PROJECTIVE NORMALITY OF FINITE GROUP QUOTIENTS

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ABSTRACT. In this paper, we prove that for any finite dimensional vector space V over an algebraically closed field k , and for any finite subgroup G of $GL(V)$ which is either solvable or is generated by pseudo reflections such that $|G|$ is a unit in k , the projective variety $\mathbb{P}(V)/G$ is projectively normal with respect to the descent of $\mathcal{O}(1)^{\otimes |G|}$.

INTRODUCTION

Let G be a finite group. Let V be a finite dimensional representation of G over a field k . In 1916, E. Noether proved that if the characteristic of k does not divide $|G|$, then the k -algebra of invariants $k[V]^G$ is finitely generated (see [8]). In 1926, she proved that the same result holds in all characteristics (see [9]). So, when k is algebraically closed, it is an interesting problem to study GIT-quotient varieties $V/G = \text{Spec}(k[V]^G)$ and $\mathbb{P}(V)/G$ (see [6] and [7]). Also, for every point x in $\mathbb{P}(V)$, the isotropy G_x acts trivially on the fiber of the line bundle $\mathcal{O}(1)^{\otimes |G|}$ over x . Hence, by Proposition (4.2), page 83 of [5], when $|G|$ is a unit in k , the line bundle $\mathcal{O}(1)^{\otimes |G|}$ descends to the quotient $\mathbb{P}(V)/G$, where $\mathcal{O}(1)$ denotes the ample generator of the Picard group of $\mathbb{P}(V)$. Let us denote it by $\mathcal{L}$. On the other hand, V/G is normal. So, it is a natural question to ask if $\mathbb{P}(V)/G$ is projectively normal with respect to the line bundle $\mathcal{L}$. More generally, an interesting question is the following.

Let G be a finite group acting on a projectively normal polarized variety $(X, \mathcal{O}(1))$, where $\mathcal{O}(1)$ is a G -linearized very ample line bundle making $X \subset \mathbb{P}(V)$ projectively normal and $\mathcal{L} \in \text{Pic}(X/G)$ is the descent of the line bundle $\mathcal{O}(1)^{\otimes |G|}$ on X . Is the polarized variety $(X/G, \mathcal{L})$ projectively normal?

In this note, we give an affirmative answer to this question in the following two cases:

- (1) G is solvable and the characteristic of the base field does not divide $|G|$;
- (2) G is a finite subgroup of $GL(V)$ generated by pseudo reflections, where V is a finite dimensional vector space over a field k of characteristic not dividing $|G|$.

Proof of the main result essentially uses an arithmetic result due to Erdős-Ginzburg-Ziv (see [2]).

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1. SOLVABLE CASE

In this section, we prove the following proposition, which can be applied to prove our main result when the group G is solvable.

Proposition 1.1. *Let G be a finite solvable group, and let V be a finite dimensional faithful representation of G over a field k of characteristic not dividing $|G|$. Let $m = |G|$, $R := \bigoplus_{d \geq 0} R_d$; $R_d := (\text{Sym}^{dm} V)^G$. Then R is generated as a k -algebra by R_1 .*

Proof. Step 1: We first prove the statement when G is cyclic of order m . Let ξ be a primitive m -th root of unity in an algebraic closure $\bar{k}$ of k . Let $F = k(\xi)$. Since F is a free k -module, we have $V^G \otimes_k F = (V \otimes_k F)^G$. Hence, we may assume that $\xi \in k$.

Let $G = \langle g \rangle$. Write $V = \bigoplus_{i=0}^{m-1} V_i$ where $V_i := \{v \in V : g.v = \xi^i.v\}$, $0 \leq i \leq m-1$. Now, let $f \in R_d$ be of the form $f = X_0 \cdot X_1 \cdots X_{m-1}$ with $X_i \in \text{Sym}^{a_i} V_i$ such that $\sum_{i=0}^{m-1} a_i = dm$. Since f is G -invariant we have

$$\sum_{i=0}^{m-1} i.a_i \equiv 0 \pmod{m}.$$

If $d = 1$, $f \in R_1$, so we may assume that $d \geq 2$. Now, consider the sequence of integers

$$\underbrace{0, \dots, 0}_{a_0 \text{ times}}, \underbrace{1, \dots, 1}_{a_1 \text{ times}}, \dots, \underbrace{m-1, \dots, m-1}_{a_{m-1} \text{ times}}.$$

Since the sequence has dm terms and $d \geq 2$, by a theorem of Erdős-Ginzburg-Ziv (see [2]), there is a subsequence with exactly m terms whose terms add up to a multiple of m . Thus there exist $f_1 \in R_1$ and $f_2 \in R_{d-1}$ such that $f = f_1 \cdot f_2$. Hence the proof follows by induction on $\deg(f)$.

Step 2: Now we assume that G is any finite solvable group of order m . We use induction on m to prove the statement. We may assume that m is not a prime number. Since G is solvable it has a normal subgroup H such that G/H is a cyclic group of prime order.

Let $W := (\text{Sym}^{|H|} V)^H$. Since H is a normal subgroup of G , both $\underbrace{W \otimes \dots \otimes W}_{d|G/H| \text{ copies}}$ and $(\text{Sym}^{d|H|} V)^H$ have natural G/H -module structures. Let $G_1 = G/H$. Since $|H| < |G|$, by induction, the homomorphism $\underbrace{W \otimes \dots \otimes W}_{d|G_1| \text{ copies}} \longrightarrow (\text{Sym}^{d|G|} V)^H$ is surjective. (1)

Claim. The natural map $(\text{Sym}^{d|G_1|} W)^{G_1} \longrightarrow (\text{Sym}^{d|G|} V)^G$ is surjective. (2)

The surjectivity of the natural map $\text{Sym}^{d, |G_1|} W \longrightarrow (\text{Sym}^{d|G|} V)^H$ of G_1 -modules follows from (1) and the following commutative diagram:

$$\begin{array}{ccc} \otimes^{d|G_1|} W & \longrightarrow & (\text{Sym}^{d|G|} V)^H \\ \downarrow & \nearrow & \\ \text{Sym}^{d, |G_1|} W & & \end{array}$$

Hence applying Reynold's operator we have the claim.

Now, consider the commutative diagram:

$$\begin{array}{ccc} \otimes^d(\mathrm{Sym}^{|G_1|}W)^{G_1} & \longrightarrow & (\mathrm{Sym}^{d,|G_1|}W)^{G_1} \\ \downarrow & & \downarrow \\ \otimes^d(\mathrm{Sym}^{|G|}V)^G & \longrightarrow & (\mathrm{Sym}^{d,|G|}V)^G \end{array}$$

The first horizontal map is surjective by step (1) and the second vertical map is surjective by (2). Thus the second horizontal map is surjective. Thus the proposition follows. $\square$

2. GROUP GENERATED BY PSEUDO REFLECTIONS

In this section, we will prove a combinatorial lemma which can be applied to prove our main result when the group G is generated by pseudo reflections.

Let $\underline{a} = (a_1, a_2, \dots, a_r) \in \mathbb{N}^r$ and $N_{\underline{a}} = \prod_{j=1}^r a_j$. Consider the semigroup

$$M_{\underline{a}} = \{(m_1, m_2, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r : \sum_{j=1}^r m_j a_j \equiv 0 \pmod{N_{\underline{a}}}\}$$

and the set

$$S_{\underline{a}} = \{(m_1, m_2, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r : \sum_{j=1}^r m_j a_j = N_{\underline{a}}\}.$$

Lemma 2.1. $M_{\underline{a}}$ is generated by $S_{\underline{a}}$ for $\underline{a} \in \mathbb{N}^r$.

Proof. Suppose $(m_1, m_2, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r$ such that

$$\sum_{j=1}^r m_j \cdot a_j = q \cdot N_{\underline{a}}, \text{ with } q \geq 2.$$

Let $n = q \cdot N_{\underline{a}}$. Choose any $n \times r$ matrix

$$A = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1r} \\ x_{21} & x_{22} & \cdots & x_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nr} \end{pmatrix}$$

with entries in $\{0, 1\}$ such that each row sum $\sum_{j=1}^r x_{i,j}$ is equal to 1 and for each $j = 1, 2, \dots, r$, the j th column sum $\sum_{i=1}^n x_{i,j}$ is $m_j a_j$.

Since $q \geq 2$, $n \geq 2a_1$, the sequence $\{x_{11}, x_{21}, \dots, x_{n1}\}$ has at least $2a_1$ terms. Hence, applying the theorem of Erdős-Ginzburg-Ziv repeatedly (see [2]) and rearranging the entries of the first column if necessary, we can assume that the n terms of the sequence can be partitioned into $\frac{n}{a_1 a_2}$ subsequences $\{x_{11}, x_{21}, \dots, x_{a_1 1}\}$, $\{x_{(a_1+1)1}, x_{(a_1+2)1}, \dots, x_{2a_1 1}\}$, $\dots$, $\{x_{(n-a_1+1)1}, x_{(n-a_1+2)1}, \dots, x_{n1}\}$, each of length a_1 , and the sum of the terms of each of these subsequences is a multiple of a_1 .

Again, consider the sequence $\{\sum_{i=1}^{a_1} x_{i2}, \sum_{i=a_1+1}^{2a_1} x_{i2}, \dots, \sum_{i=n-a_1+1}^n x_{i2}\}$. Using the same argument as above we can assume that this sequence can be partitioned into $\frac{n}{a_1 a_2}$ subsequences, each of length a_2 , and the sum of the terms of each subsequence is a multiple of a_2 .

Proceeding in this way, we can see that for each $j = 1, 2, \dots, r$, the sum of the first $N_{\underline{a}}$ -terms in the j th column of the matrix A is a multiple of a_j .

Let $b_j = \sum_{i=1}^{N_{\underline{a}}} x_{ij}$. By construction of the x_{ij} 's, b_j is a multiple of a_j for every $j = 1, 2, \dots, r$. So, for each $j = 1, 2, \dots, r$, write $b_j = a_j c_j$, with $c_j \in \mathbb{Z}_{\geq 0}$. Now, since $\sum_{j=1}^r x_{i,j} = 1$, for each $i = 1, 2, \dots, n$, we have $\sum_{j=1}^r b_j = N_{\underline{a}}$. Hence, $(c_1, c_2, \dots, c_r) \in S_{\underline{a}}$. As $m_j a_j = \sum_{i=1}^n x_{ij}$, we have $b_j \leq m_j$ for every $j = 1, 2, \dots, r$. Thus, we have $(m_1, m_2, \dots, m_r) = (c_1, c_2, \dots, c_r) + (m_1 - c_1, m_2 - c_2, \dots, m_r - c_r)$, with $\sum_{j=1}^r (m_j - c_j) a_j = (q - 1) N_{\underline{a}}$. So, the lemma follows by induction on q . $\square$

Corollary 2.2. *Let V be a finite dimensional vector space over a field k . Let G be a finite subgroup of $GL(V)$ which is generated by pseudo reflections. Further assume that the characteristic of k does not divide $|G|$. Then $R = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} R_d$, where $R_d := (Sym^{d \cdot |G|}(V^*))^G$, is generated by R_1 .*

Proof. By a theorem of Chevalley-Serre-Shephard-Todd (see [1], [4], [10], [11]), $(Sym(V^*))^G$ is a polynomial ring $K[f_1, f_2, \dots, f_r]$ where each f_i is a homogeneous polynomial of degree a_i and $\prod_{i=1}^r a_i = |G|$. Thus, the proof follows from Lemma 2.1. $\square$

3. MAIN THEOREM

We now prove our main result.

Let G be a finite group and V be a finite dimensional, faithful representation of G over an algebraically closed field of characteristic not dividing $|G|$. Let $\mathcal{O}(1)$ denote the ample generator of the Picard group of $\mathbb{P}(V)$. Let $\mathcal{L}$ denote the descent of the line bundle $\mathcal{O}(1)^{\otimes |G|}$ to the quotient $\mathbb{P}(V)/G$.

Before stating the main result, we recall the following.

1. A polarized variety $(X, \mathcal{L})$ where $\mathcal{L}$ is a very ample line bundle is said to be projectively normal if its homogeneous coordinate ring $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} (H^0(X, \mathcal{L}^{\otimes n|G|}))$ is integrally closed. For a reference, see exercise 3.18, page 23 of [3].

2. The polarized variety $(\mathbb{P}(V)/G, \mathcal{L})$ is

$$Proj\left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} (H^0(\mathbb{P}(V), \mathcal{O}(1)^{\otimes n|G|}))^G\right) = Proj\left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} (Sym^{n|G|}(V^*))^G\right).$$

For a reference, see Theorem 3.14 and page 76 of [6].

Now, let us assume that either G is solvable or is generated by pseudo reflections in $GL(V)$. Then, we have

Theorem 3.1. $\mathbb{P}(V)/G$ is projectively normal with respect to $\mathcal{L}$.

Proof. The proof follows from Proposition 1.1, Corollary 2.2, and exercise 5.14(d), Chapter II of [3]. $\square$

Remark. It is an interesting and important problem to answer the following question.

Let G be a reductive group (not necessarily finite) acting morphically on a projective variety X . Let $\mathcal{M}$ be a G -linearized very ample line bundle on X such that $\mathcal{M}$ descends to the quotient $X//G$. Let $\mathcal{L}$ be the descent. Is the polarized variety $(X//G, \mathcal{L})$ projectively normal?

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