PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 3, March 2009, Pages 955–963
S 0002-9939(08)09583-X
Article electronically published on September 5, 2008

L^p ESTIMATES FOR MAXIMAL AVERAGES ALONG ONE-VARIABLE VECTOR FIELDS IN $\mathbf{R}^2$

MICHAEL BATEMAN

(Communicated by Michael T. Lacey)

ABSTRACT. We prove a conjecture of Lacey and Li in the case that the vector field depends only on one variable. Specifically: let v be a vector field defined on the unit square such that $v(x, y) = (1, u(x))$ for some measurable $u : [0, 1] \to [0, 1]$. Let δ be a small parameter, and let $\mathcal{R}$ be the collection of rectangles R of a fixed width such that δ much of the vector field inside R is pointed in (approximately) the same direction as R. We show that the operator defined by

$$M_{\mathcal{R}} f(z) = \sup_{z \in R \in \mathcal{R}} \frac{1}{|R|} \int_R |f| \tag{0.1}$$

is bounded on L^p for $p > 1$ with constants comparable to $\frac{1}{\delta}$.

1. Introduction

In the paper [6], Lacey and Li reduce the problem of bounding in L^2 the Hilbert transform along a $C^{1+\varepsilon}$ vector field to estimating the L^p norm of a related maximal function for some $p < 2$. They have established these maximal function bounds when $p = 2$ (see [7]) and conjecture that they hold for $p > 1$. Here we prove the conjecture for vector fields of one variable. More precise statements follow.

Let v be a vector field on $\mathbf{R}^2$. We will assume $v\colon [0, 1] \times [0, 1] \to [0, 1]$, i.e., we work only in a bounded region, and we assume all vectors are of the form $v(x, y) = (1, u(x, y))$. To define the maximal operator in question we need to introduce some notation. For a rectangle R, we write $L(R)$ for its length, and $W(R)$ for its width. Let $slope(R)$ be the slope of the long side of R. (We will assume $L(R) \geq W(R)$.) We define its interval of uncertainty $EX(R)$ to be the interval of width $\frac{W(R)}{L(R)}$ centered at $slope(R)$. Let

$$V(R) = \{(x, y) \in R\colon u(x, y) \in EX(R)\}. \tag{1.1}$$

Fix $0 < \delta \leq 1$, $0 < w \ll 1$, and let

$$\widetilde{\mathcal{R}}_{\delta,w,v} = \{R\colon L(R) \leq \frac{1}{100||v||_{lip}}, W(R) = w, \text{ and } |V(R)| \geq \delta|R|\}, \tag{1.2}$$

where $||v||_{lip}$ is the Lipschitz constant of the vector field v, and where $|\,.\,|$ indicates the Euclidean measure of a set. In words: $\tilde{\mathcal{R}}_{\delta,w,v}$ is the collection of rectangles R

Received by the editors Feburary 18, 2008.
2000 *Mathematics Subject Classification.* Primary 42B25.
This work was supported in part by NSF Grant DMS0653763.

such that δ much of the vector field in R is pointed in (almost) the same direction as R.

We will consider several similar maximal operators in this paper. If $\mathcal{R}$ is a collection of rectangles, define

$$M_{\mathcal{R}}f(z) = \sup_{z\in R\in\mathcal{R}} \frac{1}{|R|}\int_R |f|. \tag{1.3}$$

The motivation for studying this operator comes from work of Lacey and Li [1], in which they prove Theorem 1.1. Define, for a sufficiently small value of β, the truncated integral operator

$$H_{v,\beta}f(z) = p.v. \int_{-\beta}^{\beta} \frac{f(z+tv(z))}{t}dt. \tag{1.4}$$

Theorem 1.1 (Lacey-Li)**.** *Suppose there is a $p < 2$ and an N such that for any Lipschitz vector field v,*

$$||M_{\tilde{\mathcal{R}}_{\delta,w,v}}||_{L^p\to L^p} \lesssim \frac{1}{\delta^N} \tag{1.5}$$

for any $0 < w \ll \frac{1}{100||v||_{lip}}$. Then if v is a $C^{1+\varepsilon}$ vector field for some $\varepsilon > 0$,

$$||H_{v,\beta}||_{L^2\to L^2} \lesssim (1+\log||v||_{C^{1+\varepsilon}})^2. \tag{1.6}$$

It is interesting to note that the boundedness of $H_{v,\beta}$ on L^2 is strong enough to prove Carleson's theorem on pointwise convergence of Fourier series, even if only known for smooth vector fields v depending on one variable. Similarly, boundedness on L^2 of the one-variable version of the Hilbert transform is a consequence of Carleson's theorem. The reader is encouraged to consult [6] for the full story. Here we prove that the hypothesis of this theorem is satisfied provided that the vector field v depends only on one variable. In fact, this additional assumption eliminates the need to assume that v has any smoothness. So now we define

$$\mathcal{R}_{\delta,w,v} = \{R\colon |V(R)| \geq \delta|R|, W(R) = w \text{ and } L(R) \leq 1\}. \tag{1.7}$$

Theorem 1.2. *Let $v\colon [0,1]\times[0,1] \to [0,1]$ be a measurable vector field depending only on the first variable; i.e., let $v(x,y) = u(x)$ for some measurable $u : [0,1] \to [0,1]$. Then*

$$||M_{\mathcal{R}_{\delta,w,v}}||_{L^p\to L^p} \lesssim \frac{1}{\delta}, \tag{1.8}$$

with constants independent of w and v.

The key observation in the proof is that one can essentially reduce the problem to the boundedness of the one-dimensional dyadic maximal operator. This appears implicity in Lemmas 3.2 and 3.3.

In section 2, we reduce the problem to a model with discrete slopes and parallelograms that project vertically to dyadic intervals. There is essentially nothing new here, and experts may wish to skim for notation. In section 3, we prove Theorem 1.2.

Finally, we mention some related work on Hilbert transforms and maximal operators along vector fields. The paper [1] by Carbery, Seeger, Wainger, and Wright is especially relevant in the situation of a one-variable vector field. For a discussion of maximal directional Hilbert transforms, see work of Kim [5] and Karagulyan [4].

2. Reductions

We begin by defining a discrete set of slopes. Let

$$S_k = \{\frac{j+\frac{1}{2}}{2^k} : j \in \{0,1,...,2^k-1\}\}. \tag{2.1}$$

Let

$$\mathcal{R}^k_{\delta,w,v} = \{R \in \mathcal{R}_{\delta,w,v} : 2^{k-1}w < L(R) \leq 2^k w \text{ and } slope(R) \in S_k\}. \tag{2.2}$$

Note that $\mathcal{R}^k_{\delta,w,v}$ is just a collection of rectangles in $\mathcal{R}_{\delta,w,v}$ whose intervals of uncertainty have size about 2^{-k}, and whose slopes are 2^{-k}-separated. Let

$$\mathcal{R}^{dis}_{\delta,w,v} = \bigcup_{k=1}^{\log\frac{1}{w}} \mathcal{R}^k_{\delta,w,v}. \tag{2.3}$$

Now we show that it is enough to consider averages over rectangles in $R^{dis}_{\delta,w,v}$.

Lemma 2.1. *For any locally integrable function f,*

$$M_{\mathcal{R}_{\delta,w,v}} f(z) \lesssim M_{\mathcal{R}^{dis}_{\frac{\delta}{10},5w,v}} f(z). \tag{2.4}$$

Proof. Let $R \in \mathcal{R}_{\delta,w,v}$ with $|EX(R)| \sim 2^{-k}$. There are two slopes s_1 and s_2 in S_k such that

$$|s_j - slope(R)| \leq |EX(R)|. \tag{2.5}$$

There are (at least) two corresponding rectangles R_1 and R_2 such that $slope(R_j) = s_j$ and such that $R \subseteq 5R_j$. Further, either $|V(R_1)| \geq \frac{\delta}{10}|R_1|$ or $|V(R_2)| \geq \frac{\delta}{10}|R_2|$. Say it holds for R_1. Then

$$\frac{1}{|R|}\int_R |f| \leq \frac{25}{|R_1|}\int_{R_1} |f|, \tag{2.6}$$

and $R_1 \in \mathcal{R}^{dis}_{\frac{\delta}{10},5w,v}$. This completes the proof. □

Hence we may restrict our attention to the discrete model. We will identify slopes with intervals. That is, we will identify $s \in S_k$ with the dyadic interval centered at s. So $S_0 = \{[0,1]\}$, $S_1 = \{[0,\frac{1}{2}),[\frac{1}{2},1]\}$, $S_2 = \{[0,\frac{1}{4}),[\frac{1}{4},\frac{1}{2}],[\frac{1}{2},\frac{3}{4}],[\frac{3}{4},1]\}$, etc. With this identification, it is clear what we mean by $s \subseteq s'$ for $s \in S_k$ and $s' \in S_{k'}$ for some $k' < k$.

We will further restrict our attention to a model in which we average over parallelograms that project vertically onto dyadic intervals. The reduction to parallelograms is trivial. Let $\mathcal{D}$ be the dyadic intervals, and let $\mathcal{D}'$ be the intervals in $\mathcal{D}$ shifted left by $\frac{1}{3}$. It is not too difficult to check (use binary expansions) that if J is an interval, then either there is a $K \in \mathcal{D}$ with $J \subseteq K$ and $|K| \leq 16|J|$, or there is a $K \in \mathcal{D}'$ with $J \subseteq K$ and $|K| \leq 16|J|$. With this observation it is clear that we may control $M_{\mathcal{R}_{\delta,w,v}}$ with two dyadic models, with comparable values of the parameter δ.

3. Proof of main theorem

We begin this section by rewriting the definition of the maximal operator under consideration, taking into account the reductions made in the previous section. This will require some new notation. Then we will state a covering lemma and indicate how it yields Theorem 1.2.

3.1. **Notation.** Fix a small number w (and for convenience, assume w is an integer power of 2). Let $u : [0,1] \to S_{\log \frac{1}{w}}$, and let $v(x,y) = (1, u(x))$. Now let $\mathcal{D}$ be the dyadic intervals contained in $[0,1]$. Let $I \in \mathcal{D}$, and let $s \in S_{\log \frac{|I|}{w}}$. (Recall that parallelograms with length $|I|$ will only have slopes defined up to an error of $\frac{w}{|I|}$; this is why we take $s \in S_{\log \frac{|I|}{w}}$.) For the remainder of the paper, we will view v, w, and δ as being fixed.

Define the popularity of a slope s in the interval I to be

$$Pop_I(s) = \frac{1}{|I|}|\{x \in I \colon u(x) \subseteq s\}|. \tag{3.1}$$

Note that the popularity of a slope s in the interval I is the proportion of points in I at which the vector field is pointed in approximately the direction s. Hence if $Pop_I(s) \geq \delta$, then any rectangle R projecting to I and pointed in the direction s will be admissible in the definition of the maximal operator, since then $V(R) \geq \delta$. Also, we again recall that slopes are viewed as intervals, hence the notation $u(x) \subseteq s$. Let

$$S(I) = \{s \in S_{\log \frac{|I|}{w}} \colon Pop_I(s) \geq \delta\}. \tag{3.2}$$

This is the set of allowable slopes for rectangles projecting to I. Given a parallelogram R, define *slope*(R) to be the slope of the long side of R, and define *int*(R) to be the projection of R onto the x-axis. We will let

$$\mathcal{R} = \{\text{parallelograms } R \colon \mathit{int}(R) \in \mathcal{D} \text{ and } \mathit{slope}(R) \in S(\mathit{int}(R))\}. \tag{3.3}$$

Because of this, all intervals considered in the rest of the paper are assumed to be dyadic. Recall that $M_{\mathcal{R}}$ is defined by

$$M_{\mathcal{R}} f(z) = \sup_{z \in R \in \mathcal{R}} \frac{1}{|R|} \int_R |f| \tag{3.4}$$

for locally integrable f. Our goal is to show that $||M_{\mathcal{R}}||_{L^p \to L^p} \lesssim \frac{1}{\delta}$.

3.2. **Statement of covering lemma.** We remark that there is nothing new about this covering lemma approach. It is similar to the proof of the Hardy-Littlewood maximal theorem from the Vitali covering lemma. A. Cordoba and R. Fefferman, and Strömberg used it several times in the 1970s in the context of maximal averages over families of rectangles. See [2], [3] and [8]. A more recent use can be found in, e.g., [7].

Lemma 3.1. *Let $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$. Let q be an integer greater than or equal to 2. Then we may write $\widetilde{\mathcal{R}}$ as the disjoint union of collections $\mathcal{R}_1$ and $\mathcal{R}_2$ such that*

$$\left| \bigcup_{R \in \mathcal{R}_2} R \right| \lesssim \sum_{R \in \mathcal{R}_1} |R| \tag{3.5}$$

and

$$\int \left(\sum_{R \in \mathcal{R}_1} \chi_R \right)^q \leq c_q \frac{1}{\delta^{q-1}} \sum_{R \in \mathcal{R}_1} |R|. \tag{3.6}$$

To see that the lemma implies Theorem 1.2, let $f \in L^p$, let $\frac{1}{p} + \frac{1}{q} = 1$, and let

$$E_\lambda = \{M_{\mathcal{R}} f > \lambda\}. \tag{3.7}$$

Then write $E_\lambda = \bigcup_{R \in \widetilde{\mathcal{R}}} R$ for some $\widetilde{\mathcal{R}} \subseteq \mathcal{R}$, where

$$\frac{1}{|R|} \int_R |f| > \lambda \tag{3.8}$$

for $R \in \widetilde{\mathcal{R}}$. We have a decomposition $\widetilde{\mathcal{R}} = \mathcal{R}_1 \sqcup \mathcal{R}_2$ as in the statement of the lemma, which gives us

$$\begin{aligned} \sum_{R \in \mathcal{R}_1} |R| &\leq \sum_{R \in \mathcal{R}_1} \frac{1}{\lambda} \int_R |f| \\ &\leq \frac{1}{\lambda} ||f||_p \left(\int \left(\sum_{R \in \mathcal{R}_1} \chi_R \right)^q \right)^{\frac{1}{q}} \\ &\lesssim \frac{1}{\lambda} ||f||_p \frac{1}{\delta^{1-\frac{1}{q}}} \left(\sum_{R \in \mathcal{R}_1} |R| \right)^{\frac{1}{q}}. \end{aligned} \tag{3.9}$$

This implies

$$\sum_{R \in \mathcal{R}_1} |R| \lesssim \frac{1}{\delta} \frac{1}{\lambda^p} ||f||_p^p. \tag{3.10}$$

This quantity obviously dominates $\left|\bigcup_{R \in \mathcal{R}_1} R\right|$, and it dominates $\left|\bigcup_{R \in \mathcal{R}_2} R\right|$ by the covering lemma. Hence

$$|E_\lambda| \lesssim \frac{1}{\delta} \frac{1}{\lambda^p} ||f||_p^p. \tag{3.11}$$

This is the weak type (p, p) estimate for $M_{\mathcal{R}}$. Since we can prove the covering lemma for arbitrarily large integers q, we can prove weak type (p, p) for any $p > 1$.

3.3. **Proof of covering lemma.**

3.3.1. *Selection procedure.* We construct the collections $\mathcal{R}_1$ and $\mathcal{R}_2$ as follows. Initialize

$$\begin{aligned} \widetilde{\mathcal{R}} &:= \widetilde{\mathcal{R}}, \\ \mathcal{R}_1 &:= \emptyset, \\ \mathcal{R}_2 &:= \emptyset. \end{aligned} \tag{3.12}$$

While $\widetilde{\mathcal{R}} \neq \emptyset$, choose $R \in \widetilde{\mathcal{R}}$ of maximal length, and update

$$\begin{aligned} \widetilde{\mathcal{R}} &:= \widetilde{\mathcal{R}} \setminus \{R\}, \\ \mathcal{R}_1 &:= \mathcal{R}_1 \cup \{R\}, \\ \mathcal{R}_2 &:= \mathcal{R}_2; \end{aligned} \tag{3.13}$$

if there is an $R' \in \widetilde{\mathcal{R}}$ such that $R' \subseteq \{\sum_{R' \in \mathcal{R}_1} \chi_{5R'} \geq 1\}$, update

$$\begin{aligned} \widetilde{\mathcal{R}} &:= \widetilde{\mathcal{R}} \setminus \{R'\}, \\ \mathcal{R}_1 &:= \mathcal{R}_1, \\ \mathcal{R}_2 &:= \mathcal{R}_2 \cup \{R'\}. \end{aligned} \tag{3.14}$$

Here, of course, by $5R'$ we mean the parallelogram with the same center and side lengths inflated by a factor of 5. We make one important observation about the parallelograms in $\mathcal{R}_1$. If $R, R' \in \mathcal{R}_1$ intersect, then they have different slopes. More precisely, if $L(R) \leq L(R')$, then $slope(R) \not\supseteq slope(R')$. For if it did, then $R \subseteq 5R'$.

(We are using the fact that $W(R) = W(R')$.) Hence R was put into the collection $\mathcal{R}_2$.

It is clear by construction and by Chebyshev that

$$\left|\bigcup_{R\in\mathcal{R}_2} R\right| \lesssim \sum_{R\in\mathcal{R}_1} |R|, \tag{3.15}$$

so it remains to prove the estimate (3.6). Note that

$$\int \left(\sum_{R\in\mathcal{R}_1} \chi_R\right)^q \lesssim \sum_{R\in\mathcal{R}_1} \int_R \left(\sum_{R'\in\mathcal{R}_1,\ int(R')\subseteq int(R)} \chi_{R'}\right)^{q-1}, \tag{3.16}$$

so if we define

$$f(x,y) = \sum_{R'\in\mathcal{R}_1,\ int(R')\subseteq int(R)} \chi_{R'}(x,y), \tag{3.17}$$

it is enough to show

$$\int_R f(x,y)^{q-1} \lesssim \frac{1}{\delta^{q-1}}|R| \tag{3.18}$$

for any $R \in \mathcal{R}_1$.

3.3.2. *Uniform estimates on rectangles.* Without loss of generality, we will assume $int(R) = [0,1]$. To prove (3.18), we will introduce some auxiliary functions. To do this, we need some more notation. The important point of this section is that we can control the two-variable function f with a function of one variable that is relatively well-behaved.

For $I \in \mathcal{D}$, we will define a set of slopes $T(I)$ as follows. The definition is inductive, starting with the largest interval and then moving to its subintervals. First for $[0,1]$, the largest interval, define

$$T([0,1]) = S([0,1]). \tag{3.19}$$

Note that $T([0,1])$ is just the set of allowable slopes for the interval $[0,1]$. (Recall that the allowable slopes for an interval are those that are at least δ-popular.) Now for smaller intervals I, we will define $T(I)$ similarly, except that we will not include slopes that have been used by an ancestor of I. (By "ancestor", we mean another dyadic interval containing I.) More precisely, having defined $T(K)$ for $K \supsetneq I$, define

$$T(I) = \{s \in S(I) \colon s \not\supseteq s' \text{ for any } s' \in T(K), K \supsetneq I\}. \tag{3.20}$$

For $s \in T(I)$, let

$$\mu_I^s = |I| Pop_I(s) = |\{x \in I \colon u(x) \subseteq s\}|; \tag{3.21}$$

otherwise, let $\mu_I^s = 0$. Now we define the auxiliary functions: let

$$g(x) = \sum_I \chi_I(x)\#(T(I)) \tag{3.22}$$

and

$$h(x) = \sum_I \sum_{s\in T(I)} \chi_I(x)\frac{\mu_I^s}{|I|}. \tag{3.23}$$

Our strategy will be to control the function f by the one-variable function g, and then to control g by the function h, which we will show to be in **BMO**. We remark that the function h looks very similar to the adjoint of a particular linearization of the one-dimensional dyadic maximal function acting on the function **1**. The proof of Lemma 3.3 below is essentially the proof that this function is in dyadic **BMO**.

Lemma 3.2. *With f and h defined above, we have $f(x,y) \leq \frac{1}{\delta}h(x)$ for every y.*

Lemma 3.3. *With h defined immediately above, $h \in \mathbf{BMO}_{dyadic}([0,1])$.*

With these two lemmas, we can easily finish the proof of Lemma 3.1. By the John-Nirenberg theorem, and the fact that $\int_{[0,1]} h(x)dx = 1$, we have $||h||_r \leq c_r$ for any $1 \leq r < \infty$. Hence

$$\begin{aligned} \int_R (f(x,y))^{q-1}\, dxdy \quad &\leq \quad \frac{1}{\delta^{q-1}} w \int_{[0,1]} h(x)^{q-1} dx \\ &\leq \quad c_q \frac{1}{\delta^{q-1}} |R|. \end{aligned} \tag{3.24}$$

This completes the proof of the covering lemma. We turn our attention to the proofs of Lemmas 3.2 and 3.3. Lemma 3.3 is simple, and not really new, so we prove it first.

Proof of Lemma 3.3 . Define $\mu_I = \sum_{s \in T(I)} \mu_I^s$. The sequence μ_I is a Carleson sequence; i.e., for any interval I,

$$\sum_{J \subseteq I} \mu_J \leq C|I|. \tag{3.25}$$

(In fact, we may take $C = 1$ here.) This holds because no x-coordinate can choose more than one slope. Note that

$$h(x) = \sum_I \sum_{s \in T(I)} \chi_I(x) \frac{\mu_I^s}{|I|} = \sum_I \chi_I(x) \frac{\mu_I}{|I|}. \tag{3.26}$$

A function of this form is called a *balayage* of the Carleson sequence μ_I, and such functions are easily shown to be in $\mathbf{BMO}_{dyadic}$. To do this, it is enough to find, for each I, a number b_I such that

$$\frac{1}{|I|} \int_I |h(x) - b_I| dx \leq C. \tag{3.27}$$

Let

$$b_I = \sum_{K \supseteq I} \frac{\mu_K}{|K|}, \tag{3.28}$$

and compute, using the fact that μ_I is a Carleson sequence:

$$\begin{aligned} \frac{1}{|I|} \int_I |h(x) - b_I| dx \quad &= \quad \frac{1}{|I|} \int_I \sum_{J \subsetneq I} \chi_J(x) \frac{\mu_J}{|J|} dx \\ &= \quad \frac{1}{|I|} \sum_{J \subsetneq I} \mu_J \\ &\leq \quad C. \end{aligned} \tag{3.29}$$

□

The proof of Lemma 3.2 is a bit more involved.

Proof of Lemma 3.2 . First note that if $s \in T(I)$, then $\frac{\mu_I^s}{|I|} \geq \delta$, so

$$g(x) \leq \frac{1}{\delta} \sum_I \sum_{s \in T(I)} \chi_I(x) \frac{\mu_I^s}{|I|} = \frac{1}{\delta} h(x). \tag{3.30}$$

It remains to show $f(x,y) \leq g(x)$ for all y.

Let

$$C(x,y) = \{R \in \mathcal{R}_1 \colon (x,y) \in R\}, \tag{3.31}$$

and let

$$r_I(x,y) = \{s \in S_{\log \frac{|I|}{w}} : \exists R \in C(x,y) \text{ with } int(R) = I \text{ and } slope(R) = s\}; \tag{3.32}$$

note that it may be empty for some I. Now we define two collections of pairs of intervals and slopes:

$$\begin{aligned} P &= \{(I,s) \colon s \in r_I(x,y)\}, \\ Q &= \{(I,s) \colon s \in T(I)\}. \end{aligned} \tag{3.33}$$

Two facts about the sets P and Q will finish the proof of Lemma 3.2.

Claim 3.4. *We have*
A. $f(x,y) = \#(P)$ and $g(x) = \#(Q)$,
B. $\#(P) \leq \#(Q)$.

Proof of Part A. If $R \in C(x,y)$, then there is some I with $int(R) = I$ and $s \in r_I(x,y)$ with $slope(R) = s$. On the other hand, given $(I,s) \in P$, there can be at most one rectangle R in $C(x,y)$ with $int(R) = I$ and $slope(R) = s$. For if there were two, then the shorter one would not be in the collection $\mathcal{R}_1$, by the construction of $\mathcal{R}_1$ and $\mathcal{R}_2$. The analogous fact for g and Q follows from the definitions of g, Q, and the collections $T(I)$. □

Proof of Part B. Of course it is enough to find an injection from P to Q.

Subclaim 3.5. *Let $(I,s) \in P$. Then there is $(J,t) \in Q$ with $I \subseteq J$ and $s \supseteq t$.*

Proof. Note that $Pop_I(s) \geq \delta$, so if there is no $(J,t) \in Q$ with $I \subsetneq J$ and $s \supseteq t$, then $(I,s) \in Q$ by definition of $T(I)$. □

Now let $\alpha \colon P \to Q$ send (I,s) to one of the elements $(J,t) \in Q$ provided by the subclaim. (Such a choice may not be unique, but this is unimportant.) The important point is this:

Subclaim 3.6. *Suppose $(I_k,s_k) \in P$ and $(J_k,t_k) \in Q$ for $k = 1,2$. Suppose $I_k \subseteq J_k$ and $s_k \supseteq t_k$ for $k = 1,2$. If $(I_1,s_1) \neq (I_2,s_2)$, then $(J_1,t_1) \neq (J_2,t_2)$.*

Note that this subclaim guarantees that the function α above is one-to-one.

Proof. Let (I_k,s_k) and (J_k,t_k) be as in the statement of the subclaim with $(I_1,s_1) \neq (I_2,s_2)$. We have both $x \in I_1$ and $x \in I_2$, so without loss of generality, we will assume $I_1 \subseteq I_2$. Since s_1, s_2 are dyadic intervals, we have that if $|s_1| \geq |s_2|$, then either $s_1 \supseteq s_2$ or $s_1 \cap s_2 = \emptyset$. We consider the following two cases:

CASE A: $s_1 \not\supseteq s_2$. By the preceding observation, we have $s_1 \cap s_2 = \emptyset$. Since $t_1 \subseteq s_1$ and $t_2 \subseteq s_2$, we have $t_1 \cap t_2 = \emptyset$, and hence $t_1 \neq t_2$.

CASE B: $s_1 \supseteq s_2$. In fact, this case is not possible. Suppose it were. Then we would have R_k, $k = 1, 2$, with $int(R_k) = I_k$, $slope(R_k) = s_k$, and $R_1 \cap R_2 \neq \emptyset$. But then $R_1 \subseteq 5R_2$, so $R_1 \notin \mathcal{R}_1$. $\square$

Acknowledgements

The author thanks Nets Katz and Xiaochun Li for helpful discussions. The author also thanks the referee for pointers to the literature.

References

1. Carbery, Anthony, Andreas Seeger, Stephen Wainger, and James Wright. *Classes of singular integral operators along variable lines*, J. Geom. Anal., vol. 9, no. 4, 1999, pp. 583–605. MR1757580 (2001g:42026)
2. Cordoba, Antonio, and Robert Fefferman. *On differentiation of integrals*, Proc. Natl. Acad. Sci. USA, vol. 74, no. 6, June 1977, pp. 2211–2213. MR0476977 (57:16522)
3. Cordoba, Antonio, and Robert Fefferman. *A geometric proof of the strong maximal theorem*, Annals of Math., 2nd Series, vol. 102, no. 1, July 1975, pp. 95–100. MR0379785 (52:690)
4. Karagulyan, Grigor. *On unboundedness of maximal operators for directional Hilbert transforms*, Proc. Amer. Math. Soc., vol. 135, no. 10, 2007, pp. 3133–3141 (electronic). MR2322743 (2008e:42044)
5. Kim, Joonil. *Sharp L^2 bound of maximal Hilbert transforms over arbitrary sets of directions.* (English summary), J. Math. Anal. Appl., vol. 335, no. 1, 2007, pp. 56–63. MR2340304
6. Lacey, Michael, and Xiaochun Li. *On a Conjecture of EM Stein on the Hilbert Transform on Vector Fields.* Available at `http://arxiv.org/abs/0704.0808`.
7. Lacey, Michael, and Xiaochun Li. *On a Lipschitz Variant of the Kakeya Maximal Function.* Available at `http://arxiv.org/abs/math/0601213`.
8. Strömberg, Jan-Olov. *Maximal functions associated to rectangles with uniformly distributed directions*, Annals of Math., 2nd Series, vol. 107, no. 2, 1978, pp. 399–402. MR0481883 (58:1978)

Department of Mathematics, Indiana University, Rawles Hall, 831 East 3rd Street, Bloomington, Indiana 47405

E-mail address: `mdbatema@indiana.edu`