ON BOUNDARY BLOW-UP PROBLEMS FOR THE COMPLEX MONGE-AMPÈRE EQUATION

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ABSTRACT. We prove the $\mathcal{C}^\infty$ regularity for some complex Monge-Ampère equations with boundary data equal to $+\infty$.

1. INTRODUCTION

Cheng and Yau in [C-Y] considered the following problem:

$$(1.1) \quad \begin{cases} \det(u_{p\bar{q}}) = ge^{Ku} & \text{in } \Omega, \\ \lim_{z \rightarrow z_0} u(z) = +\infty & \text{for every } z_0 \in \partial\Omega, \end{cases}$$

where Ω is a strictly pseudoconvex domain with $\mathcal{C}^\infty$ boundary (or some more general non-compact Kähler manifold), $K > 0$ and $g \in \mathcal{C}^\infty(\bar{\Omega})$ is a positive function. They proved that problem (1.1) has a unique plurisubharmonic solution u of class $\mathcal{C}^\infty$. This immediately gives a complete Kähler-Einstein metric on Ω .

We will consider a more general problem than (1.1), i.e.

$$(1.2) \quad \begin{cases} \det(u_{p\bar{q}}) = gf(u) & \text{in } \Omega, \\ \lim_{z \rightarrow z_0} u(z) = +\infty & \text{for every } z_0 \in \partial\Omega, \end{cases}$$

where Ω and g are as above, and $f \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^\infty((\tau, +\infty))$ for some $\tau \in \mathbb{R} \cup \{-\infty\}$ such that $f = 0$ on $(-\infty, \tau)$ and $f' > 0$ on $(\tau, +\infty)$. We will show (see Theorem 2.1) the existence of a smooth solution for functions f satisfying some technical

conditions, which are fulfilled for example if $f(t) = e^{e^{\dots e^t}}$ with $\tau = -\infty$ and if $f(t) = t^p$ for $p > n$, $f(t) = t^n + t^{n+1}$, and $f(t) = t^n(\log(t+2))^{2n}$ with $\tau = 0$.

This article is organized as follows. In section 2 we formulate the main theorem, and we give the proof of it in sections 3-5. In sections 6 and 7 we obtain something more about the case $f(t) = t^p$.

2. MAIN THEOREM

Let us first consider the following condition:

W1: the function

$$\Psi(x) = \int_x^\infty \frac{dt}{F(t)^{1/(n+1)}},$$

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where F is such that $F' = f$, is well defined for x large enough.

It is easy to check (see [G-P]) that condition W1 implies

$$(2.1) \quad \lim_{x \rightarrow +\infty} \frac{F(x)^{n/(n+1)}}{f(x)} = 0.$$

Ivarsson and Matero proved in [I-M] that if $\Omega = B = \{z : |z| < 1\}$, g depends only on $|z|$, $\tau = -\infty$ and f satisfies condition W1, then problem (1.2) has a plurisubharmonic solution $u \in \mathcal{C}^\infty(B)$. For Ω strictly pseudoconvex with the boundary of class $\mathcal{C}^\infty$ Ivarsson proved in [I] that if $\tau = -\infty$, $g \equiv 1$, f fulfills W1 and the condition $\frac{F(x)f'(x)}{f^2(x)} \geq \frac{n-1}{n+1}$ for x large enough, then problem (1.2) has a plurisubharmonic and locally Lipschitz solution u .

By [C-K-N-S] there exists a unique solution w of

$$\begin{cases} w \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega}), \\ \det(w_{p\bar{q}}) = g \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega. \end{cases}$$

Let us put

$$\Phi(x) = \int_x^\infty \frac{dt}{f(t)^{1/n}} \text{ for } x > \tau;$$

then by the condition W1, it is well defined. Let $T = (\Phi^{-1}(\sup_\Omega(-w)), +\infty)$.

To formulate the main theorem we will also consider the following two conditions:

W2:

$$\Phi(\tau) = \lim_{x \rightarrow \tau^+} \Phi(x) > \sup_\Omega(-w).$$

W3: There exist $\alpha, \beta, \gamma \geq 1$ such that on T we have

$$\begin{aligned} \alpha^{-1} f^{-\beta} &\leq f' \leq \alpha f^\beta, \\ (\log f)'' &\leq \gamma (\log f)'^2. \end{aligned}$$

Now we can formulate the main theorem:

Theorem 2.1. *Let Ω , f and g be as above. If W1-W3 are satisfied, then there exists a plurisubharmonic solution $u \in \mathcal{C}^\infty(\Omega)$ of the problem (1.2).*

Note that if $\tau = -\infty$ or $f(\tau+x) \leq Cx^n$ for x in some interval $[0, \varepsilon]$, then condition W2 is satisfied. Otherwise, for ε small enough it is satisfied in $\varepsilon\Omega = \{\varepsilon z : z \in \Omega\}$ if we also change g to $g \circ S$, where $S(z) = \frac{z}{\varepsilon}$. Theorem 3.1 in [M] claims the existence of a smooth solution to a similar problem for the real Monge-Ampère equation without any condition on f near τ . We believe however that the proof only works if f is replaced by λf , where λ is a constant such that the condition analogous to W2 holds. So it seems to be natural (in the complex and real cases) to assume something like W2. We will see that it enables us to construct a solution $u > \tau$.

We believe that assumption W3 in Theorem 2.1 is in fact superfluous. However the author was unable to prove *a priori* estimates without this condition. To see that many functions satisfy this condition, note that for positive and increasing f and $x > 1$ we have

$$\liminf_{t \rightarrow +\infty} \frac{f'(t)}{f(t)^x} = 0 \geq \liminf_{t \rightarrow +\infty} \frac{(\log f(t))''}{(\log f(t))'^x}.$$

Also, if f satisfy $W1$, then

$$\limsup_{t \rightarrow +\infty} \frac{f'(t)}{f(t)^{n-1/n}} = +\infty.$$

3. APPROXIMATING SEQUENCE

In the same way as in [M] (in the real case) we will define an increasing sequence of plurisubharmonic functions approximating the solution of problem (1.2). Let $\varphi = (-\Phi)^{-1} \circ w$, $\Omega_k = \{z \in \mathbb{C}^n : \varphi(z) < k\}$ (where k is such that the set Ω_k is a strictly pseudoconvex domain of class $\mathcal{C}^\infty$) and let u_k be the solution of the following Dirichlet problem:

$$\begin{cases} u \in \mathcal{PSH} \cap \mathcal{C}^\infty(\bar{\Omega}_k), \\ \det(u_{p\bar{q}}) = gf(u) \text{ in } \Omega_k, \\ u = k (= \varphi) \text{ in } \partial\Omega_k. \end{cases}$$

The following lemma is an easy consequence of the comparison principle.

Lemma 3.1. *We have $u_{k+1} \geq u_k \geq \varphi$.*

Proof. Let $V = (-\Psi)^{-1}$. Then $V > 0$ is smooth, $V' = f^{1/n} \circ V > 0$ and $V'' = \frac{f'}{nF^{(n-2)/n}} \circ V > 0$. So V is a smooth plurisubharmonic function, and we can compute

$$\begin{aligned} \det(\varphi_{p\bar{q}}) &= \det(V' \circ ww_{p\bar{q}} + V'' \circ ww_p w_{\bar{q}}) \\ &= gf \circ V \circ w + \frac{1}{n} gf^{1/n} f' \circ V \circ ww^{p\bar{q}} w_p w_{\bar{q}} > gf \circ \varphi. \end{aligned}$$

By the comparison principle $u_k \geq \varphi$, and from this (again using the comparison principle) we obtain $u_{k+1} \geq u_k$. $\square$

4. A PRIORI ESTIMATES

In this section we shall prove that u_k and Δu_k are locally bounded, and our estimate will be independent of k .

Differentiating (the logarithm of) the equation $\det(u_{p\bar{q}}) = gf(u)$ twice we get the following formulas, which are very useful in the theory of regularity of the Monge-Ampère equation:

$$(4.1) \quad (\log gf(u))_k = u^{p\bar{q}} u_{kp\bar{q}},$$

$$(4.2) \quad (\log gf(u))_{k\bar{k}} = u^{p\bar{q}} u_{k\bar{k}p\bar{q}} - u^{p\bar{j}} u^{i\bar{q}} u_{ki\bar{j}} u_{\bar{k}p\bar{q}},$$

where $k = 1, \dots, n$ and $(u^{p\bar{q}})$ is the inverse transposed matrix of $(u_{p\bar{q}})$. We will also use the differential operator $L = L_k = u_k^{p\bar{q}} \frac{\partial^2}{\partial z_p \partial \bar{z}_q}$. Functions u_k are plurisubharmonic; hence L is elliptic and we can apply the maximum principle.

In a similar way as in [I] (or in the real case in [M]) we will prove the following lemma:

Lemma 4.1. *There is a function $H \in \mathcal{PSH} \cap \mathcal{C}(\Omega)$, such that for every $u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$, if $\det(u_{p\bar{q}}) \geq gf(u)$, then $u \leq H$.*

Proof. Let $W = (-\Psi)^{-1}$ and $H = W \circ (Kw)$, where $K > 0$ (small enough so that H is well defined) will be specified later. Then $W > 0$ is smooth, $W' = F^{1/(n+1)} \circ W > 0$ and $W'' = \frac{f}{(n+1)F^{(n-1)/(n+1)}} \circ W > 0$. So H is a smooth plurisubharmonic function, and we can compute

$$\begin{aligned} \det(H_{p\bar{q}}) &= \det(W' \circ (Kw)Kw_{p\bar{q}} + W'' \circ (Kw)K^2w_pw_{\bar{q}}) \\ &= gK^n F^{n/(n+1)} \circ W \circ (Kw) + \frac{1}{n+1} gK^{n+1} f \circ W \circ (Kw)w^{p\bar{q}}w_pw_{\bar{q}} \\ &= gf(H) \left(\frac{K^n F^{n/(n+1)}}{f} \circ H + K^{n+1} \frac{w^{p\bar{q}}w_pw_{\bar{q}}}{n+1} \right). \end{aligned}$$

By (2.1), for K small enough we have $\det(H_{p\bar{q}}) \leq gf(H)$. So by the comparison principle we obtain $u \leq H$. $\square$

Lemma 4.2. *There exists a constant C independent of k (it may depend on Ω , f and g) such that*

$$(4.3) \quad |\nabla u_k| \leq C f^{1/n}(k).$$

Proof. Since $u_k \geq \varphi$, on $\partial\Omega_k$ we have (as in [I])

$$f^{-1/n}(u_k)|\nabla u_k| = |\nabla(\Phi \circ u_k)| \leq |\nabla(\Phi \circ \varphi)| = |\nabla w|.$$

Let $r \in \{1, \dots, 2n\}$. To estimate $(u_k)_{x_r}$, let us consider the function $\eta = (u_k)_{x_r} + K f^{1/n}(k)|z|^2$. If η attains a maximum on the boundary, then the proof is complete, so we may assume that η attains maximum at $z_0 \in \Omega_k$. We can also assume that $(u_k)_{x_r}(z_0) \geq 0$. By (4.1) and because f is increasing, we have at z_0

$$\begin{aligned} L((u_k)_{x_r} + K f^{1/n}(k)|z|^2) &= (\log(gf(u_k)))_{x_r} + K f^{1/n}(k) \sum_{p=1}^n u^{p\bar{p}} \\ &\geq (\log g)_{x_r} + nK > 0 \end{aligned}$$

for K large enough. By the maximum principle we obtain (4.3). $\square$

Lemma 4.3. *If conditions W1–W3 are fulfilled, then there are constants N and C independent of k such that*

$$\frac{\Delta u_k}{f(u_k)^N} \leq C \text{ on } \Omega_k.$$

Proof. Let $c_0, c_1, c_2, \dots$ be constants depending only on Ω , $\|w\|_{C^{2,1}}$, f and g . For simplicity in what follows we will write u instead of u_k , Ω instead of Ω_k and η instead of $\eta \circ u$ (or $\eta(u)$).

We will show that for N big enough the function

$$\Lambda = \max_{i \in \{1, \dots, n\}} \lambda_i f^{-N} e^{|z|^2},$$

where λ_i are eigenvalues of the matrix $(u_{p\bar{q}})$, is bounded. The proof will be divided into two parts: first we will estimate Λ on the boundary and then inside.

Part I: Function Λ is bounded on boundary of Ω . Let $z_0 \in \partial\Omega$. After a holomorphic change of variables (see for example the proof of Lemma 1.3 in [C-K-N-S]) we can assume that

$$\begin{aligned} z_0 &= 0, \\ w &= -x_{2n-1} + a_{p\bar{q}}z_p\bar{z}_q + o(|z|^2) \text{ for some matrix } (a_{p\bar{q}})_{p,q=1}^n > 0, \\ w_{p\bar{q}} &= a_{p\bar{q}} = 0 \text{ for } p, q < n \text{ such that } p \neq q. \end{aligned}$$

Then (as in [C-K-N-S])

$$(4.4) \quad u_{p\bar{q}} = \frac{u_{x_{2n-1}}}{w_{x_{2n-1}}} w_{p\bar{q}} \text{ for } p, q < n.$$

For $l = 1, \dots, 2n-2$ we consider

$$w_l = \pm T_l u + (u_{x_l})^2 + (u_{x_{2n}})^2 + A|z|^2 - Bx_{2n-1},$$

where $T_l = a\partial_{x_l} - b\partial_{x_{2n-1}}$, $a = \rho_{x_{2n-1}}$ and $b = \rho_{x_l}$. We have

$$\begin{aligned} u^{p\bar{q}}u_{x_{2n-1}\bar{q}} &= 2u^{p\bar{q}}u_{n\bar{q}} + iu^{p\bar{q}}u_{x_{2n}\bar{q}} = 2\delta_{pn} + iu^{p\bar{q}}u_{x_{2n}\bar{q}}, \\ u^{p\bar{q}}u_{x_{2n-1}p} &= 2u^{p\bar{q}}u_{p\bar{n}} - iu^{p\bar{q}}u_{x_{2n}p} = 2\delta_{qn} - iu^{p\bar{q}}u_{x_{2n}p}. \end{aligned}$$

Thus using (4.1) and the Schwarz inequality we can calculate

$$\begin{aligned} Lw_l &= \pm u^{p\bar{q}}(a_{p\bar{q}}u_{x_l} - b_{p\bar{q}}u_{x_{2n-1}} + a_p u_{x_l\bar{q}} - b_p u_{x_{2n-1}\bar{q}} + a_{\bar{q}}u_{x_n p} - b_{\bar{q}}u_{x_{2n-1}p}) \\ &\quad \pm T_l \log(gf) + 2u_{x_l}(\log(gf))_{x_l} + 2u^{p\bar{q}}u_{p x_l}u_{\bar{q} x_l} \\ &\quad + 2u_{x_{2n}}(\log(gf))_{x_{2n}} + 2u^{p\bar{q}}u_{p x_{2n}}u_{\bar{q} x_{2n}} + A \sum_p u^{p\bar{p}} \\ &\geq -c_1 |\nabla u| \sum_p u^{p\bar{p}} - c_2 \sum_p u^{p\bar{p}} - 2\sqrt{u^{p\bar{q}}a_p a_{\bar{q}}} \sqrt{u^{p\bar{q}}u_{x_l p}u_{x_l \bar{q}}} \\ &\quad - 2\sqrt{u^{p\bar{q}}b_p b_{\bar{q}}} \sqrt{u^{p\bar{q}}u_{x_{2n} p}u_{x_{2n} \bar{q}}} - c_3 - c_4 \frac{f'|\nabla u|}{f} - c_5 |\nabla u| - 4 \frac{f'|\nabla u|^2}{f} \\ &\quad + 2u^{p\bar{q}}u_{p x_l}u_{\bar{q} x_l} + 2u^{p\bar{q}}u_{p x_{2n}}u_{\bar{q} x_{2n}} + A \sum_p u^{p\bar{p}}. \end{aligned}$$

We have $\sum u^{p\bar{p}} \geq \frac{1}{f^{1/n}(k)}$; hence (also using the inequality between arithmetic and geometric means and Lemma 4.2) for some $A < c_6(f^{2/n}(k) + \sup_\Omega f')$ we obtain $Lw_l > 0$.

Let S_ε be the connected component of the set $\{z \in \Omega : x_{2n-1} < \varepsilon\}$ such that $0 \in \bar{S}_\varepsilon$. For well chosen (small enough and independent of k and z_0) $\varepsilon > 0$ and for every $z \in \partial S_\varepsilon$ we have

$$|z| \leq \frac{c_7}{\varepsilon} \sqrt{x_{2n-1}} = c_8 \sqrt{x_{2n-1}}.$$

Then, since $T_l u = 0$ on $\partial\Omega \cap \partial S_\varepsilon$, we have

$$\pm T_l u \leq c_9 x_{2n-1} \text{ on } \partial S_\varepsilon,$$

and by Lemma 4.2

$$(u_{x_r})^2 \leq c_{10} |\nabla u|^2 |z|^2 \leq c_{11} f^{2/n}(k) x_{2n-1} \text{ for } r \neq 2n-1.$$

Hence choosing suitable $B < c_{12}(f^{2/n}(k) + \sup_{\Omega} f')$ we obtain $w_l \leq 0$ on ∂S_{ε} . From the maximum principle $w_l \leq 0$ holds on the whole of S_{ε} . This implies that $w_{lx_{2n-1}} < c_{13}(f^{2/n}(k) + \sup_{\Omega} f')$. By Lemma 4.2 and condition W3 we obtain

$$(4.5) \quad |u_{p\bar{n}}| < c_{14}f^N \text{ for some } N > 0.$$

In our coordinates at 0 we can write the Monge-Ampère equation in the form

$$gf = \prod_k u_{k\bar{k}} - \sum_{p \neq n} u_{p\bar{n}} u_{n\bar{p}} \prod_{p \neq k \neq n} u_{k\bar{k}}.$$

Solving this and using (4.4) we can calculate

$$(4.6) \quad u_{n\bar{n}} = \sum_{p \neq n} \frac{u_{p\bar{n}} u_{n\bar{p}} w_{x_{2n-1}}}{w_{p\bar{p}} u_{x_{2n-1}}} - \prod_{k \neq n} \frac{f g u_{x_{2n-1}}}{w_{k\bar{k}} u_{x_{2n-1}}}.$$

By Lemma 3.1 there exists a constant $\nu > 0$ such that $fg > \nu$ on Ω . So using the inequality between arithmetic and geometric means, we can estimate

$$\Delta u \geq n\nu^{1/n},$$

and then from the Hopf Lemma there exists a constant $\mu > 0$ such that $u_{x_{2n-1}} > \mu$. Thus, using (4.4), (4.5) and (4.6) we conclude that

$$\Delta u < c_{15}f^N.$$

Part II: Function Λ is bounded in $\bar{\Omega}$. Assume that a maximum of the function Λ is attained at $z_0 \in \Omega$ (otherwise we are done). After a linear change of variables we can assume that at z_0 the matrix $(u_{p\bar{q}})$ is diagonal and $u_{1\bar{1}} = \max_{k \in \{1, \dots, n\}} \lambda_k$. Let $h = u_{1\bar{1}} f^{-N} e^{|z|^2}$. Then the function h also attains a maximum at z_0 and $h(z_0) = \Lambda(z_0)$. From now on all formulas are assumed to hold at z_0 . We may assume that

$$(4.7) \quad h \geq C e^{|z_0|^2}$$

for some (big enough) $C > 0$, which will be specified later, and that

$$(4.8) \quad |z_0| < \frac{1}{2N}.$$

Then (again by the inequality between arithmetic and geometric means)

$$(4.9) \quad \sum_p u^{p\bar{p}} \geq C f^{(N-1)/(n-1)}.$$

Since $h_p = 0$ for $p = 1, \dots, n$, we obtain

$$(4.10) \quad u_{p1\bar{1}} = u_{1\bar{1}} \left(N \frac{f' u_p}{f} - \bar{z}_p \right).$$

We will estimate the following expression from below:

$$(4.11) \quad L(\log h) = \frac{u^{p\bar{p}} u_{p\bar{p}1\bar{1}}}{u_{1\bar{1}}} - \frac{u^{p\bar{p}} u_{p1\bar{1}} u_{\bar{p}1\bar{1}}}{u_{1\bar{1}}^2} - N(\log f)'' u^{p\bar{p}} u_p u_{\bar{p}} - N(\log f)' + \sum_1^n u^{p\bar{p}}.$$

Using (4.2), W3, (4.10), the generalized means inequality, (4.1) and the Schwarz inequality we compute

$$\begin{aligned}
& \frac{u^{p\bar{p}}u_{p\bar{p}1\bar{1}}}{u_{1\bar{1}}} = \frac{(\log(gf))_{1\bar{1}}}{u_{1\bar{1}}} + u^{1\bar{1}}u^{p\bar{p}}u^{q\bar{q}}u_{1p\bar{q}}u_{\bar{1}\bar{p}q} \\
& \geq \frac{(\log g)_{1\bar{1}}}{u_{1\bar{1}}} + (\log f)' + \frac{(\log f)''u_1u_{\bar{1}}}{u_{1\bar{1}}} + u^{p\bar{p}}\frac{u_{p1\bar{1}}u_{\bar{p}1\bar{1}}}{u_{1\bar{1}}^2} + \sum_{p=2}^n \frac{u_{p1\bar{1}}u_{\bar{p}1\bar{1}}}{u_{p\bar{p}}u_{1\bar{1}}^2} + \sum_{p=2}^n \frac{u_{p\bar{p}1\bar{1}}u_{p\bar{p}1\bar{1}}}{u_{p\bar{p}}^2u_{1\bar{1}}} \\
& \geq -c_{16} + \frac{(\log f)''u_1u_{\bar{1}}}{u_{1\bar{1}}} + u^{p\bar{p}}\frac{u_{p1\bar{1}}u_{\bar{p}1\bar{1}}}{u_{1\bar{1}}^2} \\
& \quad + \sum_{p=2}^n u^{p\bar{p}}(N\frac{f'u_p}{f} - \bar{z}_p)(N\frac{f'u_{\bar{p}}}{f} - z_p) + \frac{1}{(n-1)u_{1\bar{1}}} |(\log gf)_1 - \frac{u_{1\bar{1}1}}{u_{1\bar{1}}}|^2 \\
& \geq -c_{16} + \frac{(\log f)''u_1u_{\bar{1}}}{u_{1\bar{1}}} + u^{p\bar{p}}\frac{u_{p1\bar{1}}u_{\bar{p}1\bar{1}}}{u_{1\bar{1}}^2} + (N^2 - 1)(\log f)'^2 \sum_{p=2}^n u^{p\bar{p}}u_pu_{\bar{p}} \\
& \quad + (\log f)'^2 \sum_{p=2}^n u^{p\bar{p}}u_pu_{\bar{p}} + \frac{1}{4} \sum_{p=2}^n u^{p\bar{p}} - 2 \sqrt{\sum_{p=2}^n ((\log f)')^2 u^{p\bar{p}}u_pu_{\bar{p}}} \sqrt{\sum_{p=2}^n u^{p\bar{p}}N^2 z_p \bar{z}_p} \\
& \quad - \frac{1}{4} \sum_{p=2}^n u^{p\bar{p}} + \sum_{p=2}^n u^{p\bar{p}}|z_p|^2 + \frac{1}{(n-1)u_{1\bar{1}}} |(\log gf)_1 - \frac{u_{1\bar{1}1}}{u_{1\bar{1}}}|^2.
\end{aligned}$$

The inequality between arithmetic and geometric means and (4.8) gives us

$$\begin{aligned}
(4.12) \quad & \frac{u^{p\bar{p}}u_{p\bar{p}1\bar{1}}}{u_{1\bar{1}}} \geq -c_{16} + u^{p\bar{p}}\frac{u_{p1\bar{1}}u_{\bar{p}1\bar{1}}}{u_{1\bar{1}}^2} + (N^2 - 1)(\log f)' \sum_{p=2}^n u^{p\bar{p}}u_pu_{\bar{p}} \\
& - \frac{1}{4} \sum_{p=1}^n u^{p\bar{p}} + \frac{1}{(n-1)u_{1\bar{1}}} |(\log gf)_1 - \frac{u_{1\bar{1}1}}{u_{1\bar{1}}}|^2 + \frac{(\log f)''u_1u_{\bar{1}}}{u_{1\bar{1}}}.
\end{aligned}$$

Two cases may occur:

A)

$$|u_1| < \frac{|(\log g)_1| + |z_1|}{(\log f)'}$$

Then by W3 we get that

$$|u_1| < c_{17}f^{\alpha+1}.$$

Therefore by (4.7) and W3 we have

$$-(\log f)''u^{1\bar{1}}u_1u_{\bar{1}} \geq -c_{18}f^{4\beta-N},$$

B)

$$|u_1| \geq \frac{|(\log g)_1| + |z_1|}{(\log f)'}$$

Then by (4.10) and W3 we have

$$\begin{aligned} & \frac{1}{(n-1)u_{1\bar{1}}} |(\log gf)_1 - \frac{u_{1\bar{1}1}}{u_{1\bar{1}}}|^2 - (\log f)'' u^{1\bar{1}} u_1 u_{\bar{1}} \\ &= \frac{u^{1\bar{1}} u_1 u_{\bar{1}}}{n-1} \left| \frac{(\log g)_1 - \bar{z}_1}{u_1 (\log f)'} + 1 - N \right|^2 - (\log f)'' u^{1\bar{1}} u_1 u_{\bar{1}} \\ &\geq (\log f)'' u^{1\bar{1}} u_1 u_{\bar{1}} \left(\frac{(N-2)^2}{n-1} - \gamma \right) \geq c_{19} f^{2\alpha-2} \left(\frac{(N-2)^2}{n-1} - \gamma \right) u^{1\bar{1}} u_1 u_{\bar{1}}. \end{aligned}$$

It is clear that in both cases for N big enough, we obtain

$$\frac{1}{(n-1)u_{1\bar{1}}} |(\log gf)_1 - \frac{u_{1\bar{1}1}}{u_{1\bar{1}}}|^2 - (N-1)(\log f)'' u^{1\bar{1}} u_1 u_{\bar{1}} \geq -c_{20},$$

so using (4.9), (4.12) and (4.7) we can conclude that for C and N big enough

$$L(\log h) > 0.$$

But this contradicts the fact that h attains a maximum at z_0 . $\square$

Note that in the above proof we obtained an *a priori* estimate on the boundary (Part I of proof) in the same way as in [C-K-N-S]. In Part II we used some ideas from [B2].

5. PROOF OF THEOREM 2.1

Theorem 2.1 is an easy consequence of *a priori* estimates.

Proof of Theorem 2.1. Using Lemma 4.1 (for all domains Ω_k), Lemma 3.1 and Lemma 4.3 we obtain that u_k converges locally uniformly to some function u and Δu is locally bounded, so u is in the Sobolev space $W^{2,p}$ for every $p > 1$. The Convergence Theorem for the complex Monge-Ampère equation gives us the fact that u is a solution of problem (1.2), so using Theorem 2.5 from [B1] we get $u \in \mathcal{C}^2(\Omega)$. Now Theorem 2.1 follows from Theorem 2.2 (again) from [B1]. $\square$

Note that in Theorem 2.1 we can (without changing the proof) assume that $w \in \mathcal{C}^{2,1}(\bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$ instead of smoothness of $\partial\Omega$.

6. A NECESSARY CONDITION

Similar to the real case (see [L-M]) for the function $f(t) = t^p$, problem (1.2) has a solution if and only if $p > n$. It follows from the following theorem:

Theorem 6.1. *If a solution of the problem*

$$(6.1) \quad \begin{cases} u \in \mathcal{PSH} \cap \mathcal{C}(\Omega), \\ \det(u_{p\bar{q}}) \leq gf(u) \text{ in } \Omega, \\ \lim_{z \rightarrow z_0} u(z) = +\infty \text{ for } z_0 \in \partial\Omega \end{cases}$$

exists, then the integral $\int \frac{1}{f^{1/n}}$ converges at $+\infty$.

Proof (similar to [M]). We can assume that $\tau < 0$ and that the function

$$\mu(x) = \int_0^x \frac{dt}{f^{1/n}(t)}$$

is not bounded. Let u be a solution of (6.1). Then for A big enough the function $\phi_A = \mu^{-1} \circ (w + A)$ is well defined. In the same way as in Lemma 3.1, for the

function φ we obtain $\det(\phi_{Ap\bar{q}}) \geq gf(\phi_A)$. From the comparison principle $u \geq \phi_A$. It is a contradiction with a free choice of large A . $\square$

7. UNIQUENESS

Ivarsson in [I] proved that if $\limsup_{t \rightarrow +\infty} \frac{-\Psi(t)}{\Psi'(t)} < +\infty$, then the solution of problem (1.2) is unique. Using his method, we will also show this for $f(t) = t^p$ where $p > n$ (in this case $\lim_{t \rightarrow +\infty} \frac{-\Psi(t)}{\Psi'(t)} = +\infty$).

Theorem 7.1. *Let u, v be plurisubharmonic solutions of problem (1.2). Assume $\tau = -\infty$, W1,*

$$(7.1) \quad \limsup_{t \rightarrow +\infty} \frac{-\Psi(t)}{t\Psi'(t)} < +\infty$$

and also

$$(7.2) \quad (\log f(t))' \geq \frac{n}{t+C}$$

for some constant $C > 0$ and $t \geq \inf_{\Omega} \min\{u, v\}$. Then $u \equiv v$.

Proof. First, we will prove that

$$(7.3) \quad \lim_{z \rightarrow \partial\Omega} \frac{u(z)}{v(z)} = 1.$$

Indeed, using (7.1), we obtain that there exist $D > 0$ such that for t big enough and $x \geq 1$ we have

$$\frac{\partial}{\partial x}(\Psi(xt)) = x\Psi'(xt) \leq -D\Psi(xt),$$

so

$$(7.4) \quad \Psi(xt) \leq \Psi(t)(e^D)^{1-x}.$$

Theorem 3.8 from [I-M] gives us

$$\lim_{z \rightarrow \partial\Omega} \frac{\Psi(u(z))}{\Psi(v(z))} = 1.$$

Thus, using (7.4) we can conclude that (7.3) follows.

Let $x \geq 1$, $u_x = x(u+C)$, $v_x = x(v+C)$ and $f_x(t) = x^n f(\frac{t}{x} - C)$ for $t \in \mathbb{R}$. Note that $u_x, v_x > 0$ and $\det(u_{xp\bar{q}}) = \det(v_{xp\bar{q}}) = gf_x(u)$ on Ω . Inequality (7.2) means that $\frac{\partial f_x(t)}{\partial x} \geq 0$ for $t \in \mathbb{R}$, so $f_x \geq f_1$. Using (7.3) we obtain that the set $S = \{u_x < v_1\}$ is relatively compact in Ω for $x > 1$. Then by the comparison principle S is empty. Hence, because $\lim_{x \rightarrow 1+} u_x = u_1$, we have $u_1 \geq v_1$, and in the same way we can obtain that $v_1 \geq u_1$. Thus the theorem follows. $\square$

Corollary 7.2. *Let $p > n$ and $f(t) = t^p$. Then there is exactly one plurisubharmonic solution of problem (1.2).*

Proof. By Theorem 2.1 we know that there is a smooth solution of problem (1.2). The definition of φ and Lemma 3.1 imply that our solution is positive. If u is a positive solution we may suitably modify the function $f(t) = t^p$ for $t < \inf_{\Omega} u$ and get a positive, increasing function on $\mathbb{R}$. Hence from Theorem 7.1 we have only one positive solution. Now it is enough to prove that every solution is positive. To do this let us consider a strictly pseudoconvex domain $\tilde{\Omega}$, such that Ω is relatively

compact in $\tilde{\Omega}$. Then on $\tilde{\Omega}$ there exists a positive plurisubharmonic solution u of the equation

$$\det(u_{p\bar{q}}) = \left(\sup_{\Omega} g\right) u^p,$$

and from the comparison principle u is dominated from above by any solution of problem (1.2) which gives the statement. $\square$

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