The total grading for the spectral sequence $E_{p,q}^r$ is $p + q$. Thus, the only possible nonzero higher differential is $d_{4,1}^2 : E_{4,1}^2 \rightarrow E_{2,2}^2$. If $d_{4,1}^2 = 0$, then the spectral sequence collapses at E^2 , the $\mathbb{Z}_2$ Betti numbers for $X(\mathbb{C})$, are

$$[1 \ 0 \ 1 \ 0 \ 3 \ 4 \ 15 \ 26 \ 22 \ 0 \ 15 \ 0 \ 1]$$

and

$$\sum_{i=0}^{12} b_i(X(\mathbb{C})) = 88.$$

If $d_{4,1}^2 \neq 0$, then the $\mathbb{Z}_2$ Betti numbers for $X(\mathbb{C})$ are

$$[1 \ 0 \ 1 \ 0 \ 2 \ 3 \ 15 \ 26 \ 22 \ 0 \ 15 \ 0 \ 1]$$

and

$$\sum_{i=0}^{12} b_i(X(\mathbb{C})) = 86.$$

In either case, we obtain

$$\sum_{i=0}^6 b_i(X(\mathbb{R})) < \sum_{i=0}^{12} b_i(X(\mathbb{C})),$$

and hence X is not maximal. $\square$

We do not know why X fails to be maximal. Nonetheless, this example does have some remarkable properties. For instance, the Fano plane is a matroid which is not realizable over any field $\mathbb{F}$ unless $\text{char } \mathbb{F} = 2$. Moreover, the polytope P_M with f -vector $(28, 126, 245, 238, 112, 21)$ is highly symmetric; the symmetry group of P_M is $\text{PGL}_3(\mathbb{F}_2)$. However at the current time, we are unable to connect these unique properties of M and P_M to the topology of X .

Other than slightly larger examples which contain X as the closure of a torus orbit, no other counterexamples are known. Hence, we have no intuition as to what features might distinguish maximal from nonmaximal toric varieties. As mentioned

above, all nonsingular complete toric varieties are maximal, and there are certainly classes of singular toric varieties which are maximal. For instance, in [How] we show that toric varieties associated to Fano polyhedra (defined in [Bat]) are maximal. This counterexample reveals that the relationship between the topology of the real and complex points of a toric variety is not as straightforward as one might hope. In the future, we would like to see a description, perhaps combinatorial, of the class of maximal toric varieties.

REFERENCES

- [Bat] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, *J. Algebraic Geom.* **3** (1994), 493–535. MR1269718 (95c:14046)
- [Bih] F. Bihan, M. Franz, C. McCrory, J. van Hamel, Is every toric variety an M-variety?, *Manuscripta Math.* **120** (2006), 217–232. MR2234250
- [Deg] A. Degtyarev, I. Itenberg, V. Kharlamov, *Real Enriques Surfaces*, Springer-Verlag, New York, 2000. MR1795406 (2001k:14100)
- [Fra] M. Franz, Maple package `torhom`, version 1.3.0, September 13, 2004, Available at <http://www-fourier.ujf-grenoble.fr/~franz/maple/torhom.html>
- [Ful] W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, Princeton, NJ, 1993. MR1234037 (94g:14028)
- [Gel] I.M. Gelfand, M. Goresky, R.D. MacPherson, V.V. Serganova, Combinatorial geometries, convex polyhedra, and Schubert cells, *Adv. Math.* **63** (1987), 301–316. MR877789 (88f:14045)
- [How] V. Hower, Hodge spaces of real toric varieties, *Collect. Math.* **59** (2008), 215–237.
- [Oxl] J.G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992. MR1207587 (94d:05033)

DEPARTMENT OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GEORGIA 30332

E-mail address: `vhower@math.gatech.edu`