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PLEIJEL'S NODAL DOMAIN THEOREM FOR FREE MEMBRANES

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Abstract. We prove an analogue of Pleijel's nodal domain theorem for piecewise analytic planar domains with Neumann boundary conditions. This confirms a conjecture made by Pleijel in 1956. The proof is a combination of Pleijel's original approach and an estimate due to Toth and Zelditch for the number of boundary zeros of Neumann eigenfunctions.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded planar domain. Let $0 \le \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \dots$ be the eigenvalues of the Laplacian on Ω with either Dirichlet or Neumann boundary conditions, and $\phi_1, \phi_2, \dots, \phi_k, \dots$ be an orthonormal basis of eigenfunctions: $\Delta\phi_k = \lambda_k \phi_k$, $k = 1, 2, \dots$. We assume that the boundary $\partial\Omega$ is sufficiently regular (say, piecewise smooth). In the Neumann case we also assume throughout the paper that the internal cone condition [EE, chapter V.4] is satisfied at the corners in order to ensure that the spectrum is discrete. According to the classical Courant's nodal domain theorem [Co] (see also [CH, p. 452]), the number n_k of nodal domains of the eigenfunction ϕ_k is at most k. In 1956, Pleijel [Pl] showed that for planar domains with Dirichlet boundary conditions Courant's bound can be asymptotically improved:

$$\limsup_{k\to\infty} \frac{n_k}{k} \le \frac{4}{j_1^2} \approx 0.691..., \tag{1.1}$$

where $j_1 \approx 2.4$ is the first zero of the Bessel function J_0. In particular, this implies that the equality in Courant's theorem is attained only for a finite number of eigenfunctions. Note that in the 1-dimensional case, the eigenfunction ϕ_k has *exactly* k nodal domains for any $k = 1, 2, \dots$.

The proof of the estimate (1.1) is an application of the Faber–Krahn inequality for the first eigenvalue ([Fa, Kr]; see also [Ch, p. 87]). In [Pl, section 7], Pleijel writes regarding the inequality (1.1) that *"...it seems highly probable that the result... is also true for free membranes."* Recall that a free membrane corresponds to the Neumann boundary value problem. The difficulty in this case is that one cannot apply the Faber–Krahn inequality to nodal domains that are adjacent to $\partial\Omega$: they

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have Neumann conditions on a part of their boundary (here and further on we say that a nodal domain D is *adjacent* to $\partial\Omega$ if length$(\partial D \cap \partial\Omega) > 0$). Later, Pleijel's result was generalized in [Pe, BM, Be] to higher-dimensional domains with Dirichlet boundary conditions and to compact closed manifolds. However, the case of domains with Neumann boundary conditions remained open.

The purpose of this paper is to prove an analogue of Pleijel's theorem for piecewise analytic planar domains with Neumann boundary conditions. We consider separately the nodal domains that are adjacent to $\partial\Omega$, and the remaining nodal domains having pure Dirichlet conditions on their boundaries. In the latter case, we use the original Pleijel's approach, while in the former we apply an estimate recently obtained by Toth–Zelditch [TZ] for the number of boundary zeros of Neumann eigenfunctions. In fact, it follows from [TZ, Theorem 2] that the number of nodal domains adjacent to $\partial\Omega$ is small compared to Courant's bound; see (2.1).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a piecewise real analytic domain with Neumann boundary conditions. Then the estimate* (1.1) *holds.*

2. Proof of Theorem 1.1

Let m_k be the number of nodal domains of the eigenfunction ϕ_k that are adjacent to $\partial\Omega$ and $l_k = n_k - m_k$ be the number of the remaining nodal domains. We show first that

$$\limsup_{k\to\infty} \frac{m_k}{k} = 0. \tag{2.1}$$

Let $\mathcal{Z}_k = \{x \in \partial\Omega : \phi_k(x) = 0\}$. According to [TZ, Theorem 2], the number $N_k = \mathrm{card}(\mathcal{Z}_k)$ of boundary zeros of the eigenfunction ϕ_k is bounded by $C\sqrt{\lambda_k}$, where $C > 0$ is a constant depending only on Ω. It is easy to check that $m_k \le N_k + s$, where s is the number of connected components of $\partial\Omega$. Indeed, the set $\partial\Omega \setminus \mathcal{Z}_k$ has at most $N_k + s$ connected components, and for each connected component $\Gamma \subset \partial\Omega \setminus \mathcal{Z}_k$ there exists a unique nodal domain D adjacent to the boundary such that $\Gamma \subset \partial D$ (note that this correspondence is not necessarily one-to-one: the boundary of a nodal domain may contain more than one connected component of $\partial\Omega \setminus \mathcal{Z}_k$). At the same time, by Weyl's law (see [Ch, p. 9])

$$\lim_{k\to\infty} \frac{\lambda_k}{k} = \frac{4\pi}{\mathrm{Area}(\Omega)}. \tag{2.2}$$

Therefore, for some constant $C_1 > 0$,

$$\limsup_{k\to\infty} \frac{m_k}{k} \le \limsup_{k\to\infty} \frac{N_k + s}{k} \le \limsup_{k\to\infty} \frac{C_1\sqrt{\lambda_k}}{\lambda_k} = 0,$$

and this completes the proof of (2.1).

Now let us show that

$$\limsup_{k\to\infty} \frac{l_k}{k} \le \frac{4}{j_1^2}. \tag{2.3}$$

The proof of this bound is similar to the proof of (1.1); see [Ch, pp. 24-25]. Indeed, let $\tilde{\Omega} \subset \Omega$ be the union of all nodal domains that are not adjacent to $\partial\Omega$. Every such domain D has pure Dirichlet conditions on the boundary, and therefore one can apply the Faber-Krahn inequality:

$$\lambda_k(\Omega)\mathrm{Area}(D) = \lambda_1(D)\mathrm{Area}(D) \ge \pi j_1^2.$$

Summing up the left- and the right-hand sides over all domains D that are not adjacent to the boundary, we get:

$$\lambda_k(\Omega)\mathrm{Area}(\tilde{\Omega}) \geq l_k \pi j_1^2.$$

Therefore, taking into account that $\mathrm{Area}(\tilde{\Omega}) < \mathrm{Area}(\Omega)$ and using (2.2), one obtains

$$\limsup_{k\to\infty} \frac{l_k}{k} \leq \limsup_{k\to\infty} \frac{\lambda_k(\Omega)\mathrm{Area}(\tilde{\Omega})}{\pi j_1^2 k} \leq \frac{4}{j_1^2}.$$

Since $n_k = m_k + l_k$, putting together (2.1) and (2.3) we complete the proof of the theorem. □

Remark 2.1. Theorem 1.1 is proved under the same assumptions as [TZ, Theorem 2]. It would be interesting to extend Theorem 1.1 to higher dimensions and to replace piecewise analyticity by a weaker condition. It is likely that the results of [TZ], and hence Theorem 1.1 as well, hold also for domains with mixed Dirichlet–Neumann boundary conditions.

Remark 2.2. It is clear from the proof of (1.1) that this estimate is *not* sharp for both Dirichlet and Neumann boundary conditions. Indeed, the Faber–Krahn inequality is an equality only for the disk, and nodal domains of an eigenfunction cannot be *all* disks at the same time. Therefore, a natural problem is to find an optimal constant in (1.1). Motivated by the results of [BGS] we suggest that for any regular bounded planar domain with either Dirichlet or Neumann boundary conditions, $\limsup_{k\to\infty} \frac{n_k}{k} \leq \frac{2}{\pi} \approx 0.636...$. If true, this estimate (which is quite close to Pleijel's bound) is sharp and is attained for the basis of separable eigenfunctions on a rectangle [Pl, SS].

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