NEW FORMULAS OF THE BERGMAN KERNELS FOR COMPLEX ELLIPSOIDS IN $\mathbb{C}^2$

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ABSTRACT. We compute the explicit formula of the Bergman kernel for a nonhomogeneous domain $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{4/q_1} + |z_2|^{4/q_2} < 1\}$ for any positive integers q_1 and q_2 . We also prove that among the domains $D_p := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2p_1} + |z_2|^{2p_2} < 1\}$ in $\mathbb{C}^2$ with $p = (p_1, p_2) \in \mathbb{N}^2$, the Bergman kernel is represented in terms of closed forms if and only if $p = (p_1, 1)$, $(1, p_2)$, or $p = (2, 2)$.

1. Introduction and statement of main results

Let D be a bounded domain in $\mathbb{C}^n$. The Bergman space $L_a^2(D)$ is the space of holomorphic square integrable functions on D . For any $z \in D$, $\Phi_z : L_a^2(D) \rightarrow \mathbb{C}$ defined by $\Phi_z(f) = f(z)$ is a linear functional on $L^2(D)$ and by the Riesz representation theorem, there is a unique $K_z(\cdot) \in L_a^2(D)$ such that $\Phi_z(f) = \langle f(\cdot), K_z(\cdot) \rangle$, namely

$$f(z) = \int_D f(w) \overline{K_z(w)} dV(w)$$

for all $f \in L_a^2(D)$. Define $K_D(z, w) = \overline{K_z(w)}$, the Bergman kernel for D . The Bergman kernel depends on the choice of D and is also represented by

$$(1.1) \quad K_D(z, w) = \sum_{\alpha} \phi_{\alpha}(z) \overline{\phi_{\alpha}(w)},$$

where $\{\phi_{\alpha}\}_{\alpha}$ is the complete orthonormal basis of the Bergman space $L_a^2(D)$. The Bergman kernel has been used as one of the main tools in the function theory of several complex variables.

Now we concentrate on computing the explicit formula of the Bergman kernel. If D is a homogeneous domain in $\mathbb{C}^n$, then Hua [13] and Yin [17] computed the explicit formula of the Bergman kernel for classical domains and exceptional domains, respectively. Recently many mathematicians [5, 6, 7, 8, 9, 10, 11, 18] have made efforts to find the explicit formulas of the Bergman kernels for nonhomogeneous domains.

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Consider the complex ellipsoids or egg domains $D_p := \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1\}$, where $p = (p_1, \dots, p_n)$ for $p_j > 0$. The precise growth estimate of the Bergman kernel near a boundary point on the complex ellipsoid was studied in [11]. However, it is not easy to get the closed forms of the Bergman kernel for D_p .

In the case when $p_1, \dots, p_n$ are reciprocals of positive integers, Zinov'ev [19] computed the Bergman kernel for D_p explicitly. What happens if each p_j is a positive integer? The only known case is when $p = (1, \dots, 1, p_n)$, $p_n > 0$, for which S. Bergman [4] and J. P. D'Angelo [6, 7] obtained the Bergman kernel.

In this paper we study the Bergman kernel for the complex ellipsoids D_p in $\mathbb{C}^2$ and compute the explicit formula of the Bergman kernel for $D_{(2,2)}$, which is not homogeneous. For complete Reinhardt domains it is a standard method for computing the Bergman kernel and Szegő kernel to use (1.1), since we can choose $\phi_\alpha(z) = z^\alpha / \|z^\alpha\|$. This method is called a series method. In this case it is convenient to introduce hypergeometric functions [1, 8]. Let $(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}$, that is, $(a)_0 = 1$, $(a)_m = a(a+1) \cdots (a+m-1)$ for $m = 1, 2, \dots$, where Γ is the Gamma function. The notation $(a)_m$ is called the Pochhammer symbols. For multi-indices $a = (a_1, \dots, a_n)$ and $m = (m_1, \dots, m_n)$, we define $(a)_m = \prod_{j=1}^n (a_j)_{m_j}$. Then one of Appell's multivariate hypergeometric functions is defined by

$$F_A^{(n)}(a; b; c; z) := \sum_m \frac{(a)_{|m|} (b)_m}{(c)_m m!} z^m,$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $a \in \mathbb{C}$, $b = (b_1, \dots, b_n)$, $c = (c_1, \dots, c_n)$, and $|m| = \sum_{j=1}^n m_j$. The series $F_A^{(n)}(a; b; c; z)$ converges when $\sum_{j=1}^n |z_j| < 1$ and diverges when $\sum_{j=1}^n |z_j| > 1$. This is a multivariate version of the classical Euler-Gauss hypergeometric function which appears in a theory of the complex plane. That is to say, $F_A^{(1)}(a; b; c; z)$ is the classical Euler-Gauss hypergeometric function ${}_2F_1(a, b; c; z) = F(a, b; c; z)$. Explicit formulas of ${}_2F_1(a, b; c; z)$ are known for specific a, b, c , and z (see [12]). If $n = 2$, then $F_A^{(2)}(a; b_1, b_2; c_1, c_2; z_1, z_2)$ is

$$\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{\Gamma(k_1+k_2+a)\Gamma(k_1+b_1)\Gamma(k_2+b_2)}{\Gamma(k_1+c_1)\Gamma(k_2+c_2)k_1!k_2!} z_1^{k_1} z_2^{k_2},$$

which is usually called $F_2(a; b_1, b_2; c_1, c_2; z_1, z_2)$. This hypergeometric series F_2 plays an important role in physics and chemistry, and Opps, Saad, and Srivastava [15] obtained some reduction and transformation formulas of F_2 in the case when the parameters a, b_1, b_2, c_1 , and c_2 are positive integers. But we need the case when the parameters c_1 and c_2 are specific rational numbers, when we intend to compute the Bergman kernel for D_p in $\mathbb{C}^2$. In this paper we obtain the Bergman kernel and the Szegő kernel for $D_{(2,2)} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^4 + |z_2|^4 < 1\}$ using the formulas of $F_2(1; 1, 1, \frac{1}{2}, \frac{1}{2}; z_1, z_2)$ and $F_2(2; 1, 1, \frac{1}{2}, \frac{1}{2}; z_1, z_2)$, which are obtained in Section 2 and Section 3.

Theorem 1.1. *The Bergman kernel K and the Szegő kernel S for $D_{(2,2)} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^4 + |z_2|^4 < 1\}$ are given by*

$$\begin{aligned} K((z_1, z_2), (w_1, w_2)) &= \frac{\zeta_1(\pi + 2 \arcsin \zeta_1) f(\zeta_1^2, \zeta_2^2)}{\pi^3(1 - \zeta_1^2)^{\frac{3}{2}}(1 - \zeta_1^2 - \zeta_2^2)^3} + \frac{\zeta_2(\pi + 2 \arcsin \zeta_2) f(\zeta_2^2, \zeta_1^2)}{\pi^3(1 - \zeta_2^2)^{\frac{3}{2}}(1 - \zeta_1^2 - \zeta_2^2)^3} \\ &\quad + \frac{8\zeta_1\zeta_2}{\pi^2(1 - \zeta_1^2 - \zeta_2^2)^3} + \frac{2g(\zeta_1^2, \zeta_2^2)}{\pi^3(1 - \zeta_1^2)(1 - \zeta_2^2)(1 - \zeta_1^2 - \zeta_2^2)^2}, \\ S((z_1, z_2), (w_1, w_2)) &= \frac{2\zeta_1\zeta_2}{\pi^2(1 - \zeta_1^2 - \zeta_2^2)^2} + \frac{2}{\pi^3(1 - \zeta_1^2 - \zeta_2^2)} \\ &\quad + \frac{\zeta_1(1 - \zeta_1^2 + \zeta_2^2)(\pi + 2 \arcsin \zeta_1)}{\pi^3(1 - \zeta_1^2 - \zeta_2^2)^2(1 - \zeta_1^2)^{\frac{1}{2}}} + \frac{\zeta_2(1 - \zeta_2^2 + \zeta_1^2)(\pi + 2 \arcsin \zeta_2)}{\pi^3(1 - \zeta_1^2 - \zeta_2^2)^2(1 - \zeta_2^2)^{\frac{1}{2}}}, \end{aligned}$$

where $\zeta_1 = z_1 \bar{w}_1$, $\zeta_2 = z_2 \bar{w}_2$, $f(a, b) = 3(1 - a)^2 + 6b(1 - a) - b^2$, and $g(a, b) = 2 - a - b - (a - b)^2$.

Very recently the author learned that H. Valencourt had also obtained a similar formula as in the above theorem by different methods, but the formula in [16] is more complicated. If we apply Bell's transformation formula [2, 3] of the Bergman kernel under proper holomorphic mappings, we obtain the following corollary.

Corollary 1.2. *The Bergman kernel K for $D_{(2/q_1, 2/q_2)} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{\frac{4}{q_1}} + |z_2|^{\frac{4}{q_2}} < 1\}$ for $q_1, q_2 \in \mathbb{N}$ is given by*

$$\begin{aligned} &K((z_1, z_2), (w_1, w_2)) \\ &= \prod_{i=1}^2 \frac{1}{q_i^2 (z_i \bar{w}_i)^{1 - \frac{1}{q_i}}} \sum_{j_1=1}^{q_1} \sum_{j_2=1}^{q_2} \bar{\omega}_1^{j_1} \bar{\omega}_2^{j_2} K_{D_{(2,2)}}((z_1^{\frac{1}{q_1}}, z_2^{\frac{1}{q_2}}), (\omega_1^{j_1} w_1^{\frac{1}{q_1}}, \omega_2^{j_2} w_2^{\frac{1}{q_2}})), \end{aligned}$$

where $K_{D_{(2,2)}}$ is the Bergman kernel for $D_{(2,2)}$ whose formula is obtained explicitly in Theorem 1.1.

The Bergman kernel is of interest in geometry. For $\alpha, \beta = 1, 2$ and $z \in D$, we define

$$g_{\alpha\beta}(z) \equiv \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log K_D(z, z).$$

The matrix $G_D(z) \equiv (g_{\alpha\beta}(z))_{\alpha, \beta=1}^2$ is called the Bergman metric on D . Then the explicit formula of the Bergman kernel enables us to investigate the boundary behavior of the *Bergman canonical invariant*, which is defined by

$$J_D(z) \equiv \frac{\det G_D(z)}{K_D(z, z)}.$$

It is clear that this quantity is invariant under the biholomorphic mappings. From Theorem 1.1 the straightforward but tedious calculation leads to the following formulas. For $(0, w) \in D_{(2,2)}$ we have

$$G_D(0, w) = \begin{pmatrix} \frac{\pi(1+|w|^2)^3(3-|w|^2)}{\sqrt{1-|w|^4}H_1(w)} & 0 \\ 0 & \frac{H_2(w)+10|w|^2H_1(w)^2}{(1-|w|^4)^2H_1(w)^2} \end{pmatrix},$$

where $H_1(w) = 2\sqrt{1-|w|^4}(2+|w|^4) + 3|w|^2(\pi + 2\arcsin|w|^2)$ and $H_2(w) = 72|w|^2(1-|w|^4)^3 + 12(1-|w|^4)^{\frac{5}{2}}(1+2|w|^4)(\pi + 2\arcsin|w|^2)$. We also note that

$$K_{\Omega}(0, w) = \frac{H_1(w)}{\pi^3(1-|w|^4)^{\frac{5}{2}}}.$$

Since we have $\lim_{|w| \rightarrow 1} H_1(w) = 6\pi$ and $\lim_{|w| \rightarrow 1} H_2(w) = 0$, we obtain

$$\lim_{|w| \rightarrow 1} J_{D_{(2,2)}}(0, w) = \frac{40}{9}\pi^2.$$

It is interesting to compare this with Corollary 1 in [14]. For the Thullen domain $D_{(2,1)} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^4 + |z_2|^2 < 1\}$, the above quantity is also $40\pi^2/9$. Moreover the boundary limits of J_D at strongly pseudoconvex boundary points are the same as $9\pi^2/2$. See the details in [14].

In Section 4 we express the new formula of the Bergman kernel for D_p in $\mathbb{C}^2$ in terms of definite integrals. Using Tchebycheff's criterion of specific indefinite integrals, we prove that among the domains D_p in $\mathbb{C}^2$ with $p = (p_1, p_2) \in \mathbb{N}^2$, the Bergman kernel is represented as closed forms only if $p = (p_1, 1), (1, p_2)$, or $p = (2, 2)$. Thus the domain $D_{(2,2)}$ in Theorem 1.1 is very important in the view of the Bergman kernel.

2. Hypergeometric functions and Bergman kernel

Recall that for any nonzero a, b, c , the hypergeometric function F in the complex plane is defined by

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(k+a)\Gamma(k+b)}{\Gamma(k+c)\Gamma(k+1)} z^k.$$

In general it is not easy to express the hypergeometric functions in terms of closed forms even in one dimension. There are very few examples such as the following:

$$F(1, 1; 1; z) = \frac{1}{1-z}, \quad F(1, 1; 2; z) = -\frac{\ln(1-z)}{z}.$$

In this section we investigate whether each $F(\alpha, 1, \gamma, x)$ for $\alpha, \gamma \in \mathbb{Q}$ can be represented in terms of elementary functions which are not infinite series. The following is our main theorem in this section.

Theorem 2.1. *Let $r_j = \frac{s_j}{p_j} \in \mathbb{Q}$ for $j = 1, 2$.*

(i) *For $|x| < 1$ we have*

$$\begin{aligned} F(r_1 + r_2, 1; r_1; x) &= \frac{1}{1-x} + \frac{r_2}{r_1} \frac{x^{1-r_1}}{(1-x)^{1+r_2}} \int_0^{x^{r_1}} \frac{dy}{(1-y^{\frac{1}{r_1}})^{1-r_2}} \\ &= \frac{1}{1-x} + \frac{p_1 r_2 x^{1-r_1}}{(1-x)^{1+r_2}} I(x^{\frac{1}{p_1}}), \end{aligned}$$

where

$$I(x) = \int_0^x \frac{t^{s_1-1}}{(1-t^{p_1})^{1-\frac{s_2}{p_2}}} dt = \int_0^{(\frac{x^{p_1}}{1-x^{p_1}})^{1/p_2}} \frac{y^{\frac{p_2}{p_1}s_1-1}}{(1+y^{p_2})^{\frac{s_1}{p_1}+\frac{s_2}{p_2}}} dy.$$

(ii) *The hypergeometric function $F(\alpha, 1; \gamma; x)$ for $\alpha, \gamma \in \mathbb{Q}$ can be represented in terms of elementary functions which are not infinite series if and only if*

$$(2.1) \quad \alpha - \gamma \in \mathbb{Z} \quad \text{or} \quad \gamma \in \mathbb{Z} \quad \text{or} \quad \alpha \in \mathbb{Z}.$$

Now we will prove (i). Note that

$$F(r_1 + r_2, 1; r_1; x) = 1 + \sum_{k=1}^{\infty} \frac{(r_1 + r_2)_k}{(r_1)_k} x^k.$$

Then we have

$$\begin{aligned} g(x) &:= \sum_{k=1}^{\infty} \frac{(r_1 + r_2)_k}{(r_1)_k} x^{p_1 k} \\ &= \frac{r_1 + r_2}{r_1} x^{p_1} + \frac{\Gamma(r_1)}{\Gamma(r_1 + r_2)} \sum_{k=2}^{\infty} \frac{\Gamma(k + r_1 + r_2)}{\Gamma(k + r_1)} x^{p_1 k} \\ &= \frac{r_1 + r_2}{r_1} x^{p_1} + \frac{\Gamma(r_1)}{\Gamma(r_1 + r_2)} \sum_{k=1}^{\infty} \frac{\Gamma(k + r_1 + r_2)}{\Gamma(k + r_1)} x^{p_1 k + p_1} \\ &\quad + \frac{r_2 \Gamma(r_1)}{\Gamma(r_1 + r_2)} \sum_{k=1}^{\infty} \frac{\Gamma(k + r_1 + r_2)}{\Gamma(k + r_1 + 1)} x^{p_1 k + p_1}. \end{aligned}$$

Since

$$\int_0^x t^{s_1-1} g(t) dt = \frac{\frac{1}{p_1} \Gamma(r_1)}{\Gamma(r_1 + r_2)} \sum_{k=1}^{\infty} \frac{\Gamma(k + r_1 + r_2)}{\Gamma(k + r_1 + 1)} x^{p_1 k + s_1},$$

we have

$$g(x) = \frac{r_1 + r_2}{r_1} x^{p_1} + x^{p_1} g(x) + p_1 r_2 x^{p_1 - s_1} \int_0^x t^{s_1-1} g(t) dt.$$

Let $G(x) = \int_0^x t^{s_1-1} g(t) dt$. Then $G'(x) = x^{s_1-1} g(x)$ and

$$G'(x) - \frac{p_1 r_2 x^{p_1-1}}{1 - x^{p_1}} G(x) = \frac{r_1 + r_2}{r_1} \frac{x^{p_1 + s_1 - 1}}{1 - x^{p_1}}.$$

If we solve this first-order linear differential equation, we have

$$G(x) = \frac{r_1 + r_2}{r_1} (1 - x^{p_1})^{-r_2} \int_0^x \frac{t^{p_1 + s_1 - 1}}{(1 - t^{p_1})^{1-r_2}} dt.$$

Note that

$$\int_0^x \frac{t^{p_1 + s_1 - 1}}{(1 - t^{p_1})^{1-r_2}} dt = -\frac{1}{p_1(r_1 + r_2)} x^{s_1} (1 - x^{p_1})^{r_2} + \frac{r_1}{r_1 + r_2} \int_0^x \frac{t^{s_1-1}}{(1 - t^{p_1})^{1-r_2}} dt.$$

Thus we have

$$G(x) = -\frac{1}{s_1} x^{s_1} + \frac{1}{(1 - x^{p_1})^{r_2}} \int_0^x \frac{t^{s_1-1}}{(1 - t^{p_1})^{1-r_2}} dt.$$

Since $F(r_1 + r_2, 1; r_1; x) = 1 + t^{1-s_1} G'(t) \big|_{t=x^{1/p_1}}$, this completes the proof of (i). To prove (ii) we need the following lemma.

Lemma 2.2 (Tchebycheff, 1853, [12]). *Let $m, n \in \mathbb{Q}$ and $p, q \in \mathbb{Z}$ with $p > 0$ and $q \neq 0$. Then the indefinite integral*

$$\int x^m (ax^n + b)^{\frac{q}{p}} dx$$

can be represented in terms of elementary functions if and only if (i) $p = 1$, (ii) $\frac{m+1}{n} \in \mathbb{Z}$, (iii) $\frac{m+1}{n} + \frac{q}{p} \in \mathbb{Z}$.

By using Tchebycheff's criterion, the indefinite integral $I(x)$ defined in Theorem 2.1 (i) is represented in terms of elementary functions if and only if $p_2 = 1$, $r_1 \in \mathbb{Z}$, $r_1 + r_2 \in \mathbb{Z}$. In other words, the hypergeometric function $F(\alpha, 1; \gamma, x)$ for nonzero rational numbers α and γ can be represented as elementary functions if and only if

$$\alpha - \gamma \in \mathbb{Z} \quad \text{or} \quad \gamma \in \mathbb{Z} \quad \text{or} \quad \alpha \in \mathbb{Z},$$

which completes the proof of (ii).

The following example will play an important role in computing the Bergman kernel for $D_{(2,2)}$.

Example 2.3. For any positive integer n , we have

$$F\left(1, 1; n + \frac{1}{2}; x\right) = \frac{1-2n}{1-x} \left\{ \sum_{k=1}^{n-1} \frac{(-1)^k}{2n-2k-1} \left(\frac{1-x}{x}\right)^k + (-1)^n \left(\frac{1-x}{x}\right)^{n-\frac{1}{2}} \arcsin x^{\frac{1}{2}} \right\}.$$

Moreover we have

$$F\left(1, 1; \frac{1}{2}; x\right) = \frac{1}{1-x} + \frac{x^{\frac{1}{2}}}{(1-x)^{\frac{3}{2}}} \arcsin x^{\frac{1}{2}}.$$

Proof. Let $n \geq 1$. If $p_1 = p_2 = 2$, $s_1 = 2n + 1$, $s_2 = -2n + 1$, then we have

$$F\left(1, 1; n + \frac{1}{2}; x\right) = \frac{1}{1-x} + \frac{(1-2n)x^{\frac{1}{2}-n}}{(1-x)^{\frac{3}{2}-n}} I(x^{\frac{1}{2}}),$$

where

$$\begin{aligned} I(x) &= \int_0^{\frac{x}{(1-x^2)^{1/2}}} \frac{y^{2n}}{1+y^2} dy \\ &= \int_0^{\frac{x}{(1-x^2)^{1/2}}} \left[\sum_{k=0}^{n-1} (-1)^{n-k+1} y^{2k} + \frac{(-1)^n}{y^2+1} \right] dy \\ &= \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}}{2k+1} \left(\frac{x^2}{1-x^2}\right)^{k+\frac{1}{2}} + (-1)^n \arcsin x. \end{aligned}$$

Thus we have

$$\begin{aligned} F\left(1, 1; n + \frac{1}{2}; x\right) &= \frac{1}{1-x} + (1-2n) \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}}{2k+1} \frac{x^{k-n+1}}{(1-x)^{k-n+2}} \\ &\quad + (-1)^n (1-2n) \frac{x^{\frac{1}{2}-n}}{(1-x)^{\frac{3}{2}-n}} \arcsin x^{\frac{1}{2}}. \end{aligned}$$

If $n = 0$, then $I(x) = \arcsin x$, and we obtain the desired result. $\square$

Now we discuss the Bergman kernel for $D_p = \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1\}$. Recall that the monomials $\phi_\alpha(z) = \frac{z^\alpha}{\|z^\alpha\|_{L^2(D_p)}}$ form a complete orthonormal basis for $L^2_a(D_p)$ in (1.1). The Bergman kernel for D_p was obtained in [7, 8], and similarly we can get the Szegő kernel.

Lemma 2.4. *Let p_j be a positive integer for $j = 1, \dots, n$. Then the Bergman kernel $K(z, w)$ for the domain D_p is*

$$\begin{aligned} K(z, w) &= \frac{\prod_{j=1}^n p_j}{\pi^n} \sum_{\alpha} \frac{\Gamma(1 + \sum_{j=1}^n \frac{\alpha_j + 1}{p_j})}{\prod_{j=1}^n \Gamma(\frac{\alpha_j + 1}{p_j})} (z\bar{w})^\alpha \\ &= \frac{\prod_{j=1}^n p_j}{\pi^n} \sum_{k_1=0}^{p_1-1} \cdots \sum_{k_n=0}^{p_n-1} (z\bar{w})^k \frac{\Gamma(1 + \sum_{j=1}^n \frac{k_j + 1}{p_j})}{\prod_{j=1}^n \Gamma(\frac{k_j + 1}{p_j})} \\ &\quad \times F_A^{(n)} \left(1 + \sum_{j=1}^n \frac{k_j + 1}{p_j}; \mathbf{1}; \frac{\mathbf{k} + \mathbf{1}}{\mathbf{p}}; (z\bar{w})^p \right) \end{aligned}$$

for all $(z, w) \in D_p \times D_p$. Moreover, the Szegő kernel $S(z, w)$ for the domain D_p is

$$\begin{aligned} S(z, w) &= \frac{\prod_{j=1}^n p_j}{2\pi^n} \sum_{\alpha} \frac{\Gamma(\sum_{j=1}^n \frac{\alpha_j + 1}{p_j})}{\prod_{j=1}^n \Gamma(\frac{\alpha_j + 1}{p_j})} (z\bar{w})^\alpha \\ &= \frac{\prod_{j=1}^n p_j}{2\pi^n} \sum_{k_1=0}^{p_1-1} \cdots \sum_{k_n=0}^{p_n-1} (z\bar{w})^k \frac{\Gamma(\sum_{j=1}^n \frac{k_j + 1}{p_j})}{\prod_{j=1}^n \Gamma(\frac{k_j + 1}{p_j})} \\ &\quad \times F_A^{(n)} \left(\sum_{j=1}^n \frac{k_j + 1}{p_j}; \mathbf{1}; \frac{\mathbf{k} + \mathbf{1}}{\mathbf{p}}; (z\bar{w})^p \right). \end{aligned}$$

Here we use the notation $\mathbf{1} = 1, \dots, 1$ and $\frac{\mathbf{k} + \mathbf{1}}{\mathbf{p}} = \frac{k_1 + 1}{p_1}, \dots, \frac{k_n + 1}{p_n}$.

In this paper consider $D_p = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2p_1} + |z_2|^{2p_2} < 1\}$ in $\mathbb{C}^2$. Then by the preceding lemma, the Bergman kernel is

$$\sum_{k_1=0}^{p_1-1} \sum_{k_2=0}^{p_2-1} c(k_1, k_2) (z\bar{w})^k F_2 \left(1 + \frac{k_1 + 1}{p_1} + \frac{k_2 + 1}{p_2}; 1, 1; \frac{k_1 + 1}{p_1}, \frac{k_2 + 1}{p_2}; (z\bar{w})^p \right),$$

where $c(k_1, k_2) = \frac{p_1 p_2}{\pi^2} \Gamma(1 + \sum_{j=1}^n \frac{k_j + 1}{p_j}) / \prod_{j=1}^n \Gamma(\frac{k_j + 1}{p_j})$. Thus we need to compute the hypergeometric function $F_2(1 + \gamma_1 + \gamma_2; 1, 1; \gamma_1, \gamma_2; x, y)$. Similarly we also have to compute $F_2(\gamma_1 + \gamma_2; 1, 1; \gamma_1, \gamma_2; x, y)$ for the Szegő kernel. One can see the following transformation formula in [8].

Lemma 2.5. *For any $\gamma_1, \gamma_2 > 0$ we have*

$$F_2(\gamma_1 + \gamma_2; 1, 1; \gamma_1, \gamma_2; x, y) = \frac{\gamma_1 + \gamma_2}{2} \sum_{l=1}^2 \frac{H_l + H_l^*}{(1 - x - y)^l} + \frac{1 + \frac{1-\gamma_1}{\gamma_1}x + \frac{1-\gamma_2}{\gamma_2}y}{(1 - x - y)^2}$$

and

$$F_2(1 + \gamma_1 + \gamma_2; 1, 1; \gamma_1, \gamma_2; x, y) = \sum_{l=0}^2 \frac{H_l + H_l^*}{(1 - x - y)^{l+1}} + \frac{2}{\gamma_1 + \gamma_2} \frac{1 + \frac{1-\gamma_1}{\gamma_1}x + \frac{1-\gamma_2}{\gamma_2}y}{(1 - x - y)^3},$$

where $H_l = c_l x^l F(\gamma_1 + \gamma_2, 1; \gamma_1 + l; x)$, $H_l^* = c_l^* y^l F(\gamma_1 + \gamma_2, 1; \gamma_2 + l; y)$ and

$$c_0 = \frac{\gamma_2 - 1}{\gamma_1 + \gamma_2}, \quad c_1 = \frac{2(\gamma_2 - 1)}{(\gamma_1 + \gamma_2)\gamma_1}, \quad c_2 = \frac{2(\gamma_2 - 1)}{(\gamma_1 + \gamma_2)\gamma_1(\gamma_1 + 1)},$$

$$c_0^* = \frac{\gamma_1 - 1}{\gamma_1 + \gamma_2}, \quad c_1^* = \frac{2(\gamma_1 - 1)}{(\gamma_1 + \gamma_2)\gamma_2}, \quad c_2^* = \frac{2(\gamma_1 - 1)}{(\gamma_1 + \gamma_2)\gamma_2(\gamma_2 + 1)}.$$

Remark 2.6. The transformation formula in the above lemma is true for all $\gamma_1, \gamma_2 > 0$. But if $\gamma_1 = 1$ or $\gamma_2 = 1$, then the above formula is too complicated to compute. Instead, we can use the following relations:

$$(2.2) \quad F_2(\alpha; \beta_1, \beta_2; \beta_1, \gamma_2; x, y) = (1-x)^{-\alpha} F\left(\alpha, \beta_2; \gamma_2; \frac{y}{1-x}\right),$$

$$(2.3) \quad F_2(\alpha; \beta_1, \beta_2; \gamma_1, \beta_2; x, y) = (1-y)^{-\alpha} F\left(\alpha, \beta_1; \gamma_1; \frac{x}{1-y}\right).$$

For example, if $\gamma_1 = 1$, then

$$F_2(2 + \gamma_2; 1, 1; 1, \gamma_2; x, y) = (1-x)^{-(2+\gamma_2)} F\left(2 + \gamma_2, 1; \gamma_2; \frac{y}{1-x}\right),$$

which is simpler than the above lemma. But Lemma 2.5 is useful when γ_1 and γ_2 are not integers.

3. Proof of Theorem 1.1

In this section we prove the main theorem for the Bergman kernel. From Lemma 2.4, the Bergman kernel for $D_{(2,2)}$ is

$$(3.1) \quad \frac{4}{\pi^2} \left\{ 2\zeta_1 \zeta_2 F_2(3; 1, 1; 1, 1; \zeta_1^2, \zeta_2^2) + \frac{3}{4} \zeta_1 F_2\left(\frac{5}{2}; 1, 1; 1, \frac{1}{2}; \zeta_1^2, \zeta_2^2\right) \right. \\ \left. + \frac{3}{4} \zeta_2 F_2\left(\frac{5}{2}; 1, 1; \frac{1}{2}, 1; \zeta_1^2, \zeta_2^2\right) + \frac{1}{\pi} F_2\left(2; 1, 1; \frac{1}{2}, \frac{1}{2}; \zeta_1^2, \zeta_2^2\right) \right\},$$

where $\zeta_j = z_j \bar{w}_j$ for $j = 1, 2$. If $k_1 = k_2 = 1$, then

$$(3.2) \quad F_2(3; 1, 1; 1, 1; x, y) = \frac{1}{(1-x)^3} F\left(3, 1; 1; \frac{y}{1-x}\right) = \frac{1}{(1-x-y)^3}.$$

If $k_1 = 1$ and $k_2 = 0$, then using (2.2), we have

$$F_2\left(\frac{5}{2}; 1, 1; 1, \frac{1}{2}; x, y\right) = \frac{1}{(1-x)^{\frac{5}{2}}} F\left(\frac{5}{2}, 1; \frac{1}{2}; \frac{y}{1-x}\right).$$

Since we have

$$F\left(\frac{5}{2}, 1; \frac{1}{2}; \zeta\right) = \sum_{k=0}^{\infty} \left(\frac{4}{3}k^2 + \frac{8}{3}k + 1\right) \zeta^k = \frac{3 + 6\zeta - \zeta^2}{3(1-\zeta)^3},$$

we obtain

$$(3.3) \quad F_2\left(\frac{5}{2}; 1, 1; 1, \frac{1}{2}; x, y\right) = \frac{f(x, y)}{3(1-x-y)^3(1-x)^{\frac{3}{2}}},$$

where $f(x, y) = 3(1-x)^2 + 6(1-x)y - y^2$.

If $k_1 = 0$ and $k_2 = 1$, then similarly we have

$$(3.4) \quad F_2\left(\frac{5}{2}; 1, 1; \frac{1}{2}, 1; x, y\right) = \frac{f(y, x)}{3(1-x-y)^3(1-y)^{\frac{3}{2}}}.$$

Finally we have to compute $F_2(2; 1, 1; \frac{1}{2}, \frac{1}{2}; x, y)$. From Example 2.3, we have

$$\begin{aligned} F\left(1, 1; \frac{1}{2}; x\right) &= \frac{1}{1-x} + \frac{x^{\frac{1}{2}}}{(1-x)^{\frac{3}{2}}} \arcsin x^{\frac{1}{2}}, \\ F\left(1, 1; \frac{3}{2}; x\right) &= \frac{\arcsin x^{\frac{1}{2}}}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}, \\ F\left(1, 1; \frac{5}{2}; x\right) &= \frac{3}{x} - \frac{3(1-x)^{\frac{1}{2}}}{x^{\frac{3}{2}}} \arcsin x^{\frac{1}{2}}. \end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned} (3.5) \quad & F_2\left(2; 1, 1; \frac{1}{2}, \frac{1}{2}; x, y\right) \\ &= \sum_{l=0}^2 \frac{1}{(1-x-y)^{l+1}} \left\{ c_l x^l F\left(1, 1; \frac{1}{2} + l, x\right) \right. \\ &\quad \left. + c_l^* y^l F\left(1, 1; \frac{1}{2} + l, y\right) \right\} + \frac{2(1+x+y)}{(1-x-y)^3} \\ &= \frac{f(x, y) x^{\frac{1}{2}} \arcsin x^{\frac{1}{2}}}{2(1-x-y)^3 (1-x)^{\frac{3}{2}}} + \frac{f(y, x) y^{\frac{1}{2}} \arcsin y^{\frac{1}{2}}}{2(1-x-y)^3 (1-y)^{\frac{3}{2}}} + \frac{g(x, y)}{2(1-x)(1-y)(1-x-y)^2}, \end{aligned}$$

where $g(x, y) = 2 - x - y - (x - y)^2$. From (3.1) to (3.5), we can obtain the Bergman kernel for $D_{(2,2)}$. Similarly we can also compute the Szegő kernel.

4. General cases

So far we have obtained the Bergman kernel for $D_{(2,2)}$. In this section we discuss whether the Bergman kernel for D_p with other $p \in \mathbb{N}^2$ can be obtained explicitly and prove that the Bergman kernel for D_p in $\mathbb{C}^2$ when $p \in \mathbb{N}^2$ is represented by means of elementary functions if and only if $p = (1, p_2), (p_1, 1), (2, 2)$.

If $p \notin \mathbb{N}^2$, then there are infinitely many domains D_p in $\mathbb{C}^2$ whose Bergman kernels are represented by means of elementary functions. For example, using Bell's transformation formula of the Bergman kernel under the proper holomorphic mappings, the Bergman kernel for $D_{(1/q_1, 1/q_2)}$ for $(q_1, q_2) \in \mathbb{N}^2$ is obtained as a finite sum in [8] and [5].

For any $p_1, p_2 \in \mathbb{N}$, we have to compute

$$F_2\left(1 + \frac{k_1 + 1}{p_1} + \frac{k_2 + 1}{p_2}; 1, 1; \frac{k_1 + 1}{p_1}, \frac{k_2 + 1}{p_2}; x, y\right),$$

where $k_1 = 0, 1, \dots, p_1 - 1$ and $k_2 = 0, 1, \dots, p_2 - 1$. Let $\gamma_1 = \frac{k_1 + 1}{p_1}$ and $\gamma_2 = \frac{k_2 + 1}{p_2}$. If we apply Lemma 2.5, then we have to compute $F(\gamma_1 + \gamma_2, 1; \gamma_1 + l; x)$ and $F(\gamma_1 + \gamma_2, 1; \gamma_2 + l; y)$ for $l = 0, 1, 2$. Combining Theorem 2.1 (i) and Lemma 2.5, we can obtain the Bergman kernel for D_p in $\mathbb{C}^n$.

Theorem 4.1. *Let $\zeta_j = z_j \bar{w}_j$ for $j = 1, 2$.*

(i) Then the Bergman kernel for D_p in $\mathbb{C}^2$ is

$$K(z, w) = \sum_{k_1=0}^{p_1-1} \sum_{k_2=0}^{p_2-1} c(k_1, k_2) \zeta_1^{k_1} \zeta_2^{k_2} \left[\sum_{l=0}^2 \left\{ c_l \left(\frac{\zeta_1^{p_1 l}}{1 - \zeta_1^{p_1}} + \frac{p_1(\gamma_2 - l) \zeta_1^{p_1(1-\gamma_1)}}{(1 - \zeta_1^{p_1})^{1+\gamma_2-l}} I_l(\zeta_1) \right) \right. \right. \\ \left. \left. + c_l^* \left(\frac{\zeta_2^{p_2 l}}{1 - \zeta_2^{p_2}} + \frac{p_2(\gamma_1 - l) \zeta_2^{p_2(1-\gamma_2)}}{(1 - \zeta_2^{p_2})^{1+\gamma_1-l}} I_l^*(\zeta_2) \right) \right\} \right. \\ \left. + \frac{c_3}{(1 - \zeta_1^{p_1} - \zeta_2^{p_2})^3} \left(1 + \frac{1 - \gamma_1}{\gamma_1} \zeta_1^{p_1} + \frac{1 - \gamma_2}{\gamma_2} \zeta_2^{p_2} \right) \right],$$

where

$$I_l(x) = \int_0^x \frac{t^{k_1+lp_1}}{(1-t^{p_1})^{1-\gamma_2+l}} dt, \quad I_l^*(y) = \int_0^y \frac{t^{k_2+lp_2}}{(1-t^{p_2})^{1-\gamma_1+l}} dt.$$

Here the constants c_l and c_l^* are the same as in Lemma 2.5.

(ii) The Bergman kernel for D_p in $\mathbb{C}^2$ when $p = (p_1, p_2) \in \mathbb{N}^2$ is represented by means of elementary functions if and only if $p = (1, p_2), (p_1, 1), (2, 2)$.

Proof. By Theorem 2.1 (i), for any $\gamma_1, \gamma_2 > 0$,

$$H_l = c_l x^l F(\gamma_1 + \gamma_2, 1, \gamma_1 + l, x) = c_l x^l \left\{ \frac{1}{1-x} + \frac{p_1(\gamma_2 - l) x^{1-\gamma_1-l}}{(1-x)^{1+\gamma_2-l}} I(x^{\frac{1}{p_1}}) \right\}.$$

Similarly we can obtain H_l^* . By Theorem 2.1 (ii) or Tchebycheff's criterion, H_l (or $I_l(x)$) and H_l^* (or $I_l^*(y)$) are represented by means of elementary functions if and only if

$$(4.1) \quad \gamma_1 \in \mathbb{Z} \quad \text{or} \quad \gamma_2 \in \mathbb{Z} \quad \text{or} \quad \gamma_1 + \gamma_2 \in \mathbb{Z}$$

for all $\gamma_1 = \frac{1}{p_1}, \frac{2}{p_1}, \dots, \frac{p_1}{p_1}$ and $\gamma_2 = \frac{1}{p_2}, \frac{2}{p_2}, \dots, \frac{p_2}{p_2}$. If $p = (1, p_2), (p_1, 1)$, then condition (4.1) is true trivially. If $p = (2, 2)$, then $\gamma_1, \gamma_2 \in \{\frac{1}{2}, 1\}$ and it is clear that condition (4.1) is true. However, if $p \neq (1, p_2), (p_1, 1), (2, 2)$, then there exists a pair (γ_1, γ_2) which does not satisfy condition (4.1). For example, if we take $\gamma_1 = \frac{1}{p_1}$ and $\gamma_2 = \frac{1}{p_2}$, then $\gamma_1 + \gamma_2$ cannot be a positive integer. Thus we prove (ii) of Theorem 4.1. $\square$

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