A SUMMABILITY CRITERION FOR STOCHASTIC INTEGRATION

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ABSTRACT. In this paper we give simple, sufficient conditions for the existence of the stochastic integral for vector-valued processes X with values in a Banach space E ; namely, X is of class (LD), and the stochastic measure I_X is bounded and strongly additive in L_E^p (in particular, if I_X is bounded in L_E^p and $c_0 \not\subseteq E$) and has bounded semivariation. The result is then applied to martingales and processes with integrable variation or semivariation. For martingales the condition of being of class (LD) is superfluous. For a square-integrable martingale with values in a Hilbert space, all the conditions are superfluous. For processes with p -integrable semivariation or p -integrable variation, the conditions of I_X to be bounded and have bounded semivariation are superfluous. For processes with 1-integrable variation, all conditions are superfluous. In a forthcoming paper, we shall extend these results to local summability. The extension needs additional nontrivial work.

PRELIMINARIES

We shall use the following definitions and notation (cf. [D-M] and [D]).

1) $(\Omega, \mathcal{F}, P)$ is a *probability space* and $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a *filtration* satisfying the usual conditions; i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$, and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$, for $t \in \mathbb{R}$.

$\mathcal{R}$ is the ring generated by the semiring of predictable rectangles of the form $\{0\} \times A$ with $A \in \mathcal{F}_0$ and $(s, t] \times A$ with $s, t \in \mathbb{R}_+$ and $A \in \mathcal{F}_s$.

$\mathcal{P}$ is the σ -algebra of predictable sets, or the *predictable σ -algebra*, and is generated by the ring $\mathcal{R}$.

2) E, F, G are Banach spaces such that $E \subset L(F, G)$ continuously; i.e.,

$$|xy| \leq |x||y|, \text{ for } x \in E \text{ and } y \in F.$$

L_E^p will denote the space $L_E^p(P)$, $1 \leq p \leq +\infty$, and L^p will denote $L_{\mathbb{R}}^p$.

3) $X : \mathbb{R}_+ \times \Omega \longrightarrow E$ is a *cadlag, adapted* process and $1 \leq p < \infty$.

We assume that $X_t \in L_E^p$ for every $t \in \mathbb{R}_+$. We say X is of class (D) if the set of random variables $\{X_T \mathbf{1}_{\{T < \infty\}}; T \text{ a stopping time}\}$ is uniformly integrable (cf. [D-M, Definition VI.20]). We say X is of class (LD) if for every $a \in \mathbb{R}$, the set of random variables $\{X_T; T : \Omega \longrightarrow [0, a] \text{ a stopping time}\}$ is uniformly integrable

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([M, Definition 13.1]). We define the additive measure $I_X : \mathcal{R} \longrightarrow L_E^p$ first for the predictable rectangles by

$$I_X(\{0\} \times A) = X_0 \mathbf{1}_A, \text{ for } A \in \mathcal{F}_0,$$

and

$$I_X((s, t] \times A) = (X_t - X_s) \mathbf{1}_A, \text{ for } A \in \mathcal{F}_s,$$

and then by additivity, for the whole ring $\mathcal{R}$.

We say that the process X is p -summable relative to (F, L_G^p) if the measure I_X can be extended to a σ -additive measure $I_X : \mathcal{P} \longrightarrow L_E^p$ with bounded semivariation relative to (F, L_G^p) . (See [D], §4, for the definition and the properties of the semivariation.) If X is p -summable, we can define the stochastic integral $H \cdot X$ for certain predictable processes $H : \mathbb{R}_+ \times \Omega \longrightarrow F$.

The stochastic integral $H \cdot X$ is itself a process, and we have $(H \cdot X)_t \in L_G^p$ for $t \in \mathbb{R}_+$ (see [D, Chapter 2]).

4) A natural problem is to give simple, sufficient conditions for the summability of X . We show in this paper that if X is of class (LD) and if I_X is bounded and strongly additive (in particular, if I_X is bounded and $c_0 \not\subseteq E$), then X is p -summable relative to $(\mathbb{R}, L_G^p)$. If I_X also has bounded semivariation relative to (F, L_G^p) , then X is p -summable relative to (F, L_G^p) (Theorem 3). We apply this summability criterion to martingales (Theorems 4 and 5), to processes with integrable semivariation (Theorem 6) and to processes with integrable variation (Theorems 7 and 8). For martingales, the condition of class (LD) is automatically satisfied. For processes with p -integrable semivariation, if the embedding $E \subset L(F, G)$ is an isometry, then I_X is automatically bounded and has bounded semivariation relative to (F, L_G^p) . For processes with p -integrable variation, the isometry of $E \subset L(F, G)$ can be removed. Further, for $p = 1$, even the condition $c_0 \not\subseteq E$ can be removed.

EXTENSION OF THE MEASURE I_X

The main part of the summability criterion is the extension of the measure I_X . This is stated in terms of strong additivity of I_X . A measure $m : \mathcal{R} \longrightarrow E$ defined on a ring $\mathcal{R}$ is said to be strongly additive if for any sequence (A_n) of disjoint sets from $\mathcal{R}$ we have $m(A_n) \longrightarrow 0$. If m is bounded and $c_0 \not\subseteq E$, then m is strongly additive ([D, Theorem 6.8]). A σ -additive measure on a σ -ring is strongly additive, but the converse is not necessarily true. For a converse property, see the extension Theorem 7.7 in [D]. For the properties of a strongly additive measure, see [D], §6.

Theorem 1. *Let $X : \mathbb{R}_+ \times \Omega \longrightarrow E$ be a cadlag, adapted process such that $X_t \in L_E^p$ for every $t \in \mathbb{R}_+$. Assume that X is of class (LD), and that I_X is bounded and strongly additive in L_E^p on $\mathcal{R}$. Then I_X can be extended to a σ -additive measure $I_X : \mathcal{P} \longrightarrow L_E^p$.*

Proof. Let $Z = L_{E^*}^\infty \subset L_{E^*}^q$, $\frac{1}{p} + \frac{1}{q} = 1$. Then Z is a norming space for L_E^p . Let $g \in Z$. Regard the real-valued measure $\langle I_X, g \rangle : \mathcal{R} \longrightarrow \mathbb{R}$ defined by

$$\langle I_X, g \rangle(B) = \langle I_X(B), g \rangle = \int \langle I_X(B), g \rangle dP \text{ for } B \in \mathcal{R},$$

where the second bracket represents the duality between L_E^p and $L_{E^*}^q$ and the bracket in the integral represents the duality between E and E^* . For each $B \in \mathcal{R}$ we have

$$|\langle I_X, g \rangle(B)| \leq \|I_X(B)\|_p \|g\|_q;$$

hence $\langle I_X, g \rangle$ is bounded on $\mathcal{R}$. Since $\langle I_X, g \rangle$ is real-valued, it has bounded variation on $\mathcal{R}$ ([D, Proposition 2.16b]). Denote by G the cadlag version of the martingale $(E[g | \mathcal{F}_t])_{t \in \mathbb{R}_+}$. Henceforth, the context will determine whether we are referring to the Banach space G or the martingale G . We have

$$G_t = E[g | \mathcal{F}_t] \text{ a.s.}$$

and

$$|G_t| \leq E[|g| | \mathcal{F}_t] \leq \|g\|_\infty \text{ a.s.}$$

Consider the real-valued process XG defined by $(XG)_t = \langle X_t, G_t \rangle$ for $t \in \mathbb{R}_+$, where the bracket represents the duality between E and E^* . The Doolléans measure μ_{XG} of XG is defined for predictable rectangles by

$$\mu_{XG}(\{0\} \times A) = E[X_0 G_0 \mathbf{1}_A], \text{ for } A \in \mathcal{F}_0,$$

and

$$\mu_{XG}((s, t] \times A) = E[(X_t G_t - X_s G_s) \mathbf{1}_A], \text{ for } A \in \mathcal{F}_s,$$

and then by additivity on the whole ring $\mathcal{R}$.

We have $\mu_{XG} = \langle I_X, g \rangle$. In fact, if $B = \{0\} \times A$ with $A \in \mathcal{F}_0$, then

$$\begin{aligned} \langle I_X, g \rangle(B) &= \int \langle I_X(B), g \rangle dP = \int_A \langle X_0, g \rangle dP \\ &= \int_A \langle X_0, G_0 \rangle dP = \int_A (XG)_0 \mathbf{1}_A dP = \mu_{XG}(\{0\} \times A) = \mu_{XG}(B). \end{aligned}$$

If $B = (s, t] \times A$ with $A \in \mathcal{F}_s$, then

$$\begin{aligned} \langle I_X, g \rangle(B) &= \int_A \langle X_t - X_s, g \rangle dP = \int_A \langle X_t, G_t \rangle dP \\ &\quad - \int_A \langle X_s, G_s \rangle dP = \mu_{XG}((s, t] \times A) = \mu_{XG}(B). \end{aligned}$$

Then by additivity, we have

$$\langle I_X, g \rangle(B) = \mu_{XG}(B) \text{ for } B \in \mathcal{R}.$$

Since I_X is bounded on $\mathcal{R}$ by hypothesis, it follows that μ_{XG} is bounded; hence it has bounded variation on $\mathcal{R}$. This means that XG is a quasimartingale ([M, Definition 8.6]). Moreover, XG is a quasimartingale of class (LD) ([M, Definition 13.1]). In fact, let $a \in \mathbb{R}_+$ and let $T : \Omega \rightarrow [0, a]$ be a simple stopping time. Then

$$|(XG)_T| = |\langle X_T, G_T \rangle| \leq |X_T| |G_T| \leq |X_T| \|g\|_\infty.$$

Since, by hypothesis, X is of class (D) on $[0, a]$, it follows that XG is of class (D) on $[0, a]$; and since $a \in \mathbb{R}_+$ was arbitrary, it follows that XG is of class (LD). We can then apply Theorem 13.3 in [M] and deduce that μ_{XG} can be extended to a σ -additive measure $\mu_{XG} : \mathcal{P} \rightarrow \mathbb{R}$; that is, the measure $\langle I_X, g \rangle$ can be extended to a σ -additive measure on $\mathcal{P}$. Since I_X is strongly additive, we can then apply the extension Theorem 7.7 in [D] and deduce that I_X can be extended to a σ -additive measure $I_X : \mathcal{P} \rightarrow L_E^p$. $\square$

Corollary 2. *Let $X : \mathbb{R}_+ \times \Omega \rightarrow E$ be a cadlag, adapted process such that $X_t \in L_E^p$ for every $t \in \mathbb{R}_+$. Assume that $c_0 \not\subseteq E$, X is of class (LD) and I_X is bounded in L_E^p on $\mathcal{R}$. Then I_X can be extended to a σ -additive measure $I_X : \mathcal{P} \rightarrow L_E^p$.*

Proof. In fact, if $c_0 \not\subseteq E$ and I_X is bounded in L_E^p on $\mathcal{R}$, then I_X is strongly additive in L_E^p on $\mathcal{R}$ ([D, Theorem 6.8]). $\square$

THE SUMMABILITY CRITERION

We can now state the summability criterion.

Theorem 3. *Let $X : \mathbb{R}_+ \times \Omega \longrightarrow E$ be a cadlag, adapted process such that $X_t \in L_E^p$ for every $t \in \mathbb{R}_+$. Assume that X is of class (LD) and I_X is bounded and strongly additive in L_E^p on $\mathcal{R}$. (In particular, this is true if X is of class (LD), $c_0 \not\subseteq E$, and I_X is bounded in L_E^p on $\mathcal{R}$.)*

a) *If we consider $E = L(\mathbb{R}, E)$, hence $L_E^p = L(F, L_G^p)$, then X is p -summable relative to $(\mathbb{R}, L_E^p)$.*

b) *If $E \subset L(F, G)$, hence $L_E^p \subset L(F, L_G^p)$, and if I_X has bounded semivariation on $\mathcal{R}$ relative to (F, L_G^p) , then X is p -summable relative to (F, L_G^p) .*

Proof. By Theorem 1 and Corollary 2, I_X can be extended to a σ -additive measure $I_X : \mathcal{P} \longrightarrow L_E^p$. Then I_X is bounded on $\mathcal{P}$ ([D, Theorem 2.14]) and I_X has bounded semi-variation on $\mathcal{P}$ relative to $(\mathbb{R}, L_E^p)$ ([D, Proposition 4.14]). It follows that X is p -summable relative to $(\mathbb{R}, L_E^p)$, which proves assertion a).

If $L_E^p \subset L(F, L_G^p)$ and if I_X has bounded semivariation on $\mathcal{R}$ relative to (F, L_G^p) , then I_X has bounded semivariation on $\mathcal{P}$ relative to (F, L_G^p) ([D, Corollary 7.6]). It follows that X is p -summable relative to (F, L_G^p) , which proves assertion b). $\square$

SUMMABILITY OF MARTINGALES

For martingales the condition of being of class (LD) is automatically satisfied. We then have the following theorem.

Theorem 4. *Let $X : \mathbb{R}_+ \times \Omega \longrightarrow E$ be a right continuous martingale such that $X_t \in L_E^p$ for every $t \in \mathbb{R}_+$. Assume $c_0 \not\subseteq E$.*

a) *If I_X is bounded in L_E^p on $\mathcal{R}$, then X is p -summable relative to $(\mathbb{R}, L_E^p)$.*

b) *If I_X is bounded in L_E^p on $\mathcal{R}$ and has bounded semivariation on $\mathcal{R}$ relative to (F, L_G^p) , then X is p -summable relative to (F, L_G^p) .*

For square-integrable martingales in Hilbert spaces, all the conditions of Theorem 3 are satisfied, and we have the following theorem.

Theorem 5. *Let $X : \mathbb{R}_+ \times \Omega \longrightarrow E \subset L(F, G)$ be a square-integrable martingale such that both E and G are Hilbert spaces. Then X is a 2-summable process relative to $L(F, L_G^2)$.*

Proof. Since E is a Hilbert space, we have $c_0 \not\subseteq E$. Since X is a square-integrable martingale, it is of class (D). Finally, I_X is bounded in L_E^2 on $\mathcal{R}$ (see the proof of Lemma 17.1 and Theorem 17.2 in [D]), and I_X has bounded semivariation on $\mathcal{R}$ relative to $L(F, L_G^2)$ (see [D, Theorem 17.5]). By Theorem 4, X is 2-summable relative to $L(F, L_G^2)$.

SUMMABILITY OF PROCESSES WITH INTEGRABLE VARIATION OR INTEGRABLE SEMIVARIATION

For the definition and properties of processes with integrable variation or integrable semivariation, we refer to [D, §§19 and 21]. For the Lebesgue-Stieltjes

integral with respect to functions with finite variation or finite semivariation, see [D, §18D and §20D].

For a cadlag, adapted process $X : \mathbb{R}_+ \times \Omega \longrightarrow E \subset L(F, G)$, we denote by $|X|$ (resp. $\tilde{X}_{F,G}$) the *variation process* (resp. the *semivariation process relative to* (F, G)) of X .

We say that X has *p-integrable variation* (resp. *p-integrable semivariation relative to* (F, G)) if the total variation $|X|_\infty$ (resp. the total semivariation $(\tilde{X}_{F,G})_\infty$) belongs to L^p .

We state first the summability property of processes with integrable semivariation. $\square$

Theorem 6. *Let $X : \mathbb{R}_+ \times \Omega \longrightarrow E \subset L(F, G)$ be a cadlag, adapted process. Assume that the embedding $E \subset L(F, G)$ is an isometry, that $c_0 \not\subset E$, and that X has p-integrable variation or p-integrable semivariation relative to $L(F, L_G^p)$. Then X is p-summable relative to $L(F, L_G^p)$.*

Proof. If X has p-integrable variation, then it has p-integrable semivariation relative to (F, G) . Therefore, we shall give the proof on the assumption that X has p-integrable semivariation relative to (F, G) . Since the embedding $E \subset L(F, G)$ is an isometry, we have (see [D, Property 4.3])

$$|X_t| \leq (\tilde{X}_{F,G})_t \leq (\tilde{X}_{F,G})_\infty \in L^p;$$

therefore, $X_t \in L^p$ for every $t \in \mathbb{R}_+$. Then, for every stopping time T we have

$$|X_T| \leq (\tilde{X}_{F,G})_\infty;$$

hence X is of class (D). It remains to prove that I_X has bounded semivariation relative to $L(F, L_G^p)$. It will follow then that I_X is bounded in L_E^p . We have

$$I_X(B)(\varpi) = \int \mathbf{1}_B(s, \varpi) dX_s(\varpi), \text{ for } B \in \mathcal{R},$$

where, for each $\varpi \in \Omega$, we used the Lebesgue-Stieltjes integral with respect to the function $s \longmapsto X_s(\varpi)$ with bounded semivariation relative to (F, G) .

For $B = \{0\} \times A$ with $A \in \mathcal{F}_0$ we have

$$I_X(B)(\varpi) = \mathbf{1}_A(\varpi) X_0(\varpi) = \int \mathbf{1}_B(s, \varpi) dX_s(\varpi),$$

and for $B = (u, v] \times A$ with $A \in \mathcal{F}_u$, we have

$$\begin{aligned} I_X(B)(\varpi) &= \mathbf{1}_A(\varpi) (X_v(\varpi) - X_u(\varpi)) \\ &= \mathbf{1}_A(\varpi) \int \mathbf{1}_{(u,v]}(s, \varpi) dX_s(\varpi) = \int \mathbf{1}_B(s, \varpi) dX_s(\varpi). \end{aligned}$$

Then the equality remains valid for every $B \in \mathcal{R}$.

Let $B_1, \dots, B_n$ be disjoint sets from $\mathcal{R}$ and $x_1, \dots, x_n$ elements from F with $|x_i| \leq 1$ for $1 \leq i \leq n$. Without loss of generality, we can assume that each B_i is of the form $B_i = (u_i, v_i] \times A_i$ with $A_i \in \mathcal{F}_{u_i}$ and that $u_1 < v_1 \leq u_2 < v_2 \leq \dots \leq u_n < v_n$.

For each $\varpi \in \Omega$ we have then, taking the supremum for all $z \in G^*$ with $|z| \leq 1$,

$$\begin{aligned} \left| \sum_i I_X(B_i)(\varpi)x_i \right| &= \left| \sum_i \mathbf{1}_{A_i}(\varpi)(X_{v_i}(\varpi) - X_{u_i}(\varpi))x_i \right| \\ &= \sup \left| \sum_i \mathbf{1}_{A_i}(\varpi) \langle (X_{v_i}(\varpi) - X_{u_i}(\varpi))x_i, z \rangle \right| \\ &\leq \sup \sum_i \mathbf{1}_{A_i}(\varpi) |\langle X_{v_i}(\varpi)x_i, z \rangle - \langle X_{u_i}(\varpi)x_i, z \rangle| \\ &\leq \sup \sum_i |((X_z)_{v_i}(\varpi) - (X_z)_{u_i}(\varpi))x_i| \\ &\leq \sup \sum_i |(X_z)_{v_i}(\varpi) - (X_z)_{u_i}(\varpi)| \leq \sup |X_z|_\infty(\varpi) = (\tilde{X}_{F,G})_\infty(\varpi). \end{aligned}$$

We used Proposition 20.7 in [D] to compute the semivariation $\tilde{X}_{F,G}$ by means of the variation $|X_z|$. Then

$$\left\| \sum_i I_X(B_i)x_i \right\|_p \leq \|(\tilde{X}_{F,G})_\infty\|_p.$$

Taking the supremum for all B_i and x_i as above we get

$$(\tilde{I}_X)_{F,L_G^p}(\mathbb{R}_+ \times \Omega) \leq \|(\tilde{X}_{F,G})_\infty\|_p < \infty;$$

i.e., I_X has bounded semivariation relative to (F, L_G^p) .

In the above set of inequalities, taking just a set B in $\mathcal{R}$ and an element x in F with $|x| \leq 1$, we get

$$|I_X(B)(\varpi)x| \leq (\tilde{X}_{F,G})_\infty(\varpi).$$

Since the embedding $E \subset L(F, G)$ is an isometry, taking the supremum for x in F with $|x| \leq 1$ we have

$$|I_X(B)(\varpi)| \leq (\tilde{X}_{F,G})_\infty(\varpi);$$

therefore

$$\|I_X(B)\|_p \leq \|(\tilde{X}_{F,G})_\infty\|_p.$$

Then I_X is bounded in L_E^p on $\mathcal{R}$. By the Summability Criterion, Theorem 3, it follows that X is p -summable relative to (F, L_G^p) . $\square$

Remark. This is a new proof of Theorem 21.12a in [D].

If X has p -integrable variation, we can remove the condition that the embedding $E \subset L(F, G)$ is an isometry.

Theorem 7. *Let $X : \mathbb{R}_+ \times \Omega \longrightarrow E \subset L(F, G)$ be a cadlag, adapted process. Assume that $c_0 \notin E$ and that X has p -integrable variation. Then, for any embedding $E \subset L(F, G)$, X is p -summable relative to $L(F, L_G^p)$.*

Proof. For every $t \in \mathbb{R}_+$ we have

$$|X_t| \leq |X|_t \leq |X|_\infty \in L^p;$$

hence $X_t \in L_E^p$. Then for every stopping time T we have

$$|X_T| \leq |X|_\infty;$$

hence X is of class (D). To prove that I_X is bounded in L_E^p , let $B \in \mathcal{R}$. Then B is a finite union of disjoint predictable rectangles $B_0 = \{0\} \times A_0$ with $A_0 \in \mathcal{F}_0$, $B_1 = (u_1, v_1] \times A_1, \dots, B_n = (u_n, v_n] \times A_n$ with $A_i \in \mathcal{F}_{u_i}$ for $1 \leq i \leq n$. We can assume that $0 = u_1 < v_1 = u_2 < v_2 = \dots = u_n < v_n$. Then, for each $\varpi \in \Omega$ we have

$$\begin{aligned} |I_X(B)(\varpi)| &= \left| \sum_i I_X(B_i)(\varpi) \right| = \left| \sum_i (X_{v_i}(\varpi) - X_{u_i}(\varpi)) \mathbf{1}_{A_i}(\varpi) \right| \\ &\leq \sum_i |X_{v_i}(\varpi) - X_{u_i}(\varpi)| \leq |X|_\infty(\varpi). \end{aligned}$$

It follows that

$$\|I_X(B)\|_p \leq \| |X|_\infty \|_p;$$

hence I_X is bounded in L_E^p on $\mathcal{R}$. Now let $E \subset L(F, G)$ be any embedding. Since X has p -integrable variation, it has p -integrable semivariation relative to (F, G) . From the proof of Theorem 6, we deduce that I_X has bounded semivariation relative to (F, L_G^p) . We can now apply Theorem 3 and deduce that X is p -summable relative to (F, L_G^p) .

If $p = 1$ and X has integrable variation, we can remove even the condition $c_0 \notin E$. $\square$

Theorem 8. *Let $X : \mathbb{R}_+ \times \Omega \longrightarrow E$ be a cadlag, adapted process. If X has integrable variation and if $E \subset L(F, G)$ is any embedding, then X is summable relative to $L(F, L_G^p)$.*

Proof. We prove as in Theorem 7 that X is of class (D) and that I_X is bounded in L_E^1 on $\mathcal{R}$. We prove now that I_X has bounded variation. Let $B_0, B_1, \dots, B_n$ be a family of disjoint sets from $\mathcal{R}$. Without loss of generality, taking $u_0 = -1, v_0 = 0$, and $\mathcal{F}_{u_0} = \mathcal{F}_0$, we can assume that $B_i = (u_i, v_i] \times A_i$ with $A_i \in \mathcal{F}_{u_i}$ for $0 \leq i \leq n$. Then, for every $\varpi \in \Omega$ we have

$$\begin{aligned} \sum_i |I_X(B_i)(\varpi)| &= \sum_i |(X_{v_i}(\varpi) - X_{u_i}(\varpi)) \mathbf{1}_{A_i}(\varpi)| \\ &\leq \sum_i |X_{v_i}(\varpi) - X_{u_i}(\varpi)| \leq |X|_\infty(\varpi). \end{aligned}$$

Then

$$\sum_i \|I_X(B_i)\|_1 \leq \| |X|_\infty \|_1.$$

Taking the supremum for all the families of sets B_i as above, we deduce that

$$|I_X|(\mathbb{R}_+ \times \Omega) \leq \| |X|_\infty \|_1;$$

hence I_X has bounded variation. Then I_X is strongly additive ([D, Theorem 6.8]). Let $E \subset L(F, G)$ be any embedding. Then

$$(\widetilde{I_X})_{F,G} \leq |I_X|;$$

hence I_X has bounded semivariation relative to (F, L_G^1) . We can apply Theorem 3b and deduce that X is summable relative to (F, L_G^1) . $\square$

Remark. This is a new proof of Theorem 19.13b) in [D].

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