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MINIMALITY OF THE BOUNDARY OF A RIGHT-ANGLED COXETER SYSTEM

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ABSTRACT. In this paper, we show that the boundary $\partial\Sigma(W,S)$ of a right-angled Coxeter system (W,S) is minimal if and only if $W_{\tilde{S}}$ is irreducible, where $W_{\tilde{S}}$ is the minimum parabolic subgroup of finite index in W. We also provide several applications and remarks. In particular, we show that for a right-angled Coxeter system (W,S), the set $\{w^\infty \mid w \in W, o(w)=\infty\}$ is dense in the boundary $\partial\Sigma(W,S)$.

1. Introduction and preliminaries

The purpose of this paper is to study dense subsets of the boundary of a Coxeter system. A *Coxeter group* is a group W having a presentation

$$\langle\, S \mid (st)^{m(s,t)} = 1 \text{ for } s,t \in S \,\rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,t) = m(t,s)$ for each $s,t \in S$,
(2) $m(s,s) = 1$ for each $s \in S$, and
(3) $m(s,t) \geq 2$ for each $s,t \in S$ with $s \neq t$.

The pair (W,S) is called a *Coxeter system*. If, in addition,

(4) $m(s,t) = 2$ or ∞ for each $s,t \in S$ with $s \neq t$,

then (W,S) is said to be *right-angled*. Let (W,S) be a Coxeter system. Then W has the *word metric* d_ℓ defined by $d_\ell(w,w') = \ell(w^{-1}w')$ for each $w,w' \in W$, where $\ell(w)$ is the word length of w with respect to S. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and is called a *parabolic subgroup*. If T is the empty set, then W_T is the trivial group. A subset $T \subset S$ is called a *spherical subset of* S if the parabolic subgroup W_T is finite.

Every Coxeter system (W,S) determines a *Davis complex* $\Sigma(W,S)$ which is a CAT(0) geodesic space ([6], [7], [8], [20]). Here the 1-skeleton of $\Sigma(W,S)$ is the Cayley graph of W with respect to S. The natural action of W on $\Sigma(W,S)$ is proper, cocompact and by isometries. If W is infinite, then $\Sigma(W,S)$ is noncompact and $\Sigma(W,S)$ can be compactified by adding its ideal boundary $\partial\Sigma(W,S)$ ([4], [7, §4]).

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This boundary $\partial\Sigma(W,S)$ is called the *boundary of* (W,S). We note that the natural action of W on $\Sigma(W,S)$ induces an action of W on $\partial\Sigma(W,S)$ by homeomorphisms.

A subset A of a space X is said to be *dense* in X if $\overline{A} = X$. A subset A of a metric space X is said to be *quasi-dense* if there exists $N > 0$ such that each point of X is N-close to some point of A. Suppose that a group G acts on a compact metric space X by homeomorphisms. Then X is said to be *minimal* if every orbit Gx is dense in X.

For a negatively curved group Γ and the boundary $\partial\Gamma$ of Γ, it is known that each orbit $\Gamma\alpha$ is dense in $\partial\Gamma$ for any $\alpha \in \partial\Gamma$, that is, $\partial\Gamma$ is minimal ([9]). We note that Coxeter groups are nonpositive curved groups and not negatively curved groups in general. Indeed, there exist examples of Coxeter systems whose boundaris are not minimal (cf. [14], [16]). The purpose of this paper is to investigate when the boundary of a Coxeter system is minimal.

In [14, Theorem 1], we have obtained a sufficient condition of a Coxeter system (W,S) such that *some* orbit of the Coxeter group W is dense in the boundary $\partial\Sigma(W,S)$. After some preliminaries in Section 2, we first show that the boundary of such a Coxeter system is minimal; that is, we prove the following theorem in Section 3.

Theorem 1. *Let (W,S) be a Coxeter system. Suppose that $W^{\{s_0\}}$ is quasi-dense in W with respect to the word metric and $o(s_0t_0) = \infty$ for some $s_0, t_0 \in S$, where $o(s_0t_0)$ is the order of s_0t_0 in W. Then*

(1) *$\partial\Sigma(W,S)$ is minimal, and*
(2) *$\{w^\infty \mid w \in W,\ o(w) = \infty\}$ is dense in $\partial\Sigma(W,S)$.*

Here $W^{\{s_0\}} = \{w \in W \mid \ell(wt) > \ell(w) \text{ for each } t \in S \setminus \{s_0\}\} \setminus \{1\}$ and w^∞ is the point of $\partial\Sigma(W,S)$ to which the sequence $\{w^i \mid i \in \mathbb{N}\} \subset \Sigma(W,S)$ converges in $\Sigma(W,S) \cup \partial\Sigma(W,S)$.

In Sections 4 and 5, we investigate right-angled Coxeter groups and we prove the following main theorem.

Theorem 2. *For a right-angled Coxeter system (W,S), the boundary $\partial\Sigma(W,S)$ is minimal if and only if $W_{\tilde{S}}$ is irreducible.*

Here for $T \subset S$, W_T is said to be *irreducible* if W_T does not split as a product $W_{T_1} \times W_{T_2}$ for any nonempty subsets T_1 and T_2 of T, and $W_{\tilde{S}}$ is the minimum parabolic subgroup of finite index in (W,S), that is, for the irreducible decomposition $W = W_{S_1} \times \cdots \times W_{S_n}$, $\tilde{S} = \bigcup\{S_i \mid W_{S_i} \text{ is infinite}\}$ ([11]).

We provide several applications of Theorem 2 in Sections 5 and 6. In particular, we show the following corollary.

Corollary 3. *For a right-angled Coxeter system (W,S), the set $\{w^\infty \mid w \in W,\ o(w) = \infty\}$ is dense in the boundary $\partial\Sigma(W,S)$.*

In Section 6, we give some remarks on dense subsets of boundaries of CAT(0) groups.

2. Lemmas on Coxeter groups

In this section, we show some lemmas for (right-angled) Coxeter groups which are used later.

We first give some definitions.

Definition 2.1. Let (W,S) be a Coxeter system and $w \in W$. A representation $w = s_1 \cdots s_l$ ($s_i \in S$) is said to be *reduced*, if $\ell(w) = l$, where $\ell(w)$ is the minimum length of a word in S which represents w.

Definition 2.2. Let (W,S) be a Coxeter system. For each $w \in W$, we define $S(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$. For a subset $T \subset S$, we also define $W^T = \{w \in W \mid S(w) = T\}$.

The following lemma is known.

Lemma 2.3 ([1], [5, p.37], [18])**.** *Let (W,S) be a Coxeter system.*

(1) *Let $w \in W$ and let $w = s_1 \cdots s_l$ be a representation. If $\ell(w) < l$, then $w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_l$ for some $1 \leq i < j \leq l$.*
(2) *For each $w \in W$ and $s \in S$, $\ell(ws)$ equals either $\ell(w)+1$ or $\ell(w)-1$, and $\ell(sw)$ also equals either $\ell(w)+1$ or $\ell(w)-1$.*
(3) *For each $w \in W$, $S(w)$ is a spherical subset of S; i.e., $W_{S(w)}$ is finite.*

We can obtain the following lemma from Lemma 2.3 (3).

Lemma 2.4. *Let (W,S) be a Coxeter system and let T be a maximal spherical subset of S. Then W^T is quasi-dense in W.*

Proof. Let $w \in W$. There exists an element w' of longest length in the coset wW_T. Then we show that $S(w') = T$.

Let $t \in T$. Since $w't \in w'W_T = wW_T$ and w' is the element of longest length in wW_T, $\ell(w't) < \ell(w')$, i.e., $t \in S(w')$. Thus $T \subset S(w')$. Now T is a maximal spherical subset of S and $S(w')$ is a spherical subset of S by Lemma 2.3 (3). Hence $S(w') = T$ and $w' \in W^T$.

Here $d_\ell(w,w') \leq \max\{\ell(v) \mid v \in W_T\}$. Hence W^T is quasi-dense in W. $\square$

Lemma 2.5 ([14, Lemma 2.3 (3)])**.** *Let (W,S) be a Coxeter system and $s,t \in S$ such that $o(st) = \infty$. Then $W^{\{s\}}t \subset W^{\{t\}}$.*

Next, we provide some lemmas for right-angled Coxeter groups. We note that right-angled Coxeter groups are *rigid*; that is, a right-angled Coxeter group determines its Coxeter system uniquely up to isomorphism ([21]).

By a consequence of Tits' solution to the word problem ([23], [5, p.50]), we can obtain the following lemma (cf. [12, Lemma 5]).

Lemma 2.6. *Let (W,S) be a right-angled Coxeter system, let $w \in W$, let $w = s_1 \cdots s_l$ be a reduced representation and let $t,t' \in S$. If $tw = t(s_1 \cdots s_l)$ is reduced and $twt' = w$, then $t = t'$ and $ts_i = s_it$ for any $i \in \{1,\dots,l\}$.*

Using Lemma 2.6, we prove the following lemma.

Lemma 2.7. *Let (W,S) be a right-angled Coxeter system, let U be a spherical subset of S, let $s_0 \in S \setminus U$ and let $T = \{t \in U \mid o(s_0t) = 2\}$. Then $W^U s_0 \subset W^{T \cup \{s_0\}}$.*

Proof. Let $w \in W^U$. To prove that $ws_0 \in W^{T\cup\{s_0\}}$, we show that $S(ws_0) = T \cup \{s_0\}$. We note that $\ell(ws_0) = \ell(w)+1$ since $s_0 \notin U = S(w)$. Hence $s_0 \in S(ws_0)$. Also for each $t \in T$, by the definition of T, $\ell(ws_0t) = \ell(wts_0) < \ell(ws_0)$, and $t \in S(ws_0)$. Thus $T \cup \{s_0\} \subset S(ws_0)$. Next we show that $S(ws_0) \subset T \cup \{s_0\}$. Let $t \in S(ws_0)$. Then $\ell(ws_0t) < \ell(ws_0)$. If $w = a_1 \dots a_l$ is a reduced representation, then by Lemma 2.3 (1),

$$ws_0t = (a_1 \dots a_l)s_0t = (a_1 \dots \hat{a}_i \dots a_l)s_0$$

for some $i \in \{1,\dots,l\}$, or $t = s_0$. By Lemma 2.6, we obtain that $s_0t = ts_0$. This implies that if $t \neq s_0$, then $\ell(wt) < \ell(w)$, i.e., $t \in S(w) = U$. Since $t \in U$ and $s_0t = ts_0$, $t \in T$. Hence $S(ws_0) \subset T \cup \{s_0\}$. Thus $S(ws_0) = T \cup \{s_0\}$ and $ws_0 \in W^{T\cup\{s_0\}}$. We obtain that $W^U s_0 \subset W^{T\cup\{s_0\}}$. □

It is well-known that a Coxeter system (W,S) is irreducible if and only if the underlying graph of its Coxeter graph is connected ([1], [5, p.23], [18, p.30]). If the Coxeter system (W,S) is right-angled, then the underlying graph of its Coxeter graph is the graph $\Gamma_\infty(W,S)$, where $\Gamma_\infty(W,S)$ is defined as follows: the vertex set of $\Gamma_\infty(W,S)$ is S and for $s,t \in S$, $\{s,t\}$ spans an edge in $\Gamma_\infty(W,S)$ if and only if $m(s,t) = \infty$. Hence we obtain the following lemma.

Lemma 2.8 (cf. [1], [5], [18])**.** *For a right-angled Coxeter system (W,S), the following statements are equivalent:*

(1) *(W,S) is irreducible.*
(2) *$\Gamma_\infty(W,S)$ is connected.*
(3) *For each $a,b \in S$ with $a \neq b$, there exists a sequence $\{a = s_1, s_2, \dots, s_n = b\} \subset S$ such that $o(s_is_{i+1}) = \infty$ for any $i \in \{1,\dots,n-1\}$.*

3. Minimality of the boundary of a Coxeter system

In this section, we show an extension of a result in [14] on minimality of the boundary of a Coxeter system.

Theorem 3.1. *Let (W,S) be a Coxeter system. Suppose that $W^{\{s_0\}}$ is quasi-dense in W and $o(s_0t_0) = \infty$ for some $s_0,t_0 \in S$. Then*

(1) *$\partial\Sigma(W,S)$ is minimal, and*
(2) *$\{w^\infty \mid w \in W,\ o(w) = \infty\}$ is dense in $\partial\Sigma(W,S)$.*

Proof. Suppose that $W^{\{s_0\}}$ is quasi-dense in W and $o(s_0t_0) = \infty$ for some $s_0,t_0 \in S$. Then we show that $W\gamma$ is dense in $\partial\Sigma(W,S)$ for any $\gamma \in \partial\Sigma(W,S)$.

Let $\gamma \in \partial\Sigma(W,S)$ and let $\{v_i\} \subset W$ be a sequence which converges to γ in $\Sigma(W,S) \cup \partial\Sigma(W,S)$. Since $W^{\{s_0\}}$ is quasi-dense in W, there exists a number $N > 0$ such that for each $v \in W$, $d_\ell(v,x) \le N$ for some $x \in W^{\{s_0\}}$. Hence for each $v \in W$, there exists $u \in W$ such that $\ell(u) \le N$ and $vu \in W^{\{s_0\}}$. For each i, there exists $u_i \in W$ such that $\ell(u_i) \le N$ and $(v_i)^{-1}u_i \in W^{\{s_0\}}$. We note that the set $\{u \in W \mid \ell(u) \le N\}$ is finite because S is finite. Hence $\{u_i \mid i \in \mathbb{N}\}$ is finite, and there exist $u \in W$ and a sequence $\{i_j \mid j \in \mathbb{N}\} \subset \mathbb{N}$ such that $u_{i_j} = u$ for any $j \in \mathbb{N}$. Then for each $j \in \mathbb{N}$, $(v_{i_j})^{-1}u_{i_j} = (v_{i_j})^{-1}u \in W^{\{s_0\}}$ and $(v_{i_j})^{-1}ut_0 \in W^{\{t_0\}}$ by Lemma 2.5, since $o(s_0t_0) = \infty$. Hence $t_0u^{-1}v_{i_j} \in (W^{\{t_0\}})^{-1}$. The sequence $\{t_0u^{-1}v_{i_j} \mid j \in \mathbb{N}\}$ converges to $t_0u^{-1}\gamma$, since $\{v_{i_j} \mid j \in \mathbb{N}\}$ converges to γ. Here we recall the proof of [14, Theorem 4.1]. If we put $x_j = t_0u^{-1}v_{i_j}$ and $\alpha = t_0u^{-1}\gamma$, then the sequence $\{x_j\} \subset (W^{\{t_0\}})^{-1}$ converges to α. By the proof of [14, Theorem 4.1], we obtain that $W\alpha$ is dense in $\partial\Sigma(W,S)$; that is, $Wt_0u^{-1}\gamma$ is dense in $\partial\Sigma(W,S)$. Hence $W\gamma$ is dense in $\partial\Sigma(W,S)$, since $Wt_0u^{-1} = W$. Thus every orbit $W\gamma$ is dense in $\partial\Sigma(W,S)$ and $\partial\Sigma(W,S)$ is minimal.

The minimality of $\partial\Sigma(W,S)$ implies that the set $\{w^\infty \mid w \in W,\ o(w) = \infty\}$ is dense in $\partial\Sigma(W,S)$ (see Proposition 6.2). □

Here we have a question whether conversely if $\partial\Sigma(W,S)$ is minimal, then $W^{\{s_0\}}$ is quasi-dense in W and $o(s_0t_0) = \infty$ for some $s_0, t_0 \in S$. The answer to this question is no in general.

For example, let $S = \{s_1, s_2, s_3\}$ and let

$$W = \langle S \mid s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^4 = (s_2s_3)^4 = (s_3s_1)^4 = 1\rangle.$$

Then W is a negatively curved group and the boundary $\partial\Sigma(W,S)$ is minimal. On the other hand, there do not exist $s_0, t_0 \in S$ such that $o(s_0t_0) = \infty$.

In Section 5, we will show that in the case (W,S) is *right-angled*, the answer to this question is yes.

4. Key lemma

In this section, we prove the following lemma, which plays a key role in the proof of the main theorem.

Lemma 4.1. *Let (W,S) be a right-angled Coxeter system such that W is infinite. If W is irreducible, then $W^{\{s_0\}}$ is quasi-dense in W for some $s_0 \in S$.*

Proof. We suppose that $W^{\{s\}}$ is not quasi-dense in W for any $s \in S$. Then we show that W is not irreducible.

Let $s_0 \in S$, let $T_1 = \{t \in S \mid o(s_0t) = 2\}$ and let $S_1 = S \setminus T_1$. If $T_1 = \emptyset$, then $o(s_0s) = \infty$ for each $s \in S \setminus \{s_0\}$; hence $W^{\{s_0\}}$ is quasi-dense in W, which contradicts the assumption. Thus $T_1 \neq \emptyset$. If $S_1 = \{s_0\}$, then $W = W_{\{s_0\}} \times W_{T_1}$; i.e., W is not irreducible. We suppose that $S_1 \neq \{s_0\}$.

Let $s_1 \in S_1 \setminus \{s_0\}$, let $T_2 = \{t \in T_1 \mid o(s_1t) = 2\}$ and let $S_2 = S \setminus T_2 = S_1 \cup (T_1 \setminus T_2)$. We note that $o(s_it) = 2$ for each $i \in \{0,1\}$ and $t \in T_2$, i.e., $W_{\{s_0,s_1\}\cup T_2} = W_{\{s_0,s_1\}} \times W_{T_2}$. Since $s_1 \in S_1 \setminus \{s_0\}$, we obtain that $o(s_0s_1) = \infty$ and $W_{\{s_0,s_1\}}$ is irreducible.

Now we show that $T_2 \neq \emptyset$. Suppose that $T_2 = \emptyset$. This means that $o(s_1t) = \infty$ for any $t \in T_1$. Let U be a maximal spherical subset of S such that $s_0 \in U$. Then $o(uv) = 2$ for each $u, v \in U$ with $u \neq v$, because (W,S) is right-angled and W_U is finite. Hence $o(s_0u) = 2$ for any $u \in U$, since $s_0 \in U$. This means that $U \subset T_1 \cup \{s_0\}$. Hence $o(s_1u) = \infty$ for any $u \in U$, because $o(s_1t) = \infty$ for any $t \in T_1$ and $o(s_0s_1) = \infty$. Thus $W^Us_1 \subset W^{\{s_1\}}$ by Lemma 2.7. Here by Lemma 2.4, W^U is quasi-dense in W, since U is a maximal spherical subset of S. Hence $W^{\{s_1\}}$ is quasi-dense in W. This contradicts the assumption. Thus we obtain that $T_2 \neq \emptyset$.

If $S_2 = \{s_0, s_1\}$, then $W = W_{\{s_0,s_1\}} \times W_{T_2}$ and W is not irreducible. We suppose that $S_2 \neq \{s_0, s_1\}$. Let $s_2 \in S_2 \setminus \{s_0, s_1\}$, let $T_3 = \{t \in T_2 \mid o(s_2t) = 2\}$ and let $S_3 = S \setminus T_3 = S_2 \cup (T_2 \setminus T_3)$.

By induction, we define s_k, T_{k+1}, S_{k+1} as follows: Let

$$\begin{aligned}
&s_k \in S_k \setminus \{s_0, \dots, s_{k-1}\},\\
&T_{k+1} = \{t \in T_k \mid o(s_kt) = 2\} \text{ and}\\
&S_{k+1} = S \setminus T_{k+1}.
\end{aligned}$$

Then $W_{\{s_0,s_1,\dots,s_k\}\cup T_{k+1}} = W_{\{s_0,s_1,\dots,s_k\}} \times W_{T_{k+1}}$. If $S_{k+1} \setminus \{s_0, s_1, \dots, s_k\} = \emptyset$, then $W = W_{S_{k+1}} \times W_{T_{k+1}}$, i.e., W is not irreducible. Here we note that $T_{k+1} \subset T_k \subset \cdots \subset T_2 \subset T_1$. If $T_k \neq \emptyset$ for each k, then by the finiteness of S, there exists a number n such that $W = W_{S_n} \times W_{T_n}$; hence W is not irreducible.

We prove the following statements by induction on k.

($\mathrm{i_k}$) $T_k \neq \emptyset$.
($\mathrm{ii_k}$) $W_{\{s_0,\dots,s_{k-1}\}}$ is irreducible.
($\mathrm{iii_k}$) There exists a spherical subset $U_k \subset T_k$ such that $W^{U_k \cup \{s_i\}}$ is quasi-dense in W for each $i \in \{0,\dots,k-1\}$.

We first consider the case $k = 2$. The statement ($\mathrm{i_2}$) $T_2 \neq \emptyset$ was proved in the above. Also ($\mathrm{ii_2}$) holds, since $W_{\{s_0,s_1\}} = W_{\{s_0\}} * W_{\{s_1\}}$ is irreducible. We show that the statement ($\mathrm{iii_2}$) holds. Let U be a maximal spherical subset of S such that $s_0 \in U$. Then W^U is quasi-dense in W by Lemma 2.4. Let $U_2 = U \cap T_2$. We note that $U_2 = \{t \in U \,|\, o(s_1 t) = 2\}$. By Lemma 2.7, $W^U s_1 \subset W^{U_2 \cup \{s_1\}}$. Hence $W^{U_2 \cup \{s_1\}}$ is quasi-dense in W. (This implies that $U_2 \neq \emptyset$ by the assumption.) Also $W^{U_2 \cup \{s_0\}}$ is quasi-dense in W, since $W^{U_2 \cup \{s_1\}} s_0 \subset W^{U_2 \cup \{s_0\}}$ by Lemma 2.7. Thus ($\mathrm{iii_2}$) holds.

We suppose that ($\mathrm{i_k}$), ($\mathrm{ii_k}$) and ($\mathrm{iii_k}$) hold for some $k \geq 2$. Then we prove that ($\mathrm{i_{k+1}}$), ($\mathrm{ii_{k+1}}$) and ($\mathrm{iii_{k+1}}$) hold.

($\mathrm{i_{k+1}}$): We show that $T_{k+1} \neq \emptyset$. Suppose that $T_{k+1} = \emptyset$. If $o(s_k s_i) = 2$ for any $i \in \{0,1,\dots,k-1\}$, then $s_k \in T_k$, which contradicts the definition of s_k. Hence $o(s_k s_{i_0}) = \infty$ for some $i_0 \in \{0,1,\dots,k-1\}$. Since $T_{k+1} = \emptyset$, $o(s_k t) = \infty$ for any $t \in T_k$. Here $U_k \subset T_k$ and $o(s_k t) = \infty$ for any $t \in U_k$. Hence $W^{U_k \cup \{s_{i_0}\}} s_k \subset W^{\{s_k\}}$ by Lemma 2.7. By ($\mathrm{iii_k}$), $W^{U_k \cup \{s_{i_0}\}}$ is quasi-dense in W. Thus $W^{\{s_k\}}$ is also quasi-dense in W, which contradicts the assumption. Hence $T_{k+1} \neq \emptyset$.

($\mathrm{ii_{k+1}}$): We show that $W_{\{s_0,\dots,s_{k-1},s_k\}}$ is irreducible. Now $o(s_k s_{i_0}) = \infty$ for some $i_0 \in \{0,1,\dots,k-1\}$ by the above argument. Also $W_{\{s_0,\dots,s_{k-1}\}}$ is irreducible by the hypothesis ($\mathrm{ii_k}$). Hence $W_{\{s_0,\dots,s_{k-1},s_k\}}$ is irreducible.

($\mathrm{iii_{k+1}}$): By ($\mathrm{iii_k}$), there exists a spherical subset $U_k \subset T_k$ such that $W^{U_k \cup \{s_i\}}$ is quasi-dense in W for each $i \in \{0,\dots,k-1\}$. We define $U_{k+1} = U_k \cap T_{k+1}$, i.e., $U_{k+1} = \{t \in U_k \,|\, o(s_k t) = 2\}$. Here $o(s_k s_{i_0}) = \infty$ for some $i_0 \in \{0,1,\dots,k-1\}$ by the above argument. Then $W^{U_k \cup \{s_{i_0}\}} s_k \subset W^{U_{k+1} \cup \{s_k\}}$ by Lemma 2.7. Hence $W^{U_{k+1} \cup \{s_k\}}$ is quasi-dense in W, since $W^{U_k \cup \{s_{i_0}\}}$ is so. Finally we show that $W^{U_{k+1} \cup \{s_i\}}$ is quasi-dense in W for each $i \in \{0,\dots,k-1,k\}$. We note that $W_{\{s_0,\dots,s_{k-1},s_k\}}$ is irreducible by ($\mathrm{ii_{k+1}}$). Hence for each $j_0 \in \{0,\dots,k-1\}$, there exists a sequence $\{s_k = a_0, a_1, \dots, a_m = s_{j_0}\} \subset \{s_i \,|\, i = 0,1,\dots,k\}$ such that $o(a_i a_{i+1}) = \infty$ by Lemma 2.8. Then by Lemma 2.7,

$$\begin{aligned} W^{U_{k+1} \cup \{s_k\}} a_1 a_2 \cdots a_m &\subset W^{U_{k+1} \cup \{a_1\}} a_2 \cdots a_m \\ &\subset \cdots \subset W^{U_{k+1} \cup \{a_m\}} = W^{U_{k+1} \cup \{s_{j_0}\}}, \end{aligned}$$

because $o(s_i u) = 2$ for any $i \in \{0,1,\dots,k-1,k\}$ and $u \in U_{k+1}$. Thus $W^{U_{k+1} \cup \{s_{j_0}\}}$ is quasi-dense in W. Hence ($\mathrm{iii_{k+1}}$) holds.

Thus by the induction on k, we can define s_{k-1}, T_k, S_k which satisfy ($\mathrm{i_k}$), ($\mathrm{ii_k}$) and ($\mathrm{iii_k}$). Since S is finite, there exists a number n such that $S_n = \{s_0, s_1, \dots, s_{n-1}\}$ and $W = W_{S_n} \times W_{T_n}$, where $T_n \neq \emptyset$. Therefore W is not irreducible. $\square$

5. Dense subsets of the boundary of a right-angled Coxeter group

We obtain the following main theorem from Theorem 3.1 and Lemma 4.1.

Theorem 5.1. *Let (W,S) be a right-angled Coxeter system such that W is infinite. Then the following statements are equivalent:*

(1) *$\partial\Sigma(W,S)$ is minimal.*

(2) *$W_{\tilde{S}}$ is irreducible.*
(3) *$W^{\{s_0\}}$ is quasi-dense in W and $o(s_0t_0) = \infty$ for some $s_0, t_0 \in S$.*
(4) *There does not exist a finite-index subgroup of W which splits as a product $W_1 \times W_2$ where each W_i is infinite.*

Proof. (3) $\Rightarrow$ (1): If the statement (3) holds, then $\partial\Sigma(W,S)$ is minimal by Theorem 3.1.

(1) $\Rightarrow$ (2): Suppose that $W_{\tilde{S}}$ is not irreducible. Let $W_{\tilde{S}} = W_{S_1} \times W_{S_2}$, where W_{S_1} and W_{S_2} are infinite. Then $\partial\Sigma(W,S) = \partial\Sigma(W_{\tilde{S}},\tilde{S})$ and $\Sigma(W_{\tilde{S}},\tilde{S}) = \Sigma(W_{S_1},S_1) \times \Sigma(W_{S_2},S_2)$. Here by [11, Theorem 4.3], $\partial\Sigma(W_{S_1},S_1)$ is W-invariant, that is, $W\partial\Sigma(W_{S_1},S_1) = \partial\Sigma(W_{S_1},S_1)$. Thus for $\alpha \in \partial\Sigma(W_{S_1},S_1)$, $\overline{W\alpha} \subset \partial\Sigma(W_{S_1},S_1)$. Hence $\partial\Sigma(W,S)$ is not minimal. In Section 6, we will give a more general proof (Theorem 6.4).

(2) $\Rightarrow$ (3): Suppose that $W_{\tilde{S}}$ is irreducible. By Lemma 4.1, $(W_{\tilde{S}})^{\{s_0\}} = W^{\{s_0\}} \cap W_{\tilde{S}}$ is quasi-dense in $W_{\tilde{S}}$ for some $s_0 \in \tilde{S}$. Here $W = W_{\tilde{S}} \times W_{S\setminus\tilde{S}}$ and $W_{S\setminus\tilde{S}}$ is finite (see [11]). Hence $W^{\{s_0\}}$ is quasi-dense in W. Since $W_{\tilde{S}}$ is irreducible, $o(s_0t_0) = \infty$ for some $t_0 \in \tilde{S}$ by Lemma 2.8. Thus the statement (3) holds.

(4) $\Rightarrow$ (2): If $W_{\tilde{S}}$ is not irreducible, then $W_{\tilde{S}}$ splits as a product $W_{\tilde{S}} = W_{A_1} \times W_{A_2}$ for some $A_i \subset \tilde{S}$, where each W_{A_i} is infinite. Here $W_{\tilde{S}}$ is a finite-index subgroup of W.

(1) $\Rightarrow$ (4): We obtain this implication from Theorem 6.4. $\square$

The following question appears in [15].

Question 5.2. Let (W,S) be a Coxeter system. Is it the case that if (W,S) is an irreducible Coxeter system, then $W\partial\Sigma(W_T,T)$ is dense in $\partial\Sigma(W,S)$ for any subset T of S such that W_T is infinite?

Theorem 5.1 implies that the answer to Question 5.2 is yes for right-angled Coxeter groups. Moreover, as an application of Theorem 5.1, we obtain the following corollary.

Corollary 5.3. *Let (W,S) be a right-angled Coxeter system and let $T \subset S$. Then the following statements are equivalent:*
(1) *$W\partial\Sigma(W_T,T)$ is dense in $\partial\Sigma(W,S)$.*
(2) *If $W = W_{S_1} \times \cdots \times W_{S_n}$ is the irreducible decomposition of W, then $W_{S_i\cap T}$ is infinite for each $i \in \{1,\dots,n\}$ such that W_{S_i} is infinite.*

Proof. (1) $\Rightarrow$ (2): Let $W = W_{S_1} \times \cdots \times W_{S_n}$ be the irreducible decomposition of W. We suppose that there exists $i_0 \in \{1,\dots,n\}$ such that $W_{S_{i_0}}$ is infinite and $W_{S_{i_0}\cap T}$ is finite. Let $A_1 = S\setminus S_{i_0}$ and $A_2 = S_{i_0}$. Then $W = W_{A_1} \times W_{A_2}$, W_{A_2} is infinite and $W_{A_2\cap T}$ is finite. We note that $\partial\Sigma(W_{A_1},A_1)$ is W-invariant by [11, Theorem 4.3]. Since $W_T = W_{A_1\cap T} \times W_{A_2\cap T}$ and $W_{A_2\cap T}$ is finite, $\partial\Sigma(W_T,T) \subset \partial\Sigma(W_{A_1},A_1)$. Thus

$$W\partial\Sigma(W_T,T) \subset W\partial\Sigma(W_{A_1},A_1) = \partial\Sigma(W_{A_1},A_1).$$

Since W_{A_2} is infinite and

$$\partial\Sigma(W,S) = \partial\Sigma(W_{A_1},A_1) * \partial\Sigma(W_{A_2},A_2),$$

$W\partial\Sigma(W_T,T)$ is not dense in $\partial\Sigma(W,S)$.

(2) $\Rightarrow$ (1): Let $W = W_{S_1} \times \cdots \times W_{S_n}$ be the irreducible decomposition of W. Suppose that (2) holds. Then we prove that (1) holds by induction on n.

We first consider the case $n = 1$. Then $W = W_{S_1}$ is irreducible. Since $W_{S_1 \cap T}$ is infinite, $\partial\Sigma(W_T, T) \neq \emptyset$. Hence $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W, S)$ by Theorem 5.1.

Next we consider the case $n > 1$. Let $A_1 = S_1 \cup \cdots \cup S_{n-1}$ and $A_2 = S_n$. Then $W = W_{A_1} \times W_{A_2}$ and $W_T = W_{A_1 \cap T} \times W_{A_2 \cap T}$. Here

$$\begin{aligned} W\partial\Sigma(W_T, T) &= W(\partial\Sigma(W_{A_1 \cap T}, A_1 \cap T) * \partial\Sigma(W_{A_2 \cap T}, A_2 \cap T)) \\ &\supset W_{A_1}\partial\Sigma(W_{A_1 \cap T}, A_1 \cap T) * W_{A_2}\partial\Sigma(W_{A_2 \cap T}, A_2 \cap T). \end{aligned}$$

By the inductive hypothesis, $W_{A_i}\partial\Sigma(W_{A_i \cap T}, A_i \cap T)$ is dense in $\partial\Sigma(W_{A_i}, A_i)$ for each $i = 1, 2$. Since

$$\partial\Sigma(W, S) = \partial\Sigma(W_{A_1}, A_1) * \partial\Sigma(W_{A_2}, A_2),$$

we obtain that $W\partial\Sigma(W_T, T)$ is dense in $\partial\Sigma(W, S)$. $\square$

Also we obtain the following corollary from Theorem 5.1. We give a proof in Section 6.

Corollary 5.4. *For a right-angled Coxeter system (W, S), the set $\{w^\infty \mid w \in W,\ o(w) = \infty\}$ is dense in the boundary $\partial\Sigma(W, S)$.*

6. Remarks on dense subsets of boundaries of CAT(0) groups

In this section, we investigate dense subsets of boundaries of CAT(0) groups. The definitions and basic properties of CAT(0) spaces and their boundaries can be found in [4]. A group Γ is called a *CAT(0) group* if Γ acts geometrically (i.e., properly and cocompactly by isometries) on some CAT(0) space. For example, a Coxeter group W acts geometrically on the Davis complex $\Sigma(W, S)$, which is a CAT(0) space, and every Coxeter group is a CAT(0) group.

We pose the following open problem.

Question 6.1. Suppose that a group Γ acts geometrically on a CAT(0) space X. Is it the case that the set $\{\gamma^\infty \mid \gamma \in \Gamma,\ o(\gamma) = \infty\}$ is dense in the boundary ∂X?

Here we note that γ^∞ is the point of the boundary ∂X to which the sequence $\{\gamma^i x_0 \mid i \in \mathbb{N}\} \subset X$ converges in $X \cup \partial X$, where $x_0 \in X$ and γ^∞ does not depend on the point x_0.

We introduce some relations between this question and the minimality of boundaries of CAT(0) groups.

We first show the following proposition.

Proposition 6.2. *Suppose that a group Γ acts geometrically on a CAT(0) space X. If there exists $\delta \in \Gamma$ such that $o(\delta) = \infty$ and $\Gamma\delta^\infty$ is dense in the boundary ∂X, then the set $\{\gamma^\infty \mid \gamma \in \Gamma,\ o(\gamma) = \infty\}$ is dense in ∂X. Hence, if the boundary ∂X is minimal, then the set $\{\gamma^\infty \mid \gamma \in \Gamma,\ o(\gamma) = \infty\}$ is dense in ∂X.*

Proof. Suppose that $\delta \in \Gamma$ such that $o(\delta) = \infty$ and $\Gamma\delta^\infty$ is dense in ∂X. Let $\alpha \in \partial X$. Since $\Gamma\delta^\infty$ is dense in ∂X, there exists a sequence $\{\gamma_i\} \subset \Gamma$ such that $\{\gamma_i\delta^\infty\}$ converges to α in ∂X. Here for $x_0 \in X$ and each i, the sequence $\{(\gamma_i\delta\gamma_i^{-1})^j x_0\}_j$ converges to $\gamma_i\delta^\infty$ in $X \cup \partial X$. Hence $(\gamma_i\delta\gamma_i^{-1})^\infty = \gamma_i\delta^\infty$ and $\{(\gamma_i\delta\gamma_i^{-1})^\infty\}_i$ converges to α in ∂X. Thus $\{\gamma^\infty \mid \gamma \in \Gamma,\ o(\gamma) = \infty\}$ is dense in ∂X.

Now we suppose that the boundary ∂X is minimal. It is known that every CAT(0) group has an element of infinite order ([22, Theorem 11]). Let $\delta \in \Gamma$ with $o(\delta) = \infty$. Then $\Gamma\delta^\infty$ is dense in ∂X because ∂X is minimal. Hence, by the above argument, the set $\{\gamma^\infty \mid \gamma \in \Gamma,\ o(\gamma) = \infty\}$ is dense in the boundary ∂X. $\square$

We obtain the following proposition from some splitting theorems for CAT(0) spaces.

Proposition 6.3. *Suppose that a group $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space X, where Γ_1 and Γ_2 are infinite. Then X contains a quasi-dense subspace $X' = X_1 \times X_2$ and there exists a product subgroup $\Gamma'_1 \times \Gamma'_2$ of finite index in Γ such that X_1 is the convex hull $C(\Gamma'_1 x_0)$ for some $x_0 \in X$ and Γ'_2 acts geometrically on X_2 by projection.*

Proof. By [13, Lemma 2.1], there exist subgroups $G_1 \times A_1$ and $G_2 \times A_2$ of finite index in Γ_1 and Γ_2 respectively such that G_1 and G_2 have finite center and A_i is isomorphic to $\mathbb{Z}^{n_i}$ for some n_i $(i = 1, 2)$.

In the case that A_i is not trivial for some $i \in \{1, 2\}$, we put $\Gamma'_1 = A_i$ and $\Gamma'_2 = G_1 \times G_2 \times A_{3-i}$. Then by [3, Proposition 1.1] and [4, Theorem II.7.1], the proposition holds.

In the case that A_1 and A_2 are trivial, we put $\Gamma'_1 = G_1$ and $\Gamma'_2 = G_2$. Here G_1 and G_2 have finite center. By [17, Theorem 2] and [19, Corollary 10], the proposition holds. Here concerning the condition in [19, Corollary 10], we note that if the CAT(0) group Γ has finite center, then there does not exist a Γ-fixed point in the boundary ∂X (cf. [17, Lemma 3.2]). □

Concerning the nonminimality of boundaries of CAT(0) groups, using Proposition 6.3, we show the following theorem.

Theorem 6.4. *Suppose that a group Γ acts geometrically on a CAT(0) space X. If Γ contains a finite-index subgroup $\Gamma_1 \times \Gamma_2$ where Γ_1 and Γ_2 are infinite, then the boundary ∂X is not minimal.*

Proof. Let $\Gamma_1 \times \Gamma_2$ be a finite-index subgroup of Γ, where Γ_1 and Γ_2 are infinite. Then $\Gamma_1 \times \Gamma_2$ acts geometrically on X. By Proposition 6.3, X contains a quasi-dense subspace $X_1 \times X_2$ and there exists a product subgroup $\Gamma'_1 \times \Gamma'_2$ of finite index in Γ such that X_1 is the convex hull $C(\Gamma'_1 x_0)$ for some $x_0 \in X$ and Γ'_2 acts geometrically on X_2 by projection.

To prove that ∂X is not minimal, we show that $\Gamma(\partial X_1)$ is not dense in ∂X.

Since $\Gamma'_1 \times \Gamma'_2$ is a subgroup of finite index in Γ, there exist a number n and $\{\delta_1, \dots, \delta_n\} \subset \Gamma$ such that $\Gamma = \bigcup_{i=1}^n \delta_i(\Gamma'_1 \times \Gamma'_2)$.

Since $X_1 = C(\Gamma'_1 x_0)$ is Γ'_1-invariant, $\Gamma'_1(\partial X_1) = \partial X_1$. For each $\gamma_2 \in \Gamma'_2$, $\gamma_2 X_1$ and X_1 are parallel by the proof of the splitting theorems ([3], [4], [17], [19]); hence $\gamma_2(\partial X_1) = \partial X_1$, that is, $\Gamma'_2(\partial X_1) = \partial X_1$. Thus $(\Gamma'_1 \times \Gamma'_2)(\partial X_1) = \partial X_1$.

Hence

$$\begin{aligned}
\Gamma(\partial X_1) &= (\bigcup_{i=1}^n \delta_i(\Gamma'_1 \times \Gamma'_2))(\partial X_1) \\
&= \bigcup_{i=1}^n (\delta_i(\Gamma'_1 \times \Gamma'_2)(\partial X_1)) \\
&= \bigcup_{i=1}^n (\delta_i(\partial X_1)).
\end{aligned}$$

Here we note that $\Gamma(\partial X_1) = \bigcup_{i=1}^n (\delta_i(\partial X_1))$ is closed. Hence

$$\begin{aligned}\dim \overline{\Gamma(\partial X_1)} &= \dim \bigcup_{i=1}^n (\delta_i(\partial X_1)) = \dim \partial X_1 \\ &< \dim(\partial X_1 \times [0,1]) \leq \dim(\partial X_1 * \partial X_2) = \dim \partial X.\end{aligned}$$

Here we note that $\dim \partial X_i$ is finite, because the boundary of a cocompact proper CAT(0) space is finite-dimensional ([22, Theorem 12]).

Thus $\Gamma(\partial X_1)$ is not dense in ∂X. This implies that ∂X is not minimal. □

The referee has pointed out that the converse of Theorem 6.4 (if Γ does not contain a finite-index subgroup $\Gamma_1 \times \Gamma_2$ where Γ_1 and Γ_2 are infinite, then the boundary ∂X is minimal) will not be true in general and that a counterexample will be supplied by the theory of lattices in semisimple groups, since an irreducible lattice on a product of two hyperbolic planes does not factor (with infinite factors) (cf. [10]).

On the other hand, Theorem 5.1 implies that the converse of Theorem 6.4 holds for right-angled Coxeter groups and their boundaries.

Let $\mathcal{A}$ be the set of all infinite CAT(0) groups Γ such that for any CAT(0) space X on which Γ acts geometrically, the set $\{\gamma^\infty \mid \gamma \in \Gamma,\ o(\gamma) = \infty\}$ is dense in ∂X.

Now we show the following proposition.

Proposition 6.5. *Suppose that $\Gamma_1, \dots, \Gamma_n \in \mathcal{A}$ and that each Γ_i does not contain a finite-index subgroup $\Gamma_{i1} \times \Gamma_{i2}$ such that Γ_{i1} and Γ_{i2} are infinite. Then $\Gamma_1 \times \cdots \times \Gamma_n \in \mathcal{A}$.*

Proof. We note that each Γ_i is either isomorphic to $\mathbb{Z}$ or has finite center by [13, Lemma 2.1]. Hence we can suppose that for some number k, Γ_i is isomorphic to $\mathbb{Z}$ for each $i \leq k$ and Γ_i has finite center for each $i > k$.

We prove that $\Gamma \in \mathcal{A}$ by induction on n.

In the case $n = 1$, it is obvious.

We consider the case $n = 2$. Suppose that $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space X. By Proposition 6.3, X contains a quasi-dense subspace $X_1 \times X_2$ such that $X_1 = C(\Gamma_1 x_0)$ for some $x_0 \in X$ and Γ_2 acts geometrically on X_2 by projection. Let $\alpha \in \partial X$. Here

$$\partial X = \partial X_1 * \partial X_2 = (\partial X_1 \times \partial X_2 \times [-\pi, \pi])/\sim .$$

Hence $\alpha = [\alpha_1, \alpha_2, \theta]$ for some $\alpha_1 \in \partial X_1$, $\alpha_2 \in \partial X_2$ and $\theta \in [-\pi, \pi]$. Now $\{\gamma^\infty \mid \gamma \in \Gamma_1,\ o(\gamma) = \infty\}$ is dense in ∂X_1 and $\{\delta^\infty \mid \delta \in \Gamma_2,\ o(\delta) = \infty\}$ is dense in ∂X_2. Hence there exist sequences $\{\gamma_i\} \subset \Gamma_1$ and $\{\delta_i\} \subset \Gamma_2$ such that $\{\gamma_i^\infty\}$ converges to α_1 and $\{\delta_i^\infty\}$ converges to α_2. Since $\langle \gamma_i, \delta_i \rangle$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, by the Flat Torus Theorem ([4, Theorem II.7.1]), $\langle \gamma_i, \delta_i \rangle$ acts geometrically on some convex hull $C(\langle \gamma_i, \delta_i \rangle x_i)$ which is isometric to the Euclidean plane. Here $C(\langle \gamma_i, \delta_i \rangle x_i) \subset X_1 \times X_2$ and

$$\{\gamma_i^{-\infty}, \gamma_i^\infty, \delta_i^{-\infty}, \delta_i^\infty\} \subset \partial(C(\langle \gamma_i, \delta_i \rangle x_i)).$$

Then there exists a sequence $\{a_{ij}\} \subset \langle \gamma_i, \delta_i \rangle$ such that $\{a_{ij}^\infty\}_j$ converges to $[\gamma_i^\infty, \delta_i^\infty, \theta]$. Here the sequence $\{[\gamma_i^\infty, \delta_i^\infty, \theta]\}_i$ converges to α. Hence

$$\alpha \in \overline{\{a_{ij}^\infty \mid i, j \in \mathbb{N}\}} \subset \overline{\{\gamma^\infty \mid \gamma \in \Gamma,\ o(\gamma) = \infty\}}.$$

Thus $\{\gamma^\infty \mid \gamma \in \Gamma,\ o(\gamma) = \infty\}$ is dense in ∂X and $\Gamma = \Gamma_1 \times \Gamma_2 \in \mathcal{A}$.

We consider the case $n > 2$. Suppose that $\Gamma = \Gamma_1 \times \cdots \times \Gamma_{n-1} \times \Gamma_n$ acts geometrically on a CAT(0) space X. Let $\tilde{\Gamma}_1 = \Gamma_1 \times \cdots \times \Gamma_{n-1}$ and $\tilde{\Gamma}_2 = \Gamma_n$. Here we can suppose that $\tilde{\Gamma}_2$ has finite center or that each Γ_i is isomorphic to $\mathbb{Z}$ for $i = 1, \dots, n$. By the inductive hypothesis and the same argument as the proof in the case $n = 2$, we obtain that $\{\gamma^\infty \,|\, \gamma \in \Gamma,\ o(\gamma) = \infty\}$ is dense in ∂X and $\Gamma = \Gamma_1 \times \cdots \times \Gamma_{n-1} \times \Gamma_n \in \mathcal{A}$. $\square$

Suppose that a group Γ acts geometrically on a CAT(0) space X. Then there exists a finite-index subgroup $\Gamma_1 \times \cdots \times \Gamma_n$ of Γ such that each Γ_i is infinite and each Γ_i does not contain a finite-index subgroup $\Gamma_{i1} \times \Gamma_{i2}$ where Γ_{i1} and Γ_{i2} are infinite. Here the decomposition process terminates and n is finite. Indeed each Γ_i is a CAT(0) group by [17, Theorem 9.1] and there exists $\gamma_i \in \Gamma_i$ with $o(\gamma_i) = \infty$ by [22, Theorem 11]. Then $\langle \gamma_1, \dots, \gamma_n \rangle \subset \Gamma$ is isomorphic to $\mathbb{Z}^n$. Here such an n is finite, because every abelian subgroup of a CAT(0) group is finitely generated ([4, Corollary II.7.6]).

Hence Proposition 6.5 implies that Question 6.1 is equivalent to the following question.

Question 6.6. For an infinite CAT(0) group Γ which does not contain a finite-index product subgroup of two infinite subgroups, does $\Gamma \in \mathcal{A}$?

Finally, we prove Corollary 5.4. Concerning Question 6.1, we obtain a positive answer for right-angled Coxeter groups and their boundaries.

Proof of Corollary 5.4. Let (W, S) be a right-angled Coxeter system and let $W = W_{S_1} \times \cdots \times W_{S_n}$ be the irreducible decomposition of W. We may suppose that W_{S_i} is infinite for any $i \leq k$ and W_{S_i} is finite for any $i > k$ for some number k. Then $\tilde{S} = S_1 \cup \cdots \cup S_k$ and

$$\Sigma(W, S) = \Sigma(W_{S_1}, S_1) \times \cdots \times \Sigma(W_{S_k}, S_k) \times \Sigma(W_{S \setminus \tilde{S}}, S \setminus \tilde{S}),$$

where $\Sigma(W_{S \setminus \tilde{S}}, S \setminus \tilde{S})$ is bounded, since $W_{S \setminus \tilde{S}}$ is finite. Hence,

$$\partial\Sigma(W, S) = \partial\Sigma(W_{S_1}, S_1) * \cdots * \partial\Sigma(W_{S_k}, S_k).$$

Here each Coxeter system (W_{S_i}, S_i) is irreducible and right-angled and $\partial\Sigma(W_{S_i}, S_i)$ is minimal by Theorem 5.1. Thus for each $i \in \{1, \dots, k\}$, the set $\{w^\infty \,|\, w \in W_i,\ o(w) = \infty\}$ is dense in $\partial\Sigma(W_{S_i}, S_i)$ by Proposition 6.2. By a similar argument to the proof of Proposition 6.5, we obtain that the set $\{w^\infty \,|\, w \in W,\ o(w) = \infty\}$ is dense in the boundary $\partial\Sigma(W, S)$. $\square$

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