# KENILWORTH 

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#### Abstract

We construct a $G_{\delta} \sigma$-ideal $\mathcal{I}$ of compact subsets of $2^{\omega}$ such that $\mathcal{I}$ contains all the singletons but there is no dense $G_{\delta}$ set $D \subseteq 2^{\omega}$ such that $\{K \subseteq D: K$ compact $\} \subseteq \mathcal{I}$. This answers a question of A. S. Kechris in the negative.


## 1. The question of Kechris

Let $X$ be a Polish space and let $\mathcal{I}$ be a $\sigma$-ideal of compact subsets of $X$. By the Dichotomy Theorem (see [4. Theorem 7, p. 268]), if $\mathcal{I}$ is $\boldsymbol{\Pi}_{1}^{1}$, then it is either $G_{\delta}$ or $\Pi_{1}^{1}$-complete. Thus, in view of the two alternatives, a $G_{\delta} \sigma$-ideal may be considered as an extremely simple object. Moreover, for compact $X$, one can refine the classification of the $G_{\delta}$ case of the Dichotomy Theorem: with the notation $\mathcal{K}(A)=\{K \subseteq A: K$ compact $\}$, a $G_{\delta} \sigma$-ideal $\mathcal{I} \subseteq \mathcal{K}(X)$ is either $\Pi_{2}^{0}$-complete or it is $\mathrm{D}_{2}\left(\boldsymbol{\Pi}_{1}^{0}\right)$-, $\boldsymbol{\Pi}_{1}^{0}$-, $\boldsymbol{\Sigma}_{1}^{0}$-complete or $\boldsymbol{\Delta}_{1}^{0}$, in which case $\mathcal{I}$ is of the form $\mathcal{K}(A)$ where $A$ is $\mathrm{D}_{2}\left(\boldsymbol{\Pi}_{1}^{0}\right), \boldsymbol{\Pi}_{1}^{0}, \boldsymbol{\Sigma}_{1}^{0}$ or $\boldsymbol{\Delta}_{1}^{0}$, respectively (see 4. Theorem 8, p. 269] or [5, Theorem 1.4, p. 486]). See also [6 for a recent study on $G_{\delta}$ ideals of compact sets.

However, a $\Pi_{2}^{0}$-complete $\sigma$-ideal is not necessarily of the form $\mathcal{K}(D)$ for a $G_{\delta}$ set $D \subseteq X$ : the $\sigma$-ideals of compact meager sets or compact Lebesgue null sets are obvious examples. On the other hand, for these two $\sigma$-ideals, which are comeager subsets of $\mathcal{K}(X)$, e.g. by the fact that they contain all the finite sets, it is at least true that they contain $\mathcal{K}(D)$ for a dense $G_{\delta}$ set $D \subseteq X$. This property turned out to be very useful (see e.g. [2] and [3), and in [5, Remark 4.17, p. 524] conditions on the construction of $G_{\delta} \sigma$-ideals were formulated which guarantee that a $G_{\delta} \sigma$-ideal $\mathcal{I}$ containing all the singletons satisfies $\mathcal{K}(D) \subseteq \mathcal{I}$ for some dense $G_{\delta}$ set $D \subseteq X$. Nevertheless, the problem, originally posed by A. S. Kechris ([2, Problem, p. 191] and [3, Problem 3, p. 121]; see also [5, Problem 6.1, p. 530]), whether this property holds for arbitrary $G_{\delta} \sigma$-ideals in $\mathcal{K}\left(2^{\omega}\right)$ remained open. The purpose of the present paper is to give a negative solution to this problem.
Theorem 1.1. There exists a $G_{\delta} \sigma$-ideal $\mathcal{I}$ of compact subsets of $2^{\omega}$ such that $\mathcal{I}$ contains all the singletons but there is no dense $G_{\delta}$ set $D \subseteq 2^{\omega}$ such that $\mathcal{K}(D) \subseteq \mathcal{I}$.

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Once such an example is found it is natural to ask whether the construction carries over to all (perfect) Polish spaces and whether we can work in the hyperspace of closed sets instead of compact sets; but most importantly, whether we can include into $\mathcal{I}$ more than just singletons and avoid other families of compact sets than just $\mathcal{K}(D)$ for dense $G_{\delta}$ sets $D \subseteq X$.

The extension of the construction to arbitrary perfect Polish spaces will be carried out in Section 3 without serious difficulties. Moreover, the construction of the example in Section 2 will allow us to put into $\mathcal{I}$ any fixed family of compact sets of the form $\mathcal{K}(Z)$, where $Z$ is a meager $\boldsymbol{\Sigma}_{2}^{0}$ set. Unfortunately we cannot prove a general theorem allowing us to extend any $\Pi_{1}^{1} \sigma$-ideal $\mathcal{J}$ to a $G_{\delta} \sigma$-ideal $\mathcal{I}$ in such a way that $\mathcal{K}(D) \nsubseteq \mathcal{J} \Longrightarrow \mathcal{K}(D) \nsubseteq \mathcal{I}$ for every $D$ in a reasonable collection of dense $G_{\delta}$ sets (see Problem (3.8). Another property of our construction is that it does not produce a $\sigma$-ideal with the covering property. A nontrivial $G_{\delta} \sigma$-ideal with the covering property would be a natural and well-understood way to give a negative answer to the question of Kechris (see [1, Problem 12, p. 137] and the comments after [5, Problem 6.1, p. 530]).

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## 2. The Example

In this section we prove the following result, which is slightly more general than Theorem 1.1. As above, $2^{\omega}$ stands for the Cantor space with its usual topology. For every $A \subseteq 2^{\omega}$ we set $\mathcal{K}(A)=\{K \subseteq A: K$ compact $\}$ and $\mathcal{L}(A)=\{K \in$ $\left.\mathcal{K}\left(2^{\omega}\right): K \cap A \neq \emptyset\right\}$. The space $\mathcal{K}\left(2^{\omega}\right)$ is endowed with the Vietoris topology, which makes it a compact Polish space (see e.g. [1, (4.25), Theorem, p. 26]). The closure and the interior of a set $A \subseteq X$ are denoted by $\mathrm{cl}_{X}(A)$ and $\operatorname{int}_{X}(A)$.

Theorem 2.1. Let $Z \subseteq 2^{\omega}$ be a meager $\boldsymbol{\Sigma}_{2}^{0}$ set. Then there exists a $G_{\delta} \sigma$-ideal $\mathcal{I}$ of compact subsets of $2^{\omega}$ such that $\mathcal{I}$ contains all the singletons and $\mathcal{K}(Z) \subseteq \mathcal{I}$, but there is no dense $G_{\delta}$ set $D \subseteq 2^{\omega}$ such that $\mathcal{K}(D) \subseteq \mathcal{I}$.

For the construction, we pursue the following simple strategy. We aim to construct the complement of our $G_{\delta} \sigma$-ideal $\mathcal{I}$. We start by finding a closed set $\mathcal{P} \subseteq \mathcal{K}\left(2^{\omega}\right)$ such that $\mathcal{P}$ does not contain countable compact sets or sets in $\mathcal{K}(Z)$, and for every closed nowhere dense set $A \subseteq 2^{\omega}$ we have that $\mathcal{L}(A) \cap \mathcal{P}$ is nowhere dense in $\mathcal{P}$. Then for every dense $G_{\delta}$ set $D \subseteq 2^{\omega}$ we have $\mathcal{P} \cap \mathcal{K}(D) \neq \emptyset$ since $\mathcal{P} \cap \mathcal{K}(D)=\mathcal{P} \backslash \mathcal{L}\left(2^{\omega} \backslash D\right)$ is comeager in $\mathcal{P}$. Thus already $\mathcal{P} \subseteq \mathcal{K}\left(2^{\omega}\right) \backslash \mathcal{I}$ guarantees $\mathcal{K}(D) \backslash \mathcal{I} \neq \emptyset$ for every dense $G_{\delta}$ set $D \subseteq 2^{\omega}$. Finding our $\mathcal{P}$ is the crucial step; once found, it is easy to accompany it with a countable collection of closed subsets of $\mathcal{K}\left(2^{\omega}\right)$ such that together with $\mathcal{P}$ they form the complement of a $\sigma$-ideal.

We recall some notation following [1]. For every $s, t \in 2^{<\omega},|s|$ denotes the length of $s$ and $s \frown t$ stands for the sequence $s(0) \ldots s(|s|-1) t(0) \ldots t(|t|-1)$. We set $N_{s}=\left\{x \in 2^{\omega}: s \subseteq x\right\}$. If $T \subseteq 2^{<\omega}$ is a tree, then
(i) the maximal branches, or terminal nodes, of $T$ are denoted by $\mathfrak{T}(T)$;
(ii) for $s \in 2^{<\omega}, T_{s}=\left\{t \in 2^{<\omega}: s \frown t \in T\right\}$ and $s^{\frown} T=\left\{t \in 2^{\omega}: \exists u \in T(t \subseteq\right.$ $\left.\left.s^{\frown} u\right)\right\} ;$
(iii) $[T]=\left\{x \in 2^{\omega}: \forall n<\omega\left(\left.x\right|_{n} \in T\right)\right\}$.

For every $n<\omega$, we identify $2^{n}$ with the maximal branches of the full binary tree $2^{<n}$, i.e. $2^{n}=\mathfrak{T}\left(2^{<n}\right)$, indexed according to the lexicographic order. If $\sigma: 2^{n} \rightarrow 2$ is given, $T(\sigma)$ is the subtree of $2^{<n}$ generated by $\bigcup \sigma^{-1}(1)$.

For every $\mathcal{P} \subseteq \mathcal{K}\left(2^{\omega}\right)$ we define

$$
\begin{equation*}
\mathcal{P}^{\uparrow}=\left\{A \in \mathcal{K}\left(2^{\omega}\right): \exists P \in \mathcal{P}(P \subseteq A)\right\} \tag{2.1}
\end{equation*}
$$

For $2 \leq n<\omega$ and $\sigma: 2^{n} \rightarrow 2$ we set (see (2.2) below)

$$
\begin{gathered}
g_{l}(\sigma)=\min \left\{i<2^{n}: \sigma(i)=1\right\} \\
g_{r}(\sigma)=2^{n}-1-\max \left\{i<2^{n}: \sigma(i)=1\right\}, \\
b(\sigma)=\max \left\{d \leq 2^{n}: \forall i \in\left[g_{l}(\sigma), g_{l}(\sigma)+d\right)(\sigma(i)=1)\right\},
\end{gathered}
$$

and let $n(s)$ denote the length of the longest sequence of consecutive 0 's in $\left[2^{n-2}\right.$, $\left.2^{n}-2^{n-2}-1\right]:$

$$
\begin{equation*}
n=5: \sigma=\underbrace{000}_{g_{l}(\sigma)=3} \underbrace{1111}_{b(\sigma)=4} 001 \underbrace{00}_{n(\sigma)=2} 11011101111011 \underbrace{000000}_{g_{r}(\sigma)=6} \tag{2.2}
\end{equation*}
$$

For every $2 \leq n<\omega$ set

$$
\begin{equation*}
\Sigma_{n}=\left\{\sigma \in 2^{2^{n}}: g_{l}(\sigma) \leq 2^{n-2}, g_{r}(\sigma) \leq 2^{n-2} \text { and } n(\sigma) \leq b(\sigma)\right\} \tag{2.3}
\end{equation*}
$$

e.g. the sequence of (2.2) is in $\Sigma_{5}$.

For every $\eta: \omega \rightarrow \omega$ a fixed increasing function, we define a $\sigma$-ideal $\mathcal{I}$ as follows. Consider the following inductive construction of a sequence of finite trees $T^{n} \subseteq 2^{<\omega}$ $(n<\omega)$. Set $T^{0}=\{\emptyset\}$, let $0<n<\omega$ and suppose that $T^{n-1}$ is already defined. For every $t \in \mathfrak{T}\left(T^{n-1}\right)$ take an arbitrary $m(t)<\omega$ satisfying $\max \{2, \eta(|t|)\} \leq m(t)$ and pick an arbitrary sequence $\sigma_{t} \in \Sigma_{m(t)}$. We define $T_{n}$ by extending $T^{n-1}$ at every $t \in \mathfrak{T}\left(T^{n-1}\right)$ by $T\left(\sigma_{t}\right)$, that is,

$$
T^{n}=\left\{u \in 2^{<\omega}: u ß t^{\frown} s, t \in \mathfrak{T}\left(T^{n-1}\right), s \in T\left(\sigma_{t}\right)\right\}
$$

Such a sequence $\left(T^{n}\right)_{n<\omega}$ is called $\eta$-admissible. A tree $T \subseteq 2^{<\omega}$ is $\eta$-admissible if $T=\bigcup_{n<\omega} T^{n}$ for some $\eta$-admissible sequence $\left(T^{n}\right)_{n<\omega}$. For every $s \in 2^{<\omega}$ set

$$
\begin{equation*}
\mathcal{P}_{s}=\left\{\left[s^{\frown} T\right]: T \text { is } \eta \text {-admissible }\right\}, \tag{2.4}
\end{equation*}
$$

and with the notation of (2.1) let

$$
\begin{equation*}
\mathcal{I}=\mathcal{K}\left(2^{\omega}\right) \backslash \bigcup_{s \in 2^{<\omega}} \mathcal{P}_{s}^{\uparrow} \tag{2.5}
\end{equation*}
$$

First we show that for every $\eta: \omega \rightarrow \omega, \mathcal{I}$ is a $G_{\delta} \sigma$-ideal containing all the singletons such that $\mathcal{K}(D) \nsubseteq \mathcal{I}$ for every dense $G_{\delta}$ set $D \subseteq 2^{\omega}$. Next we show that for every meager $\boldsymbol{\Sigma}_{2}^{0}$ set $Z \subseteq 2^{\omega}$ there is an increasing $\eta$ such that $\mathcal{K}(Z) \subseteq \mathcal{I}$. The propositions we use will be proved at the end of this section.

To see that $\mathcal{I}$ is a $G_{\delta}$ subset of $\mathcal{K}\left(2^{\omega}\right)$, we need the following observations.
Proposition 2.2. Let $X$ be a compact Polish space and let $\mathcal{P} \subseteq \mathcal{K}(X)$ be a closed set. Then $\mathcal{P}^{\uparrow} \subseteq \mathcal{K}(X)$ is closed as well.

Proposition 2.3. For every $s \in 2^{<\omega}$ and $K \in \mathcal{K}\left(2^{\omega}\right)$, if $K \in c l_{\mathcal{K}\left(2^{\omega}\right)}\left(\mathcal{P}_{s}\right)$, then either $K \in \mathcal{P}_{s}$ or $\operatorname{int}_{2 \omega}(K) \neq \emptyset$.

To see that $\mathcal{I}$ is a $G_{\delta}$ subset of $\mathcal{K}\left(2^{\omega}\right)$ observe that if $K \in \mathcal{K}\left(2^{\omega}\right)$ satisfies $\operatorname{int}_{2 \omega}(K) \neq \emptyset$, say for an $s \in 2^{<\omega}$ we have $N_{s} \subseteq K$, then by (2.4) we have $K \in \mathcal{P}_{s}^{\uparrow}$. Hence by Proposition 2.3.

$$
\begin{equation*}
\bigcup_{s \in 2^{<\omega}} \mathcal{P}_{s}^{\uparrow}=\bigcup_{s \in 2^{<\omega}}\left[\operatorname{cl}_{\mathcal{K}\left(2^{\omega}\right)}\left(\mathcal{P}_{s}\right)\right]^{\uparrow} \tag{2.6}
\end{equation*}
$$

and by Proposition 2.2 this latter is an $F_{\sigma}$ subset of $\mathcal{K}\left(2^{\omega}\right)$. Hence $\mathcal{I}$ is a $G_{\delta}$ subset of $\mathcal{K}\left(2^{\omega}\right)$, as required. Notice that there are sets $P \in \operatorname{cl}_{\mathcal{K}\left(2^{\omega}\right)}\left(\mathcal{P}_{s}\right)$ which have isolated points. This phenomenon is quite disturbing, but as we will see in Proposition 3.5 it is inevitable.

Next we observe that $\mathcal{I}$ contains all the singletons. By (2.5) it is enough to prove the following.
Proposition 2.4. Let $s \in 2^{<\omega}$ and $P \in \mathcal{P}_{s}$, say $P=\left[s^{\frown} T\right]$, where $T$ is $\eta$ admissible. Then for every $t \in T, N_{t} \cap P$ is a nonempty perfect subset of $2^{\omega}$.

We have to show that $\mathcal{I}$ is a $\sigma$-ideal. By (2.5), $\mathcal{I}$ is closed under taking compact subsets; thus we only need the following.
Proposition 2.5. Let $K \in \mathcal{K}\left(2^{\omega}\right)$ and suppose that $K=\bigcup_{i<\omega} K_{i}$ for some $K_{i} \in \mathcal{I}$ $(i<\omega)$. Then $K \in \mathcal{I}$.

To see that $\mathcal{K}(D) \nsubseteq \mathcal{I}$ for every dense $G_{\delta}$ set $D \subseteq 2^{\omega}$ we will prove the following.
Proposition 2.6. Let $A \subseteq 2^{\omega}$ be a nowhere dense closed set. Then $\mathcal{L}(A) \cap \mathcal{P}_{\emptyset}$ is nowhere dense in $\mathcal{P}_{\emptyset}$.

Let $D \subseteq 2^{\omega}$ be a dense $G_{\delta}$ set, say $D=2^{\omega} \backslash \bigcup_{n<\omega} A_{n}$, where $A_{n} \subseteq 2^{\omega}$ is a nowhere dense closed set $(n<\omega)$. We have $\mathcal{K}(D)=\mathcal{K}\left(2^{\omega}\right) \backslash \bigcup_{n<\omega} \mathcal{L}\left(A_{n}\right)$. By Proposition [2.6, $\mathcal{L}\left(A_{n}\right) \cap \mathcal{P}_{\emptyset}$ is nowhere dense in $\mathcal{P}_{\emptyset}(n<\omega)$. Since $\mathcal{L}\left(A_{n}\right)$ is closed, this implies $\mathcal{L}\left(A_{n}\right) \cap \operatorname{cl}_{\mathcal{K}\left(2^{\omega}\right)}\left(\mathcal{P}_{\emptyset}\right)$ is nowhere dense in $\mathrm{cl}_{\mathcal{K}\left(2^{\omega}\right)}\left(\mathcal{P}_{\emptyset}\right)(n<$ $\omega)$. Hence $\bigcup_{n<\omega} \mathcal{L}\left(A_{n}\right) \cap \operatorname{cl}_{\mathcal{K}\left(2^{\omega}\right)}\left(\mathcal{P}_{\emptyset}\right)$ is meager in $\operatorname{cl}_{\mathcal{K}\left(2^{\omega}\right)}\left(\mathcal{P}_{\emptyset}\right)$. This means that $\mathrm{cl}_{\mathcal{K}\left(2^{\omega}\right)}\left(\mathcal{P}_{\emptyset}\right) \cap \mathcal{K}(D) \neq \emptyset$, so by (2.6), $\mathcal{K}(D) \backslash \mathcal{I} \neq \emptyset$, as required.

To obtain an $\eta: \omega \rightarrow \omega$ which guarantees $\mathcal{K}(Z) \subseteq \mathcal{I}$, we need the following.
Proposition 2.7. Let $A \subseteq 2^{\omega}$ be a nowhere dense set and let $s \in 2^{<\omega}$ be arbitrary. Then there exists an $n(A, s)<\omega$ such that for every $n \geq n(A, s)$ and for every $\sigma \in \Sigma_{n}$ there exists a $t \in T(\sigma)$ such that $N_{s-t} \cap A=\emptyset$.

For every nowhere dense set $A ß 2^{\omega}$ and for every $s \in 2^{<\omega}$ we fix $n(A, s)<\omega$ satisfying Proposition 2.7. For every nowhere dense set $A \subseteq 2^{\omega}$ we define

$$
\begin{equation*}
\zeta_{A}: \omega \rightarrow \omega, \zeta_{A}(n)=\max \left\{n(A, s): s \in 2^{<n}\right\} \tag{2.7}
\end{equation*}
$$

Proposition 2.8. Let $Z \subseteq 2^{\omega}$ be a meager $\boldsymbol{\Sigma}_{2}^{0}$ set, say $Z=\bigcup_{n<\omega} Z_{n}$, where $Z_{n} \subseteq 2^{\omega}$ is a nowhere dense closed set $(n<\omega)$. If $\eta: \omega \rightarrow \omega$ is an increasing function and for every $n<\omega, \zeta_{Z_{k}}(m) \leq \eta(n)(k, m \leq 2 n)$, then $\mathcal{K}(Z) \subseteq \mathcal{I}$.

It remains to prove the propositions. Before doing so we observe some simple properties of our construction.

Lemma 2.9. Let $\left(n_{k}\right)_{k<\omega} \subseteq \omega$ be a strictly increasing sequence, let $\sigma_{k} \in \Sigma_{n_{k}}$ $(k<\omega)$ and suppose that for a $K \in \mathcal{K}\left(2^{\omega}\right)$ we have $\lim _{k<\omega}\left[T\left(\sigma_{k}\right)\right]=K$. Then $\operatorname{int}_{2 \omega}(K) \neq \emptyset$.

Proof. We distinguish two cases. If $\liminf _{k \rightarrow \infty} 2^{-n_{k}} b\left(\sigma_{k}\right)=0$, then by (2.3), $\liminf _{k \rightarrow \infty} 2^{-n_{k}} n\left(\sigma_{k}\right)=0$, as well. By $\lim _{k<\omega}\left[T\left(\sigma_{k}\right)\right]=K$, this implies $N_{01} \cup N_{10} \subseteq$ $K$, so $\operatorname{int}_{2}{ }^{\omega}(K) \neq \emptyset$.

Otherwise, for an $\varepsilon>0$ we have $\liminf _{k \rightarrow \infty} 2^{-n_{k}} b\left(\sigma_{k}\right)>\varepsilon$. By the fact $\left(\left[T\left(\sigma_{k}\right)\right]\right)_{k<\omega}$ is convergent, it is possible only if for $n=\left\lceil-\log _{2}(\varepsilon)\right\rceil$ there is a $t \in 2^{n}$ such that for every sufficiently large $k<\omega$ and $s \in 2^{n_{k}-n}$ we have $\sigma_{k}(t \frown s)=1$. Then by $\lim _{k<\omega}\left[T\left(\sigma_{k}\right)\right]=K, N_{t} \subseteq K$, so again $\operatorname{int}_{2} \omega(K) \neq \emptyset$, as required.

Lemma 2.10. Let $A \subseteq 2^{\omega}$ be a nowhere dense set, and let $s \in 2^{<\omega}$ and $M<\omega$ be arbitrary. Then there exists an $n<\omega$ and $a \sigma \in \Sigma_{n}$ such that $M \leq n$ and

$$
\begin{equation*}
\bigcup\left\{N_{s-t}: t \in 2^{n}, \sigma(t)=1\right\} \subseteq 2^{\omega} \backslash A . \tag{2.8}
\end{equation*}
$$

Proof. By identifying $N_{s}$ with $2^{\omega}$, it is enough to prove the statement in the $s=\emptyset$ case. Since $A$ is nowhere dense there is an $s_{0} \in 2^{<\omega}$ satisfying $00 \subseteq s_{0}$ and $N_{s_{0}} \subseteq 2^{\omega} \backslash A$. Let $n \geq\left|s_{0}\right|$ be such that $M \leq n$ and

$$
\begin{equation*}
\forall t \in 2^{\left|s_{0}\right|} \exists t^{\prime} \in 2^{n-1}\left(t \subseteq t^{\prime}, N_{t^{\prime}} \subseteq 2^{\omega} \backslash A\right) \tag{2.9}
\end{equation*}
$$

such an $n$ exists again by $A$ being nowhere dense. We define $\sigma: 2^{n} \rightarrow 2$ by setting, for every $t \in 2^{n}, \sigma(t)=1$ if and only if $N_{t} \subseteq 2^{\omega} \backslash A$. Then (2.8) holds, so it remains to show $\sigma \in \Sigma_{n}$.

By the definition of $s_{0}$ we have $g_{l}(\sigma) \leq 2^{n-2}$ and $b(\sigma) \geq 2^{n-\left|s_{0}\right|}$. By (2.9) applied to $t \in 2^{\left|s_{0}\right|}$ defined by $t(i)=1$ for every $i<\left|s_{0}\right|$, we have $\sigma\left(t^{\prime}\right)=1$ for some $t^{\prime} \in 2^{n}$ with $t^{\prime}(i)=1$ for every $i<\left|s_{0}\right|$, so

$$
g_{r}(\sigma) \leq 2^{n-\left|s_{0}\right|}-1 \leq 2^{n-2}
$$

Again by (2.9), $n(\sigma) \leq 2\left(2^{n-1-\left|s_{0}\right|}-1\right)=2^{n-\left|s_{0}\right|}-2 \leq b(\sigma)$, so the proof is complete.

Proof of Proposition 2.2, Let $\left(Q_{n}\right)_{n<\omega} \subseteq \mathcal{P}^{\uparrow}$ be convergent, $Q=\lim _{n<\omega} Q_{n}$; we show $Q \in \mathcal{P}^{\uparrow}$. Let $P_{n} \subseteq Q_{n}$ satisfy $P_{n} \in \mathcal{P}(n<\omega)$. By passing to a subsequence we can assume $\left(P_{n}\right)_{n<\omega}$ is convergent, say $P=\lim _{n<\omega} P_{n}$. Then $P \in \mathcal{P}$ and clearly $P \subseteq Q$, so the proof is complete.

Proof of Proposition [2.3. Fix $s \in 2^{<\omega}$, and let $K \in \operatorname{cl}_{\mathcal{K}\left(2^{\omega}\right)}\left(\mathcal{P}_{s}\right)$, say $[s \sim T(k)] \rightarrow K$ as $k \rightarrow \infty$, where $T(k)(k<\omega)$ are $\eta$-admissible trees, i.e. $T(k)=\bigcup_{n<\omega} T^{n}(k)$ with $\left(T^{n}(k)\right)_{n<\omega} \eta$-admissible $(k<\omega)$. We distinguish two cases. Suppose first that for every $n<\omega$ the set $\left\{T^{n}(k): k<\omega\right\}$ is finite. By a diagonalization, we can find a sequence $\left(k_{i}\right)_{i<\omega}$ and trees $T^{n}(\infty)(n<\omega)$ such that

$$
\forall n<\omega \forall^{\infty} i<\omega T^{n}\left(k_{i}\right)=T^{n}(\infty)
$$

Then $\left(T^{n}(\infty)\right)_{n<\omega}$ is $\eta$-admissible and $K=\left[\bigcup_{n<\omega} s \frown T^{n}(\infty)\right]$; hence $K \in \mathcal{P}_{s}$.
Otherwise, for an $n<\omega$ we have that $\left\{T^{n}(k): k<\omega\right\}$ is infinite. Let $n_{0}$ be the smallest such $n$. By passing to a subsequence, for each $n<n_{0}$ we can find a tree $T^{n}(\infty)$ such that for all $k<\omega$ and $n<n_{0}$ we have $T^{n}(k)=T^{n}(\infty)$.

We show that $\operatorname{int}_{2 \omega}(K) \neq \emptyset$ in this case. Set $t=\emptyset$ if $n_{0}=0$; otherwise let $t \in \mathfrak{T}\left(T^{n_{0}-1}(\infty)\right)$ be such that $\left\{T^{n_{0}}(k)_{t}: k<\omega\right\}$ is infinite. Let $n_{k}(t)<\omega$ and $\sigma_{k}(t) \in \Sigma_{n_{k}(t)}$ be the parameters for which $T^{n_{0}}(k)_{t}=T\left(\sigma_{k}(t)\right)$. By passing to a subsequence we can assume $n_{k}(t)$ is strictly increasing as $k \rightarrow \infty$. Then $([T(k)])_{k<\omega}$ is convergent implies $\left(\left[T\left(\sigma_{k}(t)\right)\right]\right)_{k<\omega}$ is convergent, as well. Thus from Lemma 2.9 we get that $\operatorname{int}_{2 \omega}\left(K \cap N_{s-t}\right) \neq \emptyset$, which completes the proof.

Proof of Proposition (2.4. By (2.4) it is enough to prove the statement for $s=\emptyset$. By the definition of $\Sigma_{n}$ in (2.3), for every $2 \leq n<\omega$ and $\sigma \in \Sigma_{n}$ the tree $T(\sigma)$ has at least one splitting node. Therefore an $\eta$-admissible tree is a perfect tree, so the statement follows.

Proof of Proposition 2.5. Let $K, K_{i} \in \mathcal{K}\left(2^{\omega}\right)(i<\omega)$ such that $K=\bigcup_{i<\omega} K_{i}$. Suppose $K \notin \mathcal{I}$, that is, $K \in \mathcal{P}_{s}^{\uparrow}$ for some $s \in 2^{<\omega}$, say $\left[s^{\sim} T\right] \subseteq K$ for some $\eta$-admissible tree $T=\bigcup_{n<\omega} T^{n}$ with $\left(T^{n}\right)_{n<\omega} \eta$-admissible. We show that $K_{i} \notin \mathcal{I}$ for some $i<\omega$.

By the Baire Category Theorem, there exists $i<\omega$ and $t \in 2^{<\omega}$ such that

$$
\begin{equation*}
\emptyset \neq[s \frown T] \cap N_{s} \frown t \subseteq K_{i} . \tag{2.10}
\end{equation*}
$$

By nonemptiness of the intersection on the left of (2.10) and by extending $t$ we can assume $t \in \mathfrak{T}\left(T^{m}\right)$ for some $m<\omega$. For every $n<\omega$ set $\hat{T}^{n}=T_{t}^{m+n}$. Since $\eta$ is increasing, $\left(\hat{T}^{n}\right)_{n<\omega}$ is $\eta$-admissible. Thus $\hat{T}=\bigcup_{n<\omega} \hat{T}^{n}$ is $\eta$-admissible as well. We have $[s \frown t \frown \hat{T}]=N_{s \frown t} \cap[T] \subseteq K_{i}$, so $K_{i} \in \mathcal{P}_{s \frown t}^{\uparrow}$. Hence $K_{i} \notin \mathcal{I}$, as stated.

Proof of Proposition 2.6. Let $P \in \mathcal{P}_{\emptyset}$ be arbitrary, say $P=[T]$, where $T=$ $\bigcup_{n<\omega} T^{n}$ with $\left(T^{n}\right)_{n<\omega} \eta$-admissible. We construct a sequence $\left(P_{k}\right)_{k<\omega} \subseteq \mathcal{P}_{\emptyset}$ such that $P_{k} \cap A=\emptyset(k<\omega)$ and $\lim _{k<\omega} P_{k}=P$. This will complete the proof.

For every $k<\omega$ we find an $\eta$-admissible sequence $\left(T^{n}(k)\right)_{n<\omega}$ satisfying
(1) $T^{n}(k)=T^{n}(n \leq k<\omega)$,
(2) $\bigcup\left\{N_{s}: s \in \mathfrak{T}\left(T^{k+1}(k)\right)\right\} \subseteq 2^{\omega} \backslash A$,
as follows. Fix $k<\omega$. For $n \leq k, T^{n}(k)$ is given by (1); observe that $\eta$-admissibility holds for $T^{n}(k)(n \leq k)$. Since $A$ is nowhere dense we can apply Lemma 2.10 for every $s \in \mathfrak{T}\left(T^{k}(k)\right)$ with $M=\eta(|s|)$. We get that $T^{k}(k)$ can be extended to a finite tree $T^{k+1}(k)$ such that 2 holds and $\eta$-admissibility holds for $T^{n}(k)(n \leq k+1)$. Finally we construct $T^{n}(k)(k+1<n<\omega)$ arbitrarily such that $\left(T^{n}(k)\right)_{n<\omega}$ is $\eta$-admissible; this is clearly possible.

Set $P_{k}=\left[\bigcup_{n<\omega} T^{n}(k)\right]$; then $P_{k} \in \mathcal{P}_{\emptyset}(k<\omega)$. By (11), $\lim _{k<\omega} P_{k}=P$, while by (21), $P_{k} \cap A=\emptyset(k<\omega)$. This completes the proof.

Proof of Proposition 2.7. Suppose the statement fails for an $s \in 2^{<\omega}$; i.e. for every $n<\omega$, there is a sequence $\sigma_{n} \in \Sigma_{n}$ satisfying $N_{s-t} \cap A \neq \emptyset$ for every $t \in T\left(\sigma_{n}\right)$. We can pass to a subsequence $\left(n_{k}\right)_{k<\omega}$ such that $\left(\left[T\left(\sigma_{n_{k}}\right)\right]\right)_{k<\omega}$ is convergent in $\mathcal{K}\left(2^{\omega}\right)$, say $K=\lim _{k<\omega}\left[s^{\frown} T\left(\sigma_{n_{k}}\right)\right]$. By Lemma 2.9, $\operatorname{int}_{2}(K) \neq \emptyset$. However, $K \subseteq \operatorname{cl}_{2} \omega(A)$, which contradicts $A$ is nowhere dense.

Proof of Proposition 2.8. Since $\mathcal{I}$ is a $\sigma$-ideal, it is enough to show $Z_{k} \in \mathcal{I}(k<\omega)$. Fix $k<\omega$ and $s \in 2^{<\omega}$. We show $Z_{k} \notin \mathcal{P}_{s}^{\uparrow}$; that is, we show $[s \frown T] \nsubseteq Z_{k}$ for every $T=\bigcup_{n<\omega} T^{n}$ with $\eta$-admissible $\left(T^{n}\right)_{n<\omega}$.

Let $n<\omega$ be such that there is a terminal node $t \in T^{n-1}$ satisfying $k \leq|t|$, $|s| \leq|t|$. By $\eta$-admissibility, the parameter $m(t)$ in the construction of $T_{t}^{n}$ satisfies $\eta(|t|) \leq m(t)$. Thus

$$
\zeta_{Z_{k}}\left(\left|s^{\frown} t\right|\right)=\zeta_{Z_{k}}(|s|+|t|) \leq \zeta_{Z_{k}}(2|t|) \leq \eta(|t|) \leq m(t) .
$$

By the definition of $\zeta_{Z_{k}}$ in (2.7) and by Proposition 2.7 we have $N_{s{ }_{s}{ }^{2}{ }_{u} \cap Z_{k}=\emptyset}$ for some $u \in T\left(\sigma_{t}\right)$. Since $t \subset u \in T^{n}$ and by Proposition 2.4, $[T] \cap N_{t \sim u} \neq \emptyset$, we have $[s \frown T] \cap N_{s \frown t \neg u} \neq \emptyset$; hence $[s \frown T] \nsubseteq Z_{k}$. This completes the proof.

## 3. Analysis

In this section we extend the construction in Section 2 for $2^{\omega}$ to perfect Polish spaces. We also discuss some particularities of our construction, i.e. the role of isolated points.
3.1. Example in perfect Polish spaces. First we handle compact perfect Polish spaces. Note that requiring the Polish space to be perfect is not a superfluous assumption since in a Polish space containing a dense open countable set, Theorem 1.1 cannot hold. Our main tool is Proposition 3.2. We set

$$
\begin{equation*}
I=\left\{\sigma \in 2^{\omega}: \forall n<\omega \exists i, j<\omega(n \leq i, j, \sigma(i)=0, \sigma(j)=1)\right\} \tag{3.1}
\end{equation*}
$$

$\mathbb{H}=\left(2^{\omega}\right)^{\omega}$ and $\mathbb{I}=I^{\omega} \subseteq \mathbb{H}$. We start with a folklore lemma.
Lemma 3.1. Let $C \subseteq[1 / 3,2 / 3]^{\omega} \subseteq[0,1]^{\omega}$ be a countable set and let $D \subseteq[0,1]^{\omega}$ be a dense $G_{\delta}$ set. Then there is an $x=\left(x_{n}\right)_{n<\omega} \in(0,1 / 3)^{\omega}$ such that

$$
C+x=\left\{\left(c_{n}+x_{n}\right)_{n<\omega}:\left(c_{n}\right)_{n<\omega} \in C\right\} \subseteq D
$$

Proof. For every $c \in C,\left\{x \in(0,1 / 3)^{\omega}: x+c \in[0,1]^{\omega} \backslash D\right\} \subseteq(0,1 / 3)^{\omega}$ is a meager set, so the statement follows from the Baire Category Theorem.

Proposition 3.2. If $X$ is a compact perfect Polish space, then there is a dense $G_{\delta}$ set $G \subseteq X$ and continuous surjective $\operatorname{map} \varphi: 2^{\omega} \rightarrow X$ such that $\varphi$ is one-to-one on $\varphi^{-1}(G)$ and for every nowhere dense closed set $Z \subseteq X$ we have that $\varphi^{-1}(Z) \subseteq 2^{\omega}$ is nowhere dense.

Proof. Let $f: 2^{\omega} \rightarrow[0,1]$ and $F: \mathbb{H} \rightarrow[0,1]^{\omega}$,

$$
f(\sigma)=\sum_{i<\omega} \frac{\sigma(i)}{2^{i+1}}, F\left(\left(\sigma_{n}\right)_{n<\omega}\right)=\left(f\left(\sigma_{n}\right)\right)_{n<\omega}\left(\sigma, \sigma_{n} \in 2^{\omega}(n<\omega)\right)
$$

be the usual continuous surjections. Since $f$ is a homeomorphism on $I$ of (3.1), $F$ is also a homeomorphism on $\mathbb{I}$ and $F(\mathbb{I}) \subseteq[0,1]^{\omega}$ is a dense $G_{\delta}$ set.

Every compact Polish space is homeomorphic to a closed subset of $[1 / 3,2 / 3]^{\omega}$ (see e.g. [1, (4.14), Theorem, p. 22]), so we regard $X$ as a subset of $[1 / 3,2 / 3]^{\omega} \subseteq[0,1]^{\omega}$. By applying Lemma 3.1 for a countable dense subset of $X$ and $D=F(\mathbb{I})$ we can assume that $X \cap F(\mathbb{I})$ is a dense $G_{\delta}$ subset of $X$.

Since $X$ is perfect, $X \cap F(\mathbb{I})$ is perfect, as well. Therefore $\operatorname{cl}_{\mathbb{H}}\left(F^{-1}(X) \cap \mathbb{I}\right)$ is a nonempty zero-dimensional compact perfect Polish space, so we have a homeomorphism $i: 2^{\omega} \rightarrow \operatorname{cl}_{\mathbb{H}}\left(F^{-1}(X) \cap \mathbb{I}\right)$. We show that $G=X \cap F(\mathbb{I})$ and $\varphi: 2^{\omega} \rightarrow X$, $\varphi=\left.F\right|_{\mathrm{cl}_{\mathbb{H}}\left(F^{-1}(X) \cap \mathbb{I}\right)} \circ i$ fulfill the requirements. Since $\varphi$ is continuous and $X \cap F(\mathbb{I}) \subseteq$ $\varphi\left(2^{\omega}\right), \varphi$ is surjective and $\varphi^{-1}(G)=i^{-1}\left(F^{-1}(X) \cap \mathbb{I}\right)$ is a dense $G_{\delta}$ set in $2^{\omega}$. Since $i$ and $\left.F\right|_{\mathbb{I}}$ are homeomorphisms, $\left.\varphi\right|_{\varphi^{-1}(G)}: \varphi^{-1}(G) \rightarrow G$ is a homeomorphism as well. Let $Z \subseteq X$ be a nowhere dense closed set. Then

$$
\begin{equation*}
\varphi^{-1}(Z)=\left(\varphi^{-1}(Z) \backslash \varphi^{-1}(G)\right) \cup \varphi^{-1}(Z \cap G) \subseteq\left(2^{\omega} \backslash \varphi^{-1}(G)\right) \cup \varphi^{-1}(Z \cap G) \tag{3.2}
\end{equation*}
$$

The first term of the union on the right of (3.2) is meager since $\varphi^{-1}(G)$ is a dense $G_{\delta}$ set in $2^{\omega}$. For the second term, we have that $Z \cap G$ is nowhere dense in $X \cap G$ and $\left.\varphi\right|_{\varphi^{-1}(G)}$ is a homeomorphism, so $\varphi^{-1}(Z \cap G)$ is nowhere dense in $\varphi^{-1}(G)$, hence in $2^{\omega}$ as well. So $\varphi^{-1}(Z) \subseteq 2^{\omega}$ is a closed meager, hence nowhere dense, set, as required.

Corollary 3.3. Let $X$ be a compact perfect Polish space and let $Z \subseteq X$ be a meager $\boldsymbol{\Sigma}_{2}^{0}$ set. Then there exists a $G_{\delta} \sigma$-ideal $\mathcal{I}$ of compact subsets of $X$ such that $\mathcal{I}$ contains all the singletons and $\mathcal{K}(Z) \subseteq \mathcal{I}$ but there is no dense $G_{\delta}$ set $D \subseteq X$ such that $\mathcal{K}(D) \subseteq \mathcal{I}$.
Proof. Let $G \subseteq X$ be the dense $G_{\delta}$ set and $\varphi: 2^{\omega} \rightarrow X$ be the map of Proposition 3.2. Let $\mathcal{I}_{2^{\omega}}$ be the $\sigma$-ideal of (2.5) with an $\eta$ as in Proposition 2.8 for the meager $\Sigma_{2}^{0}$ set $\varphi^{-1}(Z \cup(X \backslash G))$. We show that for $\mathcal{I} \beta \mathcal{K}(X)$,

$$
K \in \mathcal{I} \Leftrightarrow \varphi^{-1}(K) \in \mathcal{I}_{2^{\omega}}
$$

fulfills the requirements.
Since $\varphi$ is a continuous function, $\mathcal{I}$ is a $\sigma$-ideal of compact sets. We have $\mathcal{K}(Z \cup(X \backslash G)) ß \mathcal{I}$, and since $\varphi$ is one-to-one on $\varphi^{-1}(G),\{x\} \in \mathcal{I}$ for every $x \in G$ as well. Hence $\mathcal{I}$ contains all the singletons and $\mathcal{K}(Z) ß \mathcal{I}$.

Next let $D \subseteq X$ be a dense $G_{\delta}$ set. By the choice of $\varphi$ and $G, \varphi^{-1}(D \cap G)$ is a dense $G_{\delta}$ subset of $2^{\omega}$. By the definition of $\mathcal{I}_{2^{\omega}}$, there is a $K \in \mathcal{K}\left(\varphi^{-1}(D \cap G)\right) \backslash \mathcal{I}_{2^{\omega}}$. Since $\varphi$ is one-to-one on $\varphi^{-1}(G)$, we have $\varphi(K) \in \mathcal{K}(D) \backslash \mathcal{I}$, which shows $\mathcal{K}(D) \nsubseteq \mathcal{I}$.

It remains to show that $\mathcal{I}$ is a $\Pi_{2}^{0}$ set. Consider the map $\Phi: \mathcal{K}\left(2^{\omega}\right) \rightarrow \mathcal{K}(X)$, $\Phi(K)=\varphi(K)\left(K \in \mathcal{K}\left(2^{\omega}\right)\right)$. By the continuity of $\varphi, \Phi$ is continuous. We prove $\Phi\left(\mathcal{I}_{2^{\omega}}\right)=\mathcal{I}$; then we have that $\mathcal{I}$ is $\boldsymbol{\Sigma}_{1}^{1}$, so it is $\boldsymbol{\Pi}_{2}^{0}$ by [4, Theorem 11, p. 270].

By $\Phi\left(\varphi^{-1}(K)\right)=K(K \in \mathcal{K}(X)), \mathcal{I} \beta \Phi\left(\mathcal{I}_{2^{\omega}}\right)$. Let $K \in \Phi\left(\mathcal{I}_{2^{\omega}}\right)$, say $K=\varphi(L)$ for some $L \in \mathcal{I}_{2^{\omega}}$. Since $\varphi$ is one-to-one on $\varphi^{-1}(G)$,

$$
\varphi^{-1}(K)=\varphi^{-1}(K \cap G) \cup \varphi^{-1}(K \backslash G) ß L \cup \varphi^{-1}(X \backslash G)
$$

So by $L \in \mathcal{I}_{2^{\omega}}$ and $\mathcal{K}\left(\varphi^{-1}(X \backslash G)\right) ß \mathcal{I}_{2^{\omega}}, \varphi^{-1}(K) \in \mathcal{I}_{2^{\omega}}$ as well. So the proof is complete.

Before extending the construction to noncompact Polish spaces, we recall some notation. If $X$ is a Polish space, we denote the family of closed subsets of $X$ by $\mathcal{F}(X)$ and we endow $\mathcal{F}(X)$ with the Vietoris topology. For $D \subseteq X$ we set $\mathcal{F}(D)=\{F \in \mathcal{F}(X): F \subseteq D\}$. Then Corollary 3.3 easily yields the following.

Corollary 3.4. Let $X$ be an arbitrary perfect Polish space and let $Z \subseteq X$ be a meager $\boldsymbol{\Sigma}_{2}^{0}$ set. Then there exists a $G_{\delta} \sigma$-ideal $\mathcal{I}$ of closed subsets of $X$ such that $\mathcal{I}$ contains all the singletons and $\mathcal{F}(Z) \subseteq \mathcal{I}$ but there is no dense $G_{\delta}$ set $D \subseteq X$ such that $\mathcal{K}(D) \subseteq \mathcal{I}$.

The same result holds for compact sets instead of closed sets as well.
Proof. Let $\hat{X}$ be a Polish compactification of $X$. Let $\hat{\mathcal{I}} \subseteq \mathcal{K}(\hat{X})$ be the $\sigma$-ideal of Corollary 3.3 applied for $\hat{X}$ with the meager $\boldsymbol{\Sigma}_{2}^{0}$ set $Z \cup(\hat{X} \backslash X)$. Set $\mathcal{I}=$ $\{K \cap X: K \in \hat{\mathcal{I}}\} \subseteq \mathcal{F}(X)$. It is obvious that $\mathcal{I}$ is a hereditary $G_{\delta}$ family in $\mathcal{F}(X)$ and contains all the singletons. Since $\mathcal{K}(\hat{X} \backslash X) \subseteq \hat{\mathcal{I}}, \mathcal{I}$ is closed under taking closed countable unions. By $\mathcal{K}(Z \cup(\hat{X} \backslash X)) \subseteq \hat{\mathcal{I}}$ we have $\mathcal{F}(Z) \subseteq \mathcal{I}$. Finally $X$ is a dense $G_{\delta}$ subset of $\hat{X}$; thus for every dense $G_{\delta}$ set $D \subseteq X \subseteq \hat{X}$ we have $\{K \in \mathcal{K}(\hat{X}): K \subseteq D\} \nsubseteq \hat{\mathcal{I}}$ and hence $\mathcal{K}(D) \nsubseteq \mathcal{I}$ as well.

To have a $\sigma$-ideal of compact sets with the same properties we only have to take $\mathcal{I} \cap \mathcal{K}(X)$. This completes the proof.
3.2. The role of isolated points. It is easy to check that even if $\mathcal{P}_{s}$ contains only nonempty perfect sets, $\operatorname{cl}_{\mathcal{K}\left(2^{\omega}\right)}\left(\mathcal{P}_{s}\right)$ contains sets with isolated points. This is somehow disturbing: singletons are to be avoided, and it would be nice to construct
$\mathcal{K}\left(2^{\omega}\right) \backslash \mathcal{I}$ in such a way that it is closed under taking nonempty clopen portions. However, this is impossible. Recall $\mathcal{L}(A)=\left\{K \in \mathcal{K}\left(2^{\omega}\right): K \cap A \neq \emptyset\right\}\left(A \subseteq 2^{\omega}\right)$.
Proposition 3.5. Set

$$
\begin{equation*}
\mathcal{P}=\left\{K \in \mathcal{K}\left(2^{\omega}\right) \backslash\{\emptyset\}: K \text { is perfect }\right\} . \tag{3.3}
\end{equation*}
$$

Then $\mathcal{P}$ is a dense $G_{\delta}$ subset of $\mathcal{K}\left(2^{\omega}\right) \backslash\{\emptyset\}$, but for every closed set $\mathcal{L} \subseteq \mathcal{K}\left(2^{\omega}\right)$, if $\mathcal{L} \subseteq \mathcal{P}$, there is a nowhere dense closed set $A \subseteq 2^{\omega}$ such that $\mathcal{L} \subseteq \mathcal{L}(A)$. In particular, if $\mathcal{I} \subseteq \mathcal{K}\left(2^{\omega}\right)$ is a $G_{\delta}$ set and $\mathcal{K}\left(2^{\omega}\right) \backslash \mathcal{I} \subseteq \mathcal{P}$, then there is a dense $G_{\delta}$ set $D \subseteq 2^{\omega}$ such that $\mathcal{K}(D) \subseteq \mathcal{I}$.

Thus sets with isolated points play a crucial role in every construction, providing a negative answer to the question of Kechris. But Proposition 3.5 has another much more interesting corollary. First we need to recall the so-called covering property (see e.g. [3, Definition 9, p. 135], 5, Section 3] or [7]).
Definition 3.6. Let $X$ be a Polish space. A family $\mathcal{F}$ of closed subsets of $X$ has the covering property if for every $\boldsymbol{\Sigma}_{1}^{1}$ set $A \subseteq X$,
(1) either there is a countable subfamily $\mathcal{A} \in[\mathcal{F}]^{\leq \omega}$ such that $A \subseteq \bigcup \mathcal{A}$
(2) or there is a closed set $H \subseteq X$ such that $H \subseteq A$ and $F \cap H$ is nowhere dense in $H$ for every $F \in \mathcal{F}$.
The relevance of the covering property comes from the following observation. Let $\mathcal{J} \subseteq \mathcal{K}\left(2^{\omega}\right)$ be a $\Pi_{1}^{1} \sigma$-ideal such that for every dense $G_{\delta}$ set $D \subseteq 2^{\omega}$ we have $\mathcal{K}(D) \nsubseteq \mathcal{J}$. Consider the family

$$
\mathfrak{F}=\left\{\mathcal{L}(A): A \in \mathcal{K}\left(2^{\omega}\right) \text { is nowhere dense }\right\} \subseteq \mathcal{K}\left(\mathcal{K}\left(2^{\omega}\right)\right) .
$$

By assumption, the $\boldsymbol{\Sigma}_{1}^{1}$ set $\mathcal{K}\left(2^{\omega}\right) \backslash \mathcal{I}$ cannot be covered by the union of countably many members of $\mathfrak{F}$. So if the family $\mathfrak{F}$ had the covering property, we would have a compact set $\mathcal{P} \subseteq \mathcal{K}\left(2^{\omega}\right)$ for which $\mathcal{L}(A) \cap \mathcal{P}$ is nowhere dense in $\mathcal{P}$ for every nowhere dense set $A \subseteq 2^{\omega}$. As we have seen in Section 1 for $\mathcal{J}=\{K \in$ $\mathcal{K}\left(2^{\omega}\right): K$ is countable $\}$ the existence of such a $\mathcal{P}$ is the key to the construction of a $G_{\delta} \sigma$-ideal $\mathcal{I} \subseteq \mathcal{K}\left(2^{\omega}\right)$ such that $\mathcal{J} \subseteq \mathcal{I}$ and still for every dense $G_{\delta}$ set $D \subseteq 2^{\omega}$ we have $\mathcal{K}(D) \nsubseteq \mathcal{I}$. However, as an immediate corollary of Proposition 3.5 we have the following.

Corollary 3.7. The family $\mathfrak{F}$ does not have the covering property in the Polish space $\mathcal{K}\left(2^{\omega}\right)$.
Proof. By Proposition 3.5, the family $\mathcal{P}$ of (3.3) is a dense $G_{\delta}$ subset of $\mathcal{K}\left(2^{\omega}\right)$. It is obvious that $\mathcal{L}(A) \cap \mathcal{P}$ is nowhere dense in $\mathcal{P}$ for every nowhere dense set $A \subseteq 2^{\omega}$. In particular $\mathcal{P}$ cannot be covered by the union of countably many members of $\mathfrak{F}$. But again by Proposition 3.5, for every closed set $\mathcal{L} \subseteq \mathcal{P}$ there is a nowhere dense closed set $A \subseteq 2^{\omega}$ such that $\mathcal{L} \subseteq \mathcal{L}(A)$. Hence the covering property fails for $\mathfrak{F}$.

Hence, such a far-reaching generalization of our construction is not possible via the covering property of $\mathfrak{F}$. Since $\left\{A \in \mathcal{K}\left(2^{\omega}\right)\right.$ : $A$ is nowhere dense $\} \subseteq \mathcal{K}\left(2^{\omega}\right)$ is a $G_{\delta}$ set, it is easy to check that $\mathfrak{F}$ is a $G_{\delta}$ subset of $\mathcal{K}\left(\mathcal{K}\left(2^{\omega}\right)\right)$ as well. We note that by [5. Corollary 3.5, p. 509] every $K_{\sigma}$ family of compact sets has the covering property. Accordingly, for every $K_{\sigma}$ set $\mathcal{A} \subseteq \mathcal{K}\left(2^{\omega}\right)$ the construction of Section 1 goes through for $\mathfrak{F}_{\mathcal{A}}=\{\mathcal{L}(A): A \in \mathcal{A}\}$. Nevertheless, the following problem remains open. Just as for the Question of Kechris, we expect a possibly easy negative answer here as well.

Problem 3.8. Let $\mathcal{J} \subseteq \mathcal{K}\left(2^{\omega}\right)$ be an arbitrary $\boldsymbol{\Pi}_{1}^{1} \sigma$-ideal such that for every dense $G_{\delta}$ set $D \subseteq 2^{\omega}$ we have $\mathcal{K}(D) \nsubseteq \mathcal{J}$. Can we find a $G_{\delta} \sigma$-ideal $\mathcal{I} \subseteq \mathcal{K}\left(2^{\omega}\right)$ such that $\mathcal{J} \subseteq \mathcal{I}$ and still for every dense $G_{\delta}$ set $D \subseteq 2^{\omega}$ we have $\mathcal{K}(D) \nsubseteq \mathcal{I}$ ?

We close this paper with the proof of Proposition 3.5.

Proof of Proposition 3.5. For every $n<\omega$ set

$$
\mathcal{S}_{n}=\left\{S \in \mathcal{K}\left(2^{\omega}\right): \exists s \in 2^{n}\left(\left|S \cap N_{s}\right|=1\right)\right\} .
$$

Then $\mathcal{S}_{n} \subseteq \mathcal{K}\left(2^{\omega}\right)$ is a nowhere dense closed set $(n<\omega)$ and $\mathcal{K}\left(2^{\omega}\right) \backslash \mathcal{P}=\bigcup_{n<\omega} \mathcal{S}_{n}$. Let $\mathcal{L} \subseteq \mathcal{P}$ be a closed set. Since $\mathcal{L} \cap \mathcal{S}_{n}=\emptyset(n<\omega)$, by the definition of Vietoris open neighborhoods we have a function $\kappa: \omega \rightarrow \omega$ such that $n<\kappa(n)(n<\omega)$ and if $K \in \mathcal{K}\left(2^{\omega}\right), S \in \mathcal{S}_{n}$ satisfy

$$
\begin{equation*}
\forall s \in 2^{\kappa(n)}\left(K \cap N_{s} \neq \emptyset \Longleftrightarrow S \cap N_{s} \neq \emptyset\right) \tag{3.4}
\end{equation*}
$$

then $K \notin \mathcal{L}$.
Set $d_{0}=0, d_{n+1}=\kappa\left(d_{n}\right)(n<\omega)$ and let

$$
A=\left\{\sigma \in 2^{\omega}: \forall n<\omega \exists i \in\left[d_{n}, d_{n+1}\right)(\sigma(i) \neq 0)\right\}
$$

Then $A \subseteq 2^{\omega}$ is a nowhere dense closed set. We show $\mathcal{L} \subseteq \mathcal{L}(A)$. To this end, pick $K \in \mathcal{K}\left(2^{\omega}\right) \backslash\{\emptyset\}$ with $K \cap A=\emptyset$; we have to show $K \notin \mathcal{L}$.

For every $n<\omega$ let

$$
U_{n}=\left\{\sigma \in 2^{\omega}:\left.\sigma\right|_{\left[d_{n}, d_{n+1}\right)} \equiv 0 \text { and } \forall j<n \exists i \in\left[d_{j}, d_{j+1}\right)(\sigma(i) \neq 0)\right\}
$$

The sets $U_{n}(n<\omega)$ are open, pairwise disjoint and $2^{\omega} \backslash A=\bigcup_{n<\omega} U_{n}$. Hence there is a minimal $N<\omega$ such that $K \subseteq \bigcup_{n \leq N} U_{n}$. Set $\mathbb{O} \in 2^{\omega}, \mathbb{O}(i)=0(i<\omega)$. We define $S \subseteq 2^{\omega}$ by $S \cap A=\emptyset, S \cap U_{n}=K \cap U_{n}(n \neq N)$ and for $s \in 2^{d_{N+1}}$ with $N_{s} \subseteq U_{N}$ let

$$
S \cap N_{s}= \begin{cases}\{s \frown \mathbb{O}\} & \text { if } K \cap N_{s} \neq \emptyset  \tag{3.5}\\ \emptyset & \text { if } K \cap N_{s}=\emptyset\end{cases}
$$

Clearly, $S \in \mathcal{K}\left(2^{\omega}\right)$. By the minimality of $N$ we have $U_{N} \cap S \neq \emptyset$; hence by (3.5), $S \in \mathcal{S}_{d_{N}}$. By definition, (3.4) holds for $n=d_{N}$; hence $K \notin \mathcal{L}$, as required.

## References

[1] A. S. Kechris, Classical Descriptive Set Theory, Graduate Texts in Mathematics, 156 (Springer-Verlag, New York, 1995). MR1321597(96e:03057)
[2] A. S. Kechris, Hereditary properties of the class of closed sets of uniqueness for trigonometric series, Israel J. Math. 73 (1991), no. 2, 189-198. MR1135211 (93c:42008)
[3] A. S. Kechris, The descriptive set theory of $\sigma$-ideals of compact sets, Logic Colloq. '88 (Padova, 1988), 117-138, Stud. Logic Found. Math., 127, North-Holland, Amsterdam, 1989. MR 1015324 (90h:03032)
[4] A. S. Kechris, A. Louveau, W. H. Woodin, The structure of $\sigma$-ideals of compact sets, Trans. Amer. Math. Soc. 301 (1987), no. 1, 263-288. MR879573 (88f:03042)
[5] E. Matheron, M. Zelený, Descriptive set theory of families of small sets, Bull. Symb. Logic 13 (2007), no. 4, 482-537. MR 2369671
[6] S. Solecki, $G_{\delta}$ Ideals of Compact Sets, preprint.
[7] C. E. Uzcátegui A., The covering property for $\sigma$-ideals of compact sets, Fund. Math. 141 (1992), no. 2, 119-146. MR1183328(94a:03077)

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