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A NOTE ON EVALUATIONS OF MULTIPLE ZETA VALUES

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ABSTRACT. In this paper we give a short and simple proof of the remarkable evaluations of multiple zeta values established by D. Bowman and D. M. Bradley.

1. INTRODUCTION

The multiple zeta value (MZV) is defined by the convergent series

$$\zeta(k_1, k_2, \ldots, k_n) := \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

where $k_1, k_2, \ldots, k_n$ are positive integers and $k_1 \geq 2$. One remarkable property of MZVs is that MZVs are evaluated for some special arguments as rational multiples of powers of π^2. For example, the following evaluations were proven by many authors ([1], [5], [8]):

$$\zeta(\{2\}_m) = \frac{\pi^{2m}}{(2m+1)!} \quad (m \in \mathbb{Z}_{>0})$$

where $\{2\}_m$ denotes the m-tuple $(2, 2, \ldots, 2)$. In [8], D. Zagier conjectured the following evaluations:

$$\zeta(\{3,1\}_n) = \frac{2\pi^{4n}}{(4n+2)!} \quad (n \in \mathbb{Z}_{>0}).$$

These evaluations were proved by J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisoněk ([2], [3]). In addition, D. Bowman and D. M. Bradley proved the following theorem which contained these results:

Theorem 1 ([4])**.** *For nonnegative integers m, n, we have*

$$\sum_{\substack{j_0+j_1+\cdots+j_{2n}=m \\ j_0, j_1, \ldots, j_{2n} \geq 0}} \zeta(\{2\}_{j_0}, 3, \{2\}_{j_1}, 1, \{2\}_{j_2}, \ldots, \{2\}_{j_{2n-2}}, 3, \{2\}_{j_{2n-1}}, 1, \{2\}_{j_{2n}})$$
$$= \binom{m+2n}{m} \frac{\pi^{2m+4n}}{(2n+1) \cdot (2m+4n+1)!}.$$

In this article we provide a short and simple proof of Theorem 1 which refines the proof of Theorem 5.1 in [4].

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2. Algebraic setup

We summarize the algebraic setup of MZVs introduced by Hoffman (cf. [6], [7]). Let $\mathfrak{H} = \mathbb{Q}\langle x, y\rangle$ be the noncommutative polynomial ring in two indeterminates x, y and $\mathfrak{H}^1$ and $\mathfrak{H}^0$ its subrings $\mathbb{Q}+\mathfrak{H}y$ and $\mathbb{Q}+x\mathfrak{H}y$. We set $z_k = x^{k-1}y$ $(k = 1, 2, 3, \ldots)$. Then $\mathfrak{H}^1$ is freely generated by $\{z_k\}_{k\geq 1}$.

We define the $\mathbb{Q}$-linear map (called an evaluation map) $Z : \mathfrak{H}^0 \longrightarrow \mathbb{R}$ by $Z(1) = 1$ and

$$Z(u_1u_2\cdots u_k) = \int\cdots\int_{1>t_1>t_2>\cdots>t_k>0} \omega_{u_1}(t_1)\omega_{u_2}(t_2)\cdots\omega_{u_k}(t_k)$$

$(u_1, u_2, \ldots, u_k \in \{x, y\})$, where $\omega_x(t) = dt/t$ and $\omega_y(t) = dt/(1-t)$. As $u_1u_2\cdots u_k$ is in $\mathfrak{H}^0$, we always have $\omega_{u_1}(t) = dt/t$ and $\omega_{u_k}(t) = dt/(1-t)$, so the integral converges. By the Drinfel′d integral representation, we have

$$Z(z_{k_1}z_{k_2}\cdots z_{k_n}) = \zeta(k_1, k_2, \ldots, k_n).$$

We next define the shuffle product $⧢$ on $\mathfrak{H}$ inductively by

$$\begin{aligned} 1⧢w &= w⧢1 = w, \\ u_1w_1⧢u_2w_2 &= u_1(w_1⧢u_2w_2) + u_2(u_1w_1⧢w_2) \end{aligned}$$

$(u_1, u_2 \in \{x, y\}$ and w, w_1, w_2 are words in $\mathfrak{H})$, together with $\mathbb{Q}$-bilinearity. The shuffle product $⧢$ is commutative and associative. By the standard shuffle product identity of iterated integrals, the evaluation map Z is a homomorphism with respect to the shuffle product $⧢$:

$$Z(w_1⧢w_2) = Z(w_1)Z(w_2) \quad \big(w_1, w_2 \in \mathfrak{H}^0\big).$$

We also define the shuffle product $\widetilde{⧢}$ on $\mathfrak{H}^1$ inductively by

$$\begin{aligned} 1\widetilde{⧢}w &= w\widetilde{⧢}1 = w, \\ u_1w_1\widetilde{⧢}u_2w_2 &= u_1(w_1\widetilde{⧢}u_2w_2) + u_2(u_1w_1\widetilde{⧢}w_2) \end{aligned}$$

$(u_1, u_2 \in \{z_k\}_{k\geq 1}$ and w, w_1, w_2 are words in $\mathfrak{H}^1)$, together with $\mathbb{Q}$-bilinearity. For example, we have

$$\begin{aligned} z_m\widetilde{⧢}z_n &= z_mz_n + z_nz_m, \\ z_m\widetilde{⧢}z_nz_l &= z_mz_nz_l + z_nz_mz_l + z_nz_lz_m. \end{aligned}$$

Then Theorem 1 can be restated as follows:

$$Z\left(z_2^m\widetilde{⧢}(z_3z_1)^n\right) = \binom{m+2n}{m}\frac{\pi^{2m+4n}}{(2n+1)\cdot(2m+4n+1)!} \quad \big(m, n \in \mathbb{Z}_{\geq 0}\big).$$

3. Proof of Theorem 1

We restate Proposition 4.1 and Proposition 4.2 of [4] by using $\widetilde{⧢}$ and prove them by induction.

Proposition 2. *For integers n, N which satisfy $0 \leq n \leq N$, we have*

$$z_2^n⧢z_2^N = \sum_{k=0}^{n} 4^k\binom{N+n-2k}{n-k}\left\{z_2^{N+n-2k}\widetilde{⧢}(z_3z_1)^k\right\}, \tag{1}$$

$$z_1z_2^n⧢z_1z_2^N = 2\sum_{k=0}^{n} 4^k\binom{N+n-2k}{n-k}z_1\left\{z_2^{N+n-2k}\widetilde{⧢}z_1(z_3z_1)^k\right\}. \tag{2}$$

Proof. We prove identities (1) and (2) simultaneously by induction on n. [Step 1] The case $n=0$ of (1) is clear. We can easily prove the case $n=0$ of (2) by induction on N. [Step 2] Suppose that (1) and (2) have been proven for $n-1$. We prove (1) for n by induction on N:

$$\begin{aligned}
z_2^n ш z_2^n &= 2xy\{(xy)^{n-1} ш (xy)^n\} + 2x^2\{y(xy)^{n-1} ш y(xy)^{n-1}\} \\
&= 2\sum_{k=0}^{n-1} 4^k \binom{2n-1-2k}{n-1-k} z_2\{z_2^{2n-1-2k} \widetilde{ш} (z_3z_1)^k\} \\
&\quad + \sum_{k=0}^{n-1} 4^{k+1} \binom{2n-2-2k}{n-1-k} z_3\{z_2^{2n-2-2k} \widetilde{ш} z_1(z_3z_1)^k\} \\
&= \sum_{k=0}^{n-1} 4^k \binom{2n-2k}{n-k} z_2\{z_2^{2n-1-2k} \widetilde{ш} (z_3z_1)^k\} \\
&\quad + \sum_{k=1}^{n} 4^k \binom{2n-2k}{n-k} z_3\{z_2^{2n-2k} \widetilde{ш} z_1(z_3z_1)^{k-1}\} \\
&= \binom{2n}{n} z_2^{2n} + \sum_{k=1}^{n-1} 4^k \binom{2n-2k}{n-k} \{z_2^{2n-2k} \widetilde{ш} (z_3z_1)^k\} + 4^n (z_3z_1)^n \\
&= \sum_{k=0}^{n} 4^k \binom{2n-2k}{n-k} \{z_2^{2n-2k} \widetilde{ш} (z_3z_1)^k\}.
\end{aligned}$$

Hence (1) is true for $N=n$. Suppose that the case $N-1$ of (1) has been proven. (We may assume that $N-1 \geq n$ in the following calculation.)

$$\begin{aligned}
z_2^n ш z_2^N &= xy\{(xy)^{n-1} ш (xy)^N\} + 2x^2\{y(xy)^{n-1} ш y(xy)^{N-1}\} \\
&\quad + xy\{(xy)^n ш (xy)^{N-1}\} \\
&= \sum_{k=0}^{n-1} 4^k \binom{N+n-1-2k}{n-1-k} z_2\{z_2^{N+n-1-2k} \widetilde{ш} (z_3z_1)^k\} \\
&\quad + \sum_{k=0}^{n-1} 4^{k+1} \binom{N+n-2-2k}{n-1-k} z_3\{z_2^{N+n-2-2k} \widetilde{ш} z_1(z_3z_1)^k\} \\
&\quad + \sum_{k=0}^{n} 4^k \binom{N+n-1-2k}{n-k} z_2\{z_2^{N+n-1-2k} \widetilde{ш} (z_3z_1)^k\} \\
&= \sum_{k=0}^{n-1} 4^k \binom{N+n-2k}{n-k} z_2\{z_2^{N+n-1-2k} \widetilde{ш} (z_3z_1)^k\} \\
&\quad + \sum_{k=1}^{n} 4^k \binom{N+n-2k}{n-k} z_3\{z_2^{N+n-2k} \widetilde{ш} z_1(z_3z_1)^{k-1}\} \\
&\quad + 4^n z_2\{z_2^{N-n-1} \widetilde{ш} (z_3z_1)^n\} \\
&= \binom{N+n}{n} z_2^{N+n} + \sum_{k=1}^{n-1} 4^k \binom{N+n-2k}{n-k} \{z_2^{N+n-2k} \widetilde{ш} (z_3z_1)^k\} \\
&\quad + 4^n \{z_2^{N-n} \widetilde{ш} (z_3z_1)^n\} \\
&= \sum_{k=0}^{n} 4^k \binom{N+n-2k}{n-k} \{z_2^{N+n-2k} \widetilde{ш} (z_3z_1)^k\}.
\end{aligned}$$

Hence (1) is true for N. We can prove (2) for n by induction on N with using (1) for n. $\square$

Before proceeding to the proof of Theorem 1, we prove a key identity. Comparing coefficients of $(x+1)^{2m+4n+2} = (x^2+2x+1)^{m+2n+1}$, we have

$$\binom{2m+4n+2}{2n+1} = \sum_{k=0}^{n} 2^{2k+1} \frac{(m+2n+1)!}{(n-k)!(2k+1)!(m+n-k)!}.$$

We can transform this identity as follows:

$$\begin{aligned}(3)\qquad & \frac{1}{(2n+1)!}\frac{1}{(2m+2n+1)!} \\ & = \sum_{k=0}^{n} 4^k \binom{m+2n-2k}{n-k}\binom{m+2n}{2k}\frac{1}{(2k+1)\cdot(2m+4n+1)!}.\end{aligned}$$

Proof of Theorem 1. We prove Theorem 1 by induction on n. The case $n=0$ is well known as has been mentioned in Section 1. Suppose that the assertion has been proven up to $n-1$. Putting $N = m+n$ in (1), we have

$$\begin{aligned}& 4^n Z\left(z_2^m \widetilde{ш}(z_3 z_1)^n\right) \\ & = \frac{\pi^{2n}}{(2n+1)!}\frac{\pi^{2m+2n}}{(2m+2n+1)!} \\ & \quad - \sum_{k=0}^{n-1} 4^k \binom{m+2n-2k}{n-k}\binom{m+2n}{2k}\frac{\pi^{2m+4n}}{(2k+1)\cdot(2m+4n+1)!} \\ & = 4^n \binom{m+2n}{m}\frac{\pi^{2m+4n}}{(2n+1)\cdot(2m+4n+1)!}.\end{aligned}$$

In the first equality we have used the induction hypothesis and the formula $\zeta(\{2\}_m) = \pi^{2m}/(2m+1)!$ (the case $n=0$), and in the second equality we have used (3). $\square$

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