STRONGLY NON-DEGENERATE LIE ALGEBRAS

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ABSTRACT. Let A be a semiprime 2- and 3-torsion free non-commutative associative algebra. We show that the Lie algebra $\mathcal{D}er(A)$ of (associative) derivations of A is strongly non-degenerate, which is a strong form of semiprimeness for Lie algebras, under some additional restrictions on the center of A . This result follows from a description of the quadratic annihilator of a general Lie algebra inside appropriate Lie overalgebras. Similar results are obtained for an associative algebra A with involution and the Lie algebra $S\mathcal{D}er(A)$ of involution preserving derivations of A .

INTRODUCTION

This paper is concerned with the structure of the Lie algebra $\mathcal{D}er(A)$ of associative derivations of an associative algebra A , which we will also assume to be 2- and 3-torsion free. It was proved in [10, Theorem 4 and Theorem 2] that if A is semiprime (respectively prime), then $\mathcal{D}er(A)$ is a semiprime (respectively prime) Lie algebra. We prove below that if A is prime, this result can be strengthened to show that in fact $\mathcal{D}er(A)$ is strongly non-degenerate (see below for the precise definitions).

The key result in the paper is Theorem 2.1, which has a technical flavour. Let L be a subalgebra of a Lie algebra Q . The quadratic annihilator of L inside Q is defined as the set $\{q \in Q \mid [q, [q, L]] = 0\}$. Roughly speaking, Theorem 2.1 allows us to obtain non-zero elements in the quadratic annihilator of L in itself from non-zero elements in the quadratic annihilator of L inside Q whenever Q is a weak quotient algebra of L , i.e., $[L, q] \neq 0$ for every non-zero $q \in Q$. If L is strongly non-degenerate, then the quadratic annihilator of L inside Q coincides with the annihilator of L in Q , and both are zero (Theorem 2.2 (ii)).

Another application of Theorem 2.1 leads to the proof of the fact that if a Lie algebra L contains an essential ideal which is strongly non-degenerate, then the algebra L is itself strongly non-degenerate (Proposition 2.3 (ii)). This fact was already proved by Zel'manov in [17] by making use of the Kostrikin radical, while our proof is based on elements.

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According to the above, and in order to obtain the result announced in the abstract (Theorem 2.5), we need to produce an essential ideal inside $\mathcal{D}\text{er}(A)$ which is strongly non-degenerate. The natural candidate for this is the ideal $\text{Inn}(A)$ of the so-called inner derivations of A , which can be identified with the quotient $A/Z(A)$, known to be strongly non-degenerate under appropriate mild hypotheses. However, this ideal might fail to be essential, and this is somehow measured by the ideal I_Z of those derivations that map A into the center $Z(A)$. Our result then asserts that $\mathcal{D}\text{er}(A)/I_Z$ is strongly non-degenerate. In the particular case that the center $Z(A)$ of A does not contain associative ideals (e.g. if A is prime), one has $I_Z = 0$, and then we obtain that $\mathcal{D}\text{er}(A)$ is strongly non-degenerate.

Our arguments can subsequently be adjusted with some extra effort to the case of a $*$ -semiprime algebra A and the Lie algebra $\text{S}\mathcal{D}\text{er}(A)$ of those (associative) derivations of A that commute with the involution, that is, those $\delta \in \mathcal{D}\text{er}(A)$ such that $\delta(a^*) = (\delta(a))^*$ for every $a \in A$ (Theorem 2.8).

1. NOTATION AND PRELIMINARIES

Let Φ be a unital commutative ring. All algebras in this paper, associative or not, will be Φ -modules. Recall that a *Lie algebra* over Φ is a Φ -module L , together with a bilinear map $[\cdot, \cdot]: L \times L \rightarrow L$, denoted by $(x, y) \mapsto [x, y]$ and called the *bracket* of x and y , such that the following axioms are satisfied:

- (i) $[x, x] = 0$,
- (ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (*Jacobi identity*),

for every x, y, z in L .

The standard example is obtained by considering a (not necessarily unital) associative algebra A with its usual module structure and bracket given by $[x, y] = xy - yx$. Sometimes the notation A^- is used in order to emphasize the Lie structure of A .

Given an element x of a Lie algebra L , we may define a map $\text{ad } x: L \rightarrow L$ by $\text{ad } x(y) = [x, y]$ (which is a derivation of the Lie algebra L). We shall denote by $A(L)$ the associative subalgebra (possibly without identity) of $\text{End}(L)$ generated by the elements $\text{ad } x$ for x in L .

An element x in a Lie algebra L is an *absolute zero divisor* if $(\text{ad } x)^2 = 0$. This is equivalent to saying that $[x, [x, L]] = 0$. The algebra L is said to be *strongly non-degenerate* (according to Kostrikin) if it does not contain non-zero absolute zero divisors.

Given a Lie algebra L , we say that L is *semiprime* if we have $I^2 \neq 0$ whenever I is a non-zero ideal. It is obvious from the definitions that strongly non-degenerate Lie algebras are semiprime, but the converse does not hold (see [15, Remark 1.1]).

Next, L is said to be *prime* if $[I, J] \neq 0$ for any pair of non-zero ideals I, J of L . An ideal I of L is said to be *essential* if its intersection with any non-zero ideal is again a non-zero ideal.

For two subsets X, Y of a (Lie) algebra L we define the *annihilator of Y in X* as the set

$$\text{Ann}_X(Y) := \{x \in X \mid [x, Y] = 0\},$$

and the *quadratic annihilator of Y in X* to be the set

$$\text{QAnn}_X(Y) := \{x \in X \mid [x, [x, Y]] = 0\}.$$

When $X = L$, we write $\text{Ann}(Y)$ or $\text{Ann}_L(Y)$ (if no confusion can arise) and refer to it as the annihilator of Y . If $X = Y = L$, then $\text{Ann}(L)$ is called the *center* of L and is usually denoted by $Z(L)$. In the case that $L = A^-$ for an associative algebra A , $Z(A^-)$ agrees with the associative center Z of A . It is easy to check (by using the Jacobi identity) that $\text{Ann}(X)$ is an ideal of L whenever X is an ideal of L . Therefore, for A associative we can form the Lie algebra A^-/Z . We will be primarily interested in this type of Lie algebra and in Lie algebras that arise from associative algebras with involution. If A is associative and has an involution $*$, then the set of its *skew* elements

$$K = K_A = \{x \in A \mid x^* = -x\}$$

is a subalgebra of A^- . The center $Z(K)$ of the Lie algebra K will be for brevity denoted by Z_K , and we will be interested in the Lie algebra K/Z_K .

The notion of the quadratic annihilator of an (arbitrary) algebra – defined in a similar way as we have done for Lie algebras – plays an important role; see, for example, Smirnov's paper [16]. Let us remark here that the quadratic annihilator need not be closed under sums in the case of an associative product (for an example, see [16]). The same phenomenon occurs in the Lie context, as is shown in the examples below.

Examples 1.1. (1) Let F be any field and let $L = \mathfrak{t}(3, F)$ be the Lie algebra of upper triangular matrices (see e.g. [8]). Then $\text{QAnn}(L) = \{a(e_{11} + e_{22} + e_{33}) + be_{13} + ce_{23} \mid a, b, c \in F\} \cup \{a(e_{11} + e_{22} + e_{33}) + be_{12} + ce_{13} \mid a, b, c \in F\}$, where, as usual, e_{ij} denotes the matrix in $M_3(F)$ whose entries are all zero except for the one in row i and column j . So $\text{QAnn}(L)$ is not closed under sums.

(2) Now, for L as in (1), consider the Lie algebra $\overline{L} := L/Z$. Then

$$\text{QAnn}(\overline{L}) = \{\overline{ae_{13} + be_{23}} \mid a, b \in F\} \cup \{\overline{ae_{12} + be_{13}} \mid a, b \in F\},$$

where $\overline{x}$ denotes the class of an element x in L . Again we have that the quadratic annihilator of this algebra $\overline{L}$ is not closed under sums.

Let $L \subseteq Q$ be Lie algebras. When $0 \neq [L, q] \subseteq L$ for every non-zero $q \in Q$, we say that Q is a *weak algebra of quotients* of L (see [15]). The notion of an algebra of quotients of an algebra (associative or not necessarily associative) has a long history and is an active research area, especially in recent years, following its development in the Lie and Jordan contexts. In the seminal paper [15] the second author initiated the study of algebras of quotients of Lie algebras, by adapting some ideas from the associative and also Jordan ([13]) contexts. She introduced the notion of a general (abstract) algebra of quotients of a Lie algebra, and also the notion of the maximal algebra of quotients $Q_m(L)$ of a semiprime Lie algebra L . Follow-up results can be found in [14, 4, 2].

Let B be a subalgebra of an associative algebra A . A linear map $\delta : B \rightarrow A$ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in B$. By a derivation of A we simply mean a derivation from A into A . Let $\text{Der}(A)$ denote the set of all derivations of A . It is clear that $\text{Der}(A)$ becomes a Φ -module under natural operations and it also becomes a Lie algebra by putting $[\delta, \mu] = \delta\mu - \mu\delta$ for every δ, μ in $\text{Der}(A)$. Any element x of A determines a map $\text{ad } x : A \rightarrow A$ defined by $\text{ad } x(y) = [x, y]$, which is a derivation of A . For every Lie ideal U of A , the

restriction of the map $\text{ad} : A \rightarrow \mathcal{D}\text{er}(A)$ to U , namely

$$\begin{aligned} U &\rightarrow \mathcal{D}\text{er}(A) \\ y &\mapsto \text{ad } y, \end{aligned}$$

defines a Lie algebra homomorphism with kernel $\text{Ann}_U(A)$, which allows us to identify $U/\text{Ann}_U(A)$ with the subalgebra $\text{ad } (U)$ of $\mathcal{D}\text{er}(A)$. For any $y \in U$ and $\delta \in \mathcal{D}\text{er}(A)$, $[\delta, \text{ad } y] = \text{ad } \delta(y)$; hence $\text{ad } (U)$ is an ideal of $\mathcal{D}\text{er}(A)$ whenever $\delta(U) \subseteq U$ for every $\delta \in \mathcal{D}\text{er}(A)$. The ideal $\text{ad } (A)$ of $\mathcal{D}\text{er}(A)$ is usually denoted by $\text{Inn}(A)$ and its elements are called *inner derivations* of A . Note that $A^-/Z \cong \text{Inn}(A)$.

Now let A be an associative algebra with involution $*$. The set

$$\text{SDer}(A) = \{\delta \in \mathcal{D}\text{er}(A) \mid \delta(x^*) = \delta(x)^* \text{ for all } x \in A\}$$

is a Lie subalgebra of $\mathcal{D}\text{er}(A)$. Denote by $\text{ad } (K)$ the set of Lie derivations $\text{ad } x : A \rightarrow A$ with x in K .

In what follows we will assume that 2 and 3 are invertible elements in Φ .

2. THE RESULTS

By an *extension of Lie algebras* $L \subseteq Q$ we will mean that L is a (Lie) subalgebra of the Lie algebra Q . Let $L \subseteq Q$ be an extension of Lie algebras and let $A_Q(L)$ be the associative subalgebra of $A(Q)$ generated by $\{\text{ad } x : x \in L\}$.

For an extension $L \subseteq Q$ of Lie algebras, the condition $\text{Ann}_L(Q) = 0$ ensures that the map $L \rightarrow A(Q)$ given by $x \mapsto \text{ad } x$ is a monomorphism of Lie algebras. Examples of extensions where $\text{Ann}_L(Q) = 0$ are the dense ones (see [3] for the definition of a dense extension and [14] for examples).

Theorem 2.1. *Let $L \subseteq Q$ be an extension of Lie algebras such that the map $L \rightarrow A(Q)$, $x \mapsto \text{ad } x$ is a monomorphism of Lie algebras. Let $a \in \text{QAnn}_Q(L)$. Then, for each $u \in L$ satisfying $x := [a, u] \in L$, we have that $z := [x, [x, v]]$ is in $\text{QAnn}_L(L)$ for every $v \in L$.*

Proof. In order to ease the notation in our computations, we shall temporarily get rid of the prefix ad and use capital letters X, Y , etc., instead of $\text{ad } x, \text{ad } y$, etc. Because of our assumption, we shall also identify an element x of L with its corresponding operator $X = \text{ad } x$ in $A(Q)$. An equation involving commutators on L is then translated into the corresponding equation with capital letters and commutators in $A(Q)$.

Let a be in $\text{QAnn}_Q(L)$. Then

$$(2.1) \quad [a, [a, y]] = 0 \text{ for every } y \in L.$$

This implies that $[A, [A, Y]] = 0$ for every $Y \in \text{ad } (L) \subseteq A(Q)$; hence

$$(2.2) \quad A^2Y + YA^2 - 2AYA = 0 \text{ for every } Y \in \text{ad } (L).$$

By (2.1) we have

$$(2.3) \quad A^2 = 0 \text{ on } L,$$

and by (2.2) and $\frac{1}{2} \in \Phi$,

$$(2.4) \quad AYA = 0 \text{ on } L \text{ for every } Y \in \text{ad } (L).$$

For $x = [a, u] \in L$, with $u \in L$, we have that

$$(2.5) \quad X^2 = (AU - UA)^2 = AU AU - AU^2A - UA^2U + UAU A = -AU^2A$$

on L , by using (2.3) and (2.4).

Note that $X^3 \stackrel{(2.5)}{=} (-AU^2A)(AU - UA) = -AU^2A^2U + AU^2AUA = 0$ by (2.3) and (2.4). Thus:

$$(2.6) \quad X^3 = 0 \text{ on } L.$$

We may now apply [11, Lemma 1.5.6] to obtain:

$$(2.7) \quad XYX^2 = X^2YX \text{ on } L, \text{ for every } Y \in \text{ad}(L), \text{ and}$$

$$(2.8) \quad (YX^2)^2 = X^2Y^2X^2 \text{ on } L, \text{ for every } Y \in \text{ad}(L).$$

Now,

$$(2.9) \quad \begin{aligned} X^2Y^2X^2 &= (YX^2)^2 \text{ (by (2.8))} \\ &= YX^2YX^2 \\ &= Y(XYX^2)X \text{ (by (2.7))} \\ &= 0 \text{ (by (2.6)).} \end{aligned}$$

Taking $z := [x, [x, v]]$, with $v \in L$, we compute that, on L ,

$$\begin{aligned} Z^2 &= (X^2V + VX^2 - 2XVX)(X^2V + VX^2 - 2XVX) \\ &= X^2VX^2V + VX^4V - 2XVX^3V \\ &\quad + X^2V^2X^2 + VX^2VX^2 - 2XVXVX^2 \\ &\quad - 2X^2VXVX - 2VX^3VX + 4XVX^2VX \\ &= X^2VX^2V + VX^2VX^2 \\ &\quad - 2XV(XVX^2) - 2(X^2VX)VX + 4XVX^2VX \text{ (by (2.9) with } Y=V \text{ and (2.6))} \\ &= X^2VX^2V + VX^2VX^2 \text{ (by (2.7) twice with } Y=V) \\ &= X^2VX^2V \text{ (by (2.9) with } Y=V) \\ &= (X^2VX)XV = (XVX^2)XV \text{ (by (2.7) with } Y=V) \\ &= 0 \text{ (by (2.6)),} \end{aligned}$$

which completes the proof of the theorem. $\square$

Theorem 2.2. *Let $L \subseteq Q$ be an extension of Lie algebras with Q a weak algebra of quotients of L and L strongly non-degenerate. Then*

- (i) Q is strongly non-degenerate ([15, Proposition 2.7(iii)]).
- (ii) $\text{Ann}_Q(L) = \text{QAnn}_Q(L) = 0$.

Proof. (i). Suppose that there exists a non-zero element $a \in \text{QAnn}_Q(Q)$, and choose $u \in L$ satisfying $0 \neq x := [a, u] \in L$. Since L is strongly non-degenerate, $z := [x, [x, v]] \neq 0$ for some $v \in L$. But, by Theorem 2.1, z must be zero, a contradiction.

(ii). $\text{Ann}_Q(L) = 0$ because Q is a weak algebra of quotients of L . For $a \in \text{QAnn}_Q(L)$, we have $z := [x, [x, v]] \in \text{QAnn}_L(L)$ whenever $x := [a, u] \in L$, with $u \in L$ (Theorem 2.1). Now L being strongly non-degenerate implies $\text{QAnn}_L(L) = 0$, whence x is zero. Since Q is a weak algebra of quotients of L , we obtain that $a = 0$. $\square$

Statement (ii) in the result below was proved by Zelmanov on [17, Corollary 2 on p. 543] for strongly non-degenerate Lie algebras, using the Kostrikin radical. Our proof here is based on elements.

Proposition 2.3. *Let I be a strongly non-degenerate ideal of a Lie algebra L . Then*

- (i) $\text{Ann}_L(I) = \text{QAnn}_L(I)$.
- (ii) *If $\text{Ann}_L(I) = 0$, then the algebra L is strongly non-degenerate.*

Proof. (i). Clearly, $\text{Ann}_L(I) \subseteq \text{QAnn}_L(I)$. Conversely, consider $a \in \text{QAnn}_L(I)$. The strong non-degeneracy assumption on I implies that the map $I \rightarrow A(L)$ given by $y \mapsto \text{ad } y$ is a monomorphism of Lie algebras. Since I is a strongly non-degenerate ideal of L , Theorem 2.1 implies that for every $u \in I$, the element $x = [a, u]$ is in $\text{QAnn}_I(I) = 0$, hence $a \in \text{Ann}_L(I)$.

(ii). In this case we have that L is a weak algebra of quotients of I . Apply Theorem 2.2 to obtain that L must be strongly non-degenerate too. $\square$

If A is an associative algebra, since every derivation maps Z to Z , the set

$$I_Z = \{\delta \in \text{Der}(A) \mid \delta(A) \subseteq Z\}$$

is easily seen to be a Lie ideal of $\text{Der}(A)$ that contains the center of $\text{Der}(A)$. Indeed, for every $\delta \in Z(\text{Der}(A))$ and each $a \in A$, we have that $0 = [\delta, \text{ad } a] = \text{ad } (\delta(a))$, hence $\delta(a) \in Z$. Moreover, under certain conditions, $\text{Inn}(A)$ can be seen as an essential ideal of $\text{Der}(A)/I_Z$.

Lemma 2.4. *Let A be a semiprime non-commutative associative algebra. Then*

- (i) $\text{Inn}(A)$ is (isomorphic to) an essential ideal of $\text{Der}(A)/I_Z$, where I_Z is defined as before.
- (ii) *If Z does not contain non-zero associative ideals (in particular, if A is prime), then $I_Z = 0$.*

Proof. (i). The map

$$\begin{array}{ccc} \text{Inn}(A) & \rightarrow & \text{Der}(A)/I_Z \\ \text{ad } a & \mapsto & \overline{\text{ad } a} \end{array}$$

is a monomorphism of Lie algebras. This follows from the fact that

$$(\dagger) \quad [[a, A], A] = 0, \text{ with } a \in A, \text{ implies } a \in Z.$$

Indeed, $[[a, A], A] = 0$ implies, by [6, Sublemma on p. 5], that $[a, A] = 0$, that is, $\text{ad } a = 0$.

This allows us to identify $\text{Inn}(A)$ with its image inside $\text{Der}(A)/I_Z$. The formula $[\delta, \text{ad } a] = \text{ad } \delta(a)$, where $a \in A$ and $\delta \in \text{Der}(A)$, ensures that $\text{Inn}(A)$ is indeed an ideal of $\text{Der}(A)/I_Z$.

Now let J/I_Z be a non-zero ideal of $\text{Der}(A)$ and consider $\delta \in J \setminus I_Z$, that is, $[\delta(A), A] \neq 0$. A second application of $(\dagger)$ allows us to conclude that $[[\delta(A), A], A] \neq 0$. Take a in A such that $[\delta(a), A] \not\subseteq Z$. Then $\overline{0} \neq \overline{[\delta, \text{ad } a]} = \overline{\text{ad } \delta(a)}$, and thus $\text{Inn}(A)$ is essential in $\text{Der}(A)/I_Z$.

(ii). Take $\delta \in I_Z$ and $d \in \text{Der}(A)$. Put $\mu = [\delta, d]$. For every pair of elements $a, b \in A$ we have $\mu([a, b]) = [\mu(a), b] + [a, \mu(b)]$. Note that $[\mu(a), b] = [\delta d(a), b] - [d\delta(a), b] = -[d\delta(a), b] = 0$, because $d(Z) \subseteq Z$, and analogously $[a, \mu(b)] = 0$. It follows from this that $\mu([A, A]) = 0$.

Now let I be a non-central Lie ideal of A , and take y in $I \setminus Z$. Then $[y, A] \neq 0$ and by $(\dagger)$, we get $0 \neq [[y, A], A] \subseteq I \cap [A, A]$. Thus $[A, A]$ intersects non-trivially every non-central Lie ideal of A .

We claim that the subalgebra $\langle [A, A] \rangle$ generated by $[A, A]$ contains an essential associative ideal of A . Herstein's [7, Theorem 3] implies that $\langle [A, A] \rangle$ contains a non-zero associative ideal. By Zorn's Lemma, it is possible to find M that is maximal among all the associative ideals contained in $\langle [A, A] \rangle$. If $\text{Ann}(M)$ were non-zero, we get from what we have just proved that $\text{Ann}(M) \cap [A, A] \neq 0$. Again by [7, Theorem 3], since Z does not contain non-zero associative ideals, we have that $\langle \text{Ann}(M) \cap [A, A] \rangle$ contains a non-zero associative ideal J . Notice that J is not contained in M , since otherwise $J \subseteq M \cap \text{Ann}(M)$, which is zero because M is non-zero and A is semiprime. Then $M \oplus J \subseteq M \oplus \text{Ann}(M) \cap \langle [A, A] \rangle$ and $M \subsetneq M \oplus J$, which contradicts the maximality of M .

By the first part of the proof we have $\mu([A, A]) = 0$, and so $\mu(I) = 0$, where I is an essential ideal of A contained in $\langle [A, A] \rangle$. This implies that $\mu = 0$. For, if $\mu(a) \neq 0$ for some $a \in A$, then the essentiality of I implies that there exists $y \in I$ such that $y\mu(a) \neq 0$. But $y\mu(a) = \mu(ya) - \mu(y)a = 0$, a contradiction. $\square$

Theorem 2.5. *Let A be a semiprime non-commutative associative algebra. Then*

- (i) *$\text{Der}(A)/I_Z$ is a strongly non-degenerate Lie algebra.*
- (ii) *If Z does not contain non-zero associative ideals (in particular, if A is prime), then $\text{Der}(A)$ is a strongly non-degenerate Lie algebra.*

Proof. (i). Use [5, Lemma 5.2], Proposition 2.3 (ii) and Lemma 2.4 (i).

(ii) follows from (i) and statement (ii) in Lemma 2.4. $\square$

We now consider the case where our associative algebra A has an involution $*$. Under some additional mild assumptions on A in order to rule out algebras of low degrees, we obtain similar results on the non-degeneracy of $\text{SDer}(A)$. Rather than proving them in full, we just indicate which changes are needed to adjust Lemma 2.4 and Theorem 2.5 to the current setting.

Recall that if A is a semiprime associative algebra, the *extended centroid* $\mathcal{C} = \mathcal{C}(A)$ of A is defined as the center of the two-sided right ring of quotients. It also coincides with the center of $Q_s(A)$, the symmetric ring of quotients. For every x in an algebra A we define $\deg(x)$ as the degree of algebraicity of x over the extended centroid $\mathcal{C}$, provided that x is algebraic. If x is not algebraic, then we define $\deg(x) = \infty$. Put $\deg(A) = \sup\{\deg(x) \mid x \in A\}$. It is well-known that $\deg(A) < \infty$ if and only if A is a PI algebra. Furthermore, it is known that $\deg(A) = n < \infty$ if and only if A satisfies the standard polynomial identity of degree $2n$ but does not satisfy any polynomial identity of degree $< 2n$, and this is further equivalent to the condition that A can be embedded into the matrix algebra $\mathbb{M}_n(F)$ for some field F (one can take, say, F to be the algebraic closure of $\mathcal{C}$) but cannot be embedded into $\mathbb{M}_{n-1}(R)$ for any commutative algebra R .

(a) The analogue of $(\dagger)$ in Lemma 2.4 is as follows:

If A is a semiprime algebra with involution, then if a belongs to the set of skew elements K in A and $[[a, K], K] = 0$, we get $[a, K] = 0$. To prove this, suppose $a \in K$ is such that $[a, K] \neq 0$. This means that $\bar{a} \neq 0$, where $\bar{a}$ denotes the class of a in K/Z_K . But then $[\bar{a}, K/Z_K] \neq 0$, since K/Z_K is semiprime by [9, Theorem 3], that is, $[[a, K], K] \neq 0$.

- (b) The use of [7, Theorem 3] in the proof of statement (ii) in Lemma 2.4 must be changed to [12, Lemmas 2 and 3]. In order to apply these results, certain restrictions on the degree of the algebra are needed. Concretely, we need $\deg(A/I) > 2$ for every $*$ -prime ideal I of A .

Recall that an ideal I in an algebra A with involution $*$ is a $*$ -ideal if I is invariant under the involution, that is, $I^* = I$. The algebra A is said to be $*$ -prime if the product of two non-zero $*$ -ideals is again non-zero. A $*$ -ideal I is said to be $*$ -prime if A/I is a $*$ -prime algebra. The definition of a $*$ -semiprime algebra is analogous.

Define the following Lie ideal of $\text{SDer}(A)$:

$$I_{K,Z} = \{\delta \in \text{SDer}(A) \mid \delta(K) \subseteq Z\}.$$

In the current context, our Lemma 2.4 then takes the following form.

Lemma 2.6. *Let A be a semiprime non-commutative associative algebra with involution $*$. Then*

- (i) *$\text{Inn}(K)$ is (isomorphic to) an essential ideal of $\text{SDer}(A)/I_{K,Z}$, where $I_{K,Z}$ is defined as before.*
- (ii) *If $Z(K)$ does not contain non-zero associative $*$ -ideals (in particular, if A is $*$ -prime), then $I_{K,Z} = 0$.*

The analogue of [5, Lemma 5.2] (used in the proof of statement (i) in Theorem 2.5) is the proposition below, which again requires conditions on the degree of the algebra. In particular, it generalizes [1, Theorem 2.13]. Recall that an involution $*$ in an associative algebra A is said to be of the *first kind* if it is the identity on the centroid of A . Otherwise it is called an involution of the *second kind*.

Proposition 2.7. *Let A be a $*$ -semiprime algebra. Assume either that the involution is of the second kind or that it is of the first kind and $\deg(A/I) > 2$ for every $*$ -prime ideal I of A . Then $[k, [k, K]] \subseteq Z(A)$, with $k \in K$, implies $k \in Z(A)$. In particular, $K/(K \cap Z)$ is a strongly non-degenerate Lie algebra.*

Proof. Let I be a $*$ -ideal of A . It is clear that A/I also becomes a $*$ -algebra with the natural involution.

On the other hand, if $\bar{x}$ denotes the class of an element x in A/I and $\bar{K} = \{\bar{k} \mid k \in K\}$, we have that $\bar{K} = K_{A/I}$. The containment $\bar{K} \subseteq K_{A/I}$ is clear. For the converse, take $\bar{a}$ in $K_{A/I}$ and let $y \in I$ be such that $a^* + a = y$. Then $(a^* - \frac{1}{2}y)^* = y - a - \frac{1}{2}y = -a + \frac{1}{2}y$; that is, $a - \frac{1}{2}y \in K$, and so $\bar{a} = \overline{a - \frac{1}{2}y} \in \bar{K}$.

Now, consider $k \in K$ satisfying $[k, [k, K]] \subseteq Z$. In particular, $(\text{ad } k)^3(t) = 0$ for every $t \in K$. Arguing as in the proof of [5, Lemma 5.2], we obtain

$$(\dagger) \quad (\text{ad } k)^2(t) = 0.$$

Let $\{I_\alpha\}_{\alpha \in \Lambda}$ be the collection of all $*$ -prime ideals of A . Since A is $*$ -semiprime, $\bigcap_{\alpha \in \Lambda} I_\alpha = 0$. Suppose $[\bar{k}, A/I_\beta] \neq 0$ for some $\beta \in \Lambda$. Since A/I_β is a $*$ -prime algebra we may apply [2, Lemma 5.4] in order to conclude that $Z(\bar{K}) = Z(A/I_\beta) \cap \bar{K}$, and hence $[\bar{k}, \bar{K}] \neq 0$. Use [2, Theorem 5.3] if $*$: $A/I_\beta \rightarrow A/I_\beta$ is of the first

kind or [1, Theorem 2.13] if the involution is of the second kind to conclude that $[\overline{k}, [\overline{k}, \overline{K}]] \neq 0$, in contradiction to $(\dagger)$. As a consequence, $[\overline{k}, A/I_\alpha] = 0$ for every $\alpha \in \Lambda$, that is, $[k, A] \subseteq \bigcap_{\alpha \in \Lambda} I_\alpha = 0$. $\square$

Finally, the involutive version of Theorem 2.5 is the following:

Theorem 2.8. *Let A be a $*$ -semiprime non-commutative associative algebra with $\deg(A/I) > 2$ for every $*$ -prime ideal I of A . Then*

- (i) $\text{SDer}(A)/I_{K,Z}$ is a strongly non-degenerate Lie algebra.
- (ii) If $Z(K)$ does not contain non-zero associative $*$ -ideals (in particular, if A is $*$ -prime), then $\text{SDer}(A)$ is a strongly non-degenerate Lie algebra.

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