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COMPLETIONS OF QUANTUM COORDINATE RINGS

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Abstract. Given an iterated skew polynomial ring $C[y_1;\tau_1,\delta_1]\dots[y_n;\tau_n,\delta_n]$ over a complete local ring C with maximal ideal $\mathfrak{m}$, we prove, under suitable assumptions, that the completion at the ideal $\mathfrak{m}+\langle y_1,y_2,\dots,y_n\rangle$ is an iterated skew power series ring. Under further conditions, the completion becomes a local, noetherian, Auslander regular domain. Applicable examples include quantum matrices, quantum symplectic spaces, and quantum Euclidean spaces.

1. Introduction

Let R be a ring equipped with a skew derivation (τ,δ). The skew power series ring $R[[y;\tau]]$, when $\delta=0$, is a well-known, classical object (cf. [5], [11]). The skew power series ring $R[[y;\tau,\delta]]$, when $\delta\neq 0$, has more recently appeared in quantum algebras (cf. [8, §4], [9, §4]) and in noncommutative Iwasawa theory (cf. [13], [14]). In this paper, we study iterated skew power series rings as completions of iterated skew polynomial rings. Our approach builds on the work of Venjakob in [14].

Our main result can be stated as follows: *Let*

$$R_n = C[y_1;\tau_1,\delta_1]\dots[y_l;\tau_l,\delta_l]\dots[y_n;\tau_n,\delta_n] \quad (n\geq 1)$$

be an iterated skew polynomial ring, where C is a complete local ring with maximal ideal $\mathfrak{m}$, and where C is stable under each skew derivation (τ_l,δ_l). For each $1\leq l\leq n$, set $I_{l-1}=\mathfrak{m}+\langle y_1,\dots,y_{l-1}\rangle$, and assume that $\tau_l(I_{l-1})\subseteq I_{l-1}$, $\delta_l(R_{l-1})\subseteq I_{l-1}$, and $\delta_l(I_{l-1})\subseteq I_{l-1}^2$. Then there exists an iterated skew power series ring

$$S_n = C[[y_1;\hat\tau_1,\hat\delta_1]]\dots[[y_l;\hat\tau_l,\hat\delta_l]]\dots[[y_n;\hat\tau_n,\hat\delta_n]],$$

such that $\hat\tau_l|_{R_{l-1}}=\tau_l$ and $\hat\delta_l|_{R_{l-1}}=\delta_l$, for $1\leq l\leq n$. Moreover, S_n is the completion of R_n at the ideal $\mathfrak{m}+\langle y_1,\dots,y_n\rangle$.

The paper is organized as follows: Section 2 reviews some preliminary results and proves the main result. Section 3 applies the main result to certain quantum coordinate rings, including quantum matrices, quantum symplectic spaces, and quantum Euclidean spaces.

Throughout, all rings are unital.

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2. Main result

Let R be a ring, τ a ring endomorphism of R and δ a left τ-derivation, that is, $\delta : R \to R$ is an additive map for which $\delta(rs) = \tau(r)\delta(s) + \delta(r)s$ for all $r,\, s \in R$. The pair of maps $(\tau,\, \delta)$ is called a *skew derivation* on R. To start, we recall the structure of the skew power series ring in one variable, following Venjakob [14].

2.1. Let S be the additive group of formal power series in y,

$$\sum_i r_i y^i = \sum_{i=0}^{\infty} r_i y^i,$$

with coefficients r_i in R. Using the commutation rule $yr = \tau(r)y + \delta(r)$, for $r \in R$, we wish to write the product of two arbitrary elements in S as

$$\left(\sum_i r_i y^i\right)\left(\sum_j s_j y^j\right) = \sum_n^{\infty} \sum_{j=0}^{n} \sum_{i=n-j}^{\infty} r_i (y^i s_j)_{n-j} y^n,$$

where each $(y^n r)_i$, for $0 \le i \le n$, denotes an element in R such that

$$y^n r = \sum_{i=0}^{n} (y^n r)_i y^i,$$

for $n \ge 0$. However, it is not always the case that

$$\sum_{j=0}^{n} \sum_{i=n-j}^{\infty} r_i (y^i s_j)_{n-j}$$

is well defined in R. If, under some additional restrictions (see subsection 2.3), the multiplication formula is well defined for any two power series in S, we will say that S is a *well-defined skew power series ring*, and write $S = R[[y; \tau, \delta]]$.

2.2. By a *local ring* we will always mean a ring R such that the quotient ring by the Jacobson radical $J(R)$ is simple artinian. In particular, a local ring has a unique maximal ideal which is equal to the Jacobson radical. Let R be a local ring with maximal ideal $\mathfrak{m}$. We will always equip R with the $\mathfrak{m}$-adic topology. By the associated graded ring $\operatorname{gr} R$, we will always mean with respect to the $\mathfrak{m}$-adic filtration, that is:

$$\operatorname{gr} R = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \cdots .$$

We will refer to R as a *complete local ring* if R is also *complete* (i.e., Cauchy sequences converge in the $\mathfrak{m}$-adic topology) and *separated* (i.e., the $\mathfrak{m}$-adic topology is Hausdorff).

2.3. Let R be a complete local ring with maximal ideal $\mathfrak{m}$ and with skew derivation (τ, δ). As in [14], we assume that $\tau(\mathfrak{m}) \subseteq \mathfrak{m}$, $\delta(R) \subseteq \mathfrak{m}$ and $\delta(\mathfrak{m}) \subseteq \mathfrak{m}^2$. In [14, Lemma 2.1], Venjakob proved, under these assumptions, that $S = R[[y; \tau, \delta]]$ is a well-defined skew power series ring. The following properties of S are also proved in, or easily deduced from, Venjakob's work in [14, §2].

(i) Any element $\sum_i r_i y^i$ is a unit (in S) if and only if the constant term r_0 is a unit in R. In particular, any element in $1 - \langle \mathfrak{m},\, y\rangle$ is a unit, and so the Jacobson radical $J(S) = \langle \mathfrak{m},\, y\rangle$. Hence, in view of the isomorphism $S/J(S) \cong R/\mathfrak{m}$, S is a local ring.

(ii) The $\langle \mathfrak{m},\, y\rangle$-adic filtration on S is complete and separated.

(iii) There is a canonical isomorphism $\operatorname{gr} S \cong (\operatorname{gr} R)[\bar{y};\, \bar{\tau}]$. Assume further that $\bar{\tau}$ is an automorphism. Then, S is right (respectively left) noetherian if $\operatorname{gr} R$ is right (respectively left) noetherian, S is a domain if $\operatorname{gr} R$ is a domain, and S is Auslander regular if the same holds for $\operatorname{gr} R$; see [14, Corollary 2.10] (cf. [10, Chap. III, Theorem 2.2.5], [10, Chap. III, Theorem 3.4.6 (1)]).

(iv) Now suppose that $\operatorname{gr} R$ is right noetherian and that $\bar{\tau}$ is an automorphism. Concerning right global dimension, it follows that $\operatorname{rgl} S \leq \operatorname{rgl} \operatorname{gr} R{+}1$. As far as right Krull dimension is concerned, $\operatorname{rKdim} \operatorname{gr}(S) = \operatorname{rKdim} \operatorname{gr} R{+}1$, by [6, Theorem 15.19]. Moreover, $\operatorname{rKdim} S \leq \operatorname{rKdim} \operatorname{gr} S$, as S is a complete filtered ring and $\operatorname{gr} S$ is right noetherian; see [12, D.IV.5]. Therefore $\operatorname{rKdim} S \leq \operatorname{rKdim} \operatorname{gr} R + 1$.

2.4. Contained within S is the skew polynomial ring $T = R[y;\, \tau, \delta]$. Following subsection 2.3 (ii), both S and T are endowed with a Hausdorff $\langle \mathfrak{m},\, y\rangle$-adic topology. Of course, T is a dense subring of S in this topology. Therefore, S is the completion of T with respect to the $\langle \mathfrak{m},\, y\rangle$-adic filtration, following [1, Theorem 3.3.5].

The remainder of this section is devoted to the main result. First we set up a suitable iterated skew polynomial ring. Then we construct an iterated skew power series ring, by extending skew derivations.

2.5. **Setup.** Let C be a complete local ring with maximal ideal $\mathfrak{m}$. Set $R_0 = C$, and let

$$R_n = C[y_1; \tau_1, \delta_1] \dots [y_l; \tau_l, \delta_l] \dots [y_n; \tau_n, \delta_n]$$

be an iterated skew polynomial ring with skew derivations (τ_l, δ_l) of R_{l-1}, for $1 \leq l \leq n$. For each $1 \leq l \leq n$, set

$$I_{l-1} = \mathfrak{m} + \langle y_1, \dots, y_{l-1}\rangle \subseteq R_{l-1},$$

and assume that

$$\tau_l(I_{l-1}) \subseteq I_{l-1}, \quad \delta_l(R_{l-1}) \subseteq I_{l-1}, \quad \text{and} \quad \delta_l(I_{l-1}) \subseteq I_{l-1}^2.$$

We will also need the following notation.

2.6. (i) Let $1 \leq l \leq n+1$. A nonzero monomial $c_{i_1,\dots,i_{l-1}} y_1^{i_1} \cdots y_{l-1}^{i_{l-1}}$ in R_{l-1} is said to be in *normal form*. We will write

$$c_{\underline{i}} Y_{l-1}^{\underline{i}}$$

for $c_{i_1,\dots,i_{l-1}} y_1^{i_1} \cdots y_{l-1}^{i_{l-1}}$, where $\underline{i} = (i_1, \dots, i_{l-1}) \in \mathbb{N}^{l-1}$.

(ii) We now introduce the notion of degree that we will use for nonzero monomials in normal form. Let $1 \leq l \leq n+1$, and let $c_{\underline{i}} Y_{l-1}^{\underline{i}} \in R_{l-1}$. Then there exists a largest integer k such that $c_{\underline{i}} \in \mathfrak{m}^k$. Set

$$s(c_{\underline{i}}, \underline{i}) = k + i_1 + i_2 + \dots + i_{l-1}.$$

We will refer to $s(c_{\underline{i}}, \underline{i})$ as the *degree* of $c_{\underline{i}} Y_{l-1}^{\underline{i}}$.

(iii) Let $1 \leq l \leq n$, and let $c_{\underline{i}} Y_{l-1}^{\underline{i}}$ and $d_{\underline{j}} Y_{l-1}^{\underline{j}}$ be two nonzero monomials in R_{l-1}. Then $c_{\underline{i}} Y_{l-1}^{\underline{i}} \cdot d_{\underline{j}} Y_{l-1}^{\underline{j}}$ is 0 or a sum of monomials each with degree $\geq s(c_{\underline{i}}, \underline{i}) + s(d_{\underline{j}}, \underline{j})$. An inductive argument shows that each of the polynomials $\tau_l \left(c_{\underline{i}} Y_{l-1}^{\underline{i}}\right)$ and $\delta_l \left(c_{\underline{i}} Y_{l-1}^{\underline{i}}\right)$ is 0 or a finite sum of monomials each with degree $\geq s(c_{\underline{i}}, \underline{i})$.

(iv) Let $1 \leq l \leq n$. By a *formal power series* in $y_1, \ldots, y_l$ over C, we will mean an infinite series

$$f = \sum_{\underline{i}} c_{\underline{i}} Y_l^{\underline{i}},$$

where the $c_{\underline{i}}$ are elements in C and where $\underline{i} \in \mathbb{N}^l$. Note that each monomial $c_{\underline{i}} Y_l^{\underline{i}}$ is in normal form. The set of all formal power series in $y_1, \ldots, y_l$ over C forms an abelian group, which we will denote as A_l.

2.7. Let $1 \leq l \leq n$.

(i) Given a power series $f = \sum_{\underline{i}} c_{\underline{i}} Y_l^{\underline{i}} \in A_l$, we can always write

$$f = \sum_{k=0}^{\infty} \sum_{s(c_{\underline{i}}, \underline{i}) = k} c_{\underline{i}} Y_l^{\underline{i}},$$

after regrouping the monomials appearing in f (if necessary). Note that for each k, the sum

$$\sum_{s(c_{\underline{i}}, \underline{i}) = k} c_{\underline{i}} Y_l^{\underline{i}}$$

is finite and possibly equal to 0.

(ii) On the other hand, let

$$g = G_0 + G_1 + \ldots + G_k + \ldots,$$

where each G_k is 0 or a finite sum of monomials in R_l all with degree k. Then g is a well-defined (in the above sense) formal power series in A_l. To see this, suppose that

$$G_k = \sum_{\underline{j} \in M_k} c_{\underline{j}}^{(k)} Y_l^{\underline{j}},$$

where $c_{\underline{j}}^{(k)} \in C$ and where $M_k \subseteq \mathbb{N}^l$, for $k = 0, 1, \ldots$. We will set $c_{\underline{i}}^{(k)} = 0$ when $\underline{i} \notin M_k$. Now, for a fixed $\underline{j}$, the sum

$$c_{\underline{j}}^{(0)} + c_{\underline{j}}^{(1)} + \ldots + c_{\underline{j}}^{(k)} + \ldots$$

might contain infinitely many terms. But each $c_{\underline{j}}^{(k)}$ is such that the degree of $c_{\underline{j}}^{(k)} Y_l^{\underline{j}}$ is equal to k. Hence, the preceding sum is convergent in the $\mathfrak{m}$-adic topology. Therefore,

$$g = G_0 + G_1 + \ldots + G_k + \ldots = \sum_{\underline{j} \in \cup M_k} \left(c_{\underline{j}}^{(0)} + c_{\underline{j}}^{(1)} + \ldots + c_{\underline{j}}^{(k)} + \ldots \right) Y_l^{\underline{j}}$$

is a formal power series in A_l with all coefficients in C well defined.

2.8. **Theorem.** *Retain the notation and assumptions in setup* 2.5. *Let* $S_0 = C$. *Then there exists an iterated skew power series ring*

$$S_n = C[[y_1; \hat{\tau}_1, \hat{\delta}_1]] \ldots [[y_l; \hat{\tau}_l, \hat{\delta}_l]] \ldots [[y_n; \hat{\tau}_n, \hat{\delta}_n]],$$

where each $(\hat{\tau}_l, \hat{\delta}_l)$ *is a skew derivation on* S_{l-1} *with* $\hat{\tau}_l|_{R_{l-1}} = \tau_l$ *and* $\hat{\delta}_l|_{R_{l-1}} = \delta_l$, *for* $1 \leq l \leq n$. *Moreover,* S_n *is a complete local ring with maximal ideal* $\mathfrak{m}_n = \mathfrak{m} + \langle y_1, \ldots, y_n \rangle$. *(We will refer to* S_n *as the* power series extension *of* R_n.*)*

Proof. Following subsection 2.3, the ring $C[[y_1;\tau_1,\delta_1]]$ is well defined and we may take $S_1 = C[[y_1;\tau_1,\delta_1]]$. In the notation of subsection 2.6, S_1 is the abelian group A_1 equipped with a well-defined multiplication restricting to the original multiplication in R_1. Our goal is to show that each abelian group A_l becomes an iterated skew power series ring. In the first step of the proof, we extend the pair of maps τ_l and δ_l to A_{l-1} for all $1 < l \leq n$. Then, by induction, we will show that each (τ_l, δ_l) extends to a skew derivation on S_{l-1} and that each A_l forms a ring S_l.

To start, let $f = \sum_{\underline{i}} c_{\underline{i}} Y_{l-1}^{\underline{i}}$ be a power series in A_{l-1}. As in subsection 2.7 (i), we can write

$$f = \sum_{k=0}^{\infty} F_k, \qquad \text{where} \quad F_k := \sum_{s(c_{\underline{i}},\underline{i})=k} c_{\underline{i}} Y_{l-1}^{\underline{i}} \text{ (possibly equal to 0).}$$

Our goal now is to extend τ_l and δ_l to A_{l-1}. For $k = 0, 1, 2, \ldots$, if $\tau_l(F_k) \neq 0$, then

$$\tau_l(F_k) = \sum_{\underline{j}\in T_k} t_{\underline{j}}^{(k)} Y_{l-1}^{\underline{j}}$$

for some subset $T_k \subseteq \mathbb{N}^{l-1}$ and some $t_{\underline{j}}^{(k)} \in C$. Next, let

$$G_m = \sum_{k=0}^{\infty} \sum_{\underline{j}\in N_{m,k}} t_{\underline{j}}^{(k)} Y_{l-1}^{\underline{j}},$$

where

$$N_{m,k} = \{\underline{j} \in T_k \mid \text{the degree of } t_{\underline{j}}^{(k)} Y_{l-1}^{\underline{j}} \text{ is } m\}.$$

In other words, we regroup the monomials appearing in $\sum_k \tau_l(F_k)$ by their degrees. Then

$$\tau_l(F_0) + \tau_l(F_1) + \ldots + \tau_l(F_k) + \ldots \;=\; G_0 + G_1 + \ldots + G_m + \ldots.$$

It follows from subsection 2.6 (iii) that any nonzero $\tau_l(F_k)$ is a finite sum and that each $t_{\underline{j}}^{(k)} Y_{l-1}^{\underline{j}}$ has degree $\geq k$. Hence each G_m is a finite sum by the construction. Recall from subsection 2.7 (ii) that

$$G_0 + G_1 + \ldots + G_m + \ldots$$

is a formal power series in A_{l-1}. Therefore,

$$\sum_{k=0}^{\infty} \tau_l\left(F_k\right) \in A_{l-1}.$$

Using the same argument (replacing τ_l with δ_l), we also have

$$\sum_{k=0}^{\infty} \delta_l\left(F_k\right) \in A_{l-1}.$$

Then, for $1 \leq l \leq n$ and $f = \sum_{\underline{i}} c_{\underline{i}} Y_{l-1}^{\underline{i}} \in A_{l-1}$, we extend τ_l and δ_l by setting up the following maps:

$$(2.1) \quad \hat{\tau}_l(f) = \sum_{k=0}^{\infty} \tau_l \left(\sum_{s(c_{\underline{i}},\underline{i})=k} c_{\underline{i}} Y_{l-1}^{\underline{i}} \right) \quad \text{and} \quad \hat{\delta}_l(f) = \sum_{k=0}^{\infty} \delta_l \left(\sum_{s(c_{\underline{i}},\underline{i})=k} c_{\underline{i}} Y_{l-1}^{\underline{i}} \right).$$

It is clear that $\hat{\tau}_l|_{R_{l-1}} = \tau_l$ and $\hat{\delta}_l|_{R_{l-1}} = \delta_l$, for all $1 \leq l \leq n$.

Now, let $n \geq 2$. Assume that the abelian group A_{n-1} is a well-defined power series ring, which we will denote as S_{n-1}, and also assume that S_{n-1} is a complete local ring with maximal ideal $\mathfrak{m}_{n-1} = \mathfrak{m} + \langle y_1, \ldots, y_{n-1}\rangle$. Next we show that $(\hat{\tau}_n, \hat{\delta}_n)$, from (2.1), is a skew derivation on S_{n-1}; that is, $\hat{\tau}_n$ is an automorphism of S_{n-1} and $\hat{\delta}_n$ is a left $\hat{\tau}_n$-derivation.

Let t be a positive integer. Choose two arbitrary elements a and b in S_{n-1}. Write $a = a_t + a'_t$ and $b = b_t + b'_t$, where a_t (respectively b_t) is the sum of the monomials appearing in a (respectively b) with degree $\leq t$. Then it follows from (2.1) that

$$\hat{\tau}_n(a) = \hat{\tau}_n(a_t) + \hat{\tau}_n(a'_t) \quad \text{and} \quad \hat{\tau}_n(b) = \hat{\tau}_n(b_t) + \hat{\tau}_n(b'_t).$$

Therefore, we have

$$\hat{\tau}_n(ab) = \tau_n\,(a_t \cdot b_t) + \hat{\tau}_n(a'_t \cdot b_t + a_t \cdot b'_t + a'_t \cdot b'_t), \quad \text{and}$$
$$\hat{\tau}_n(a) \cdot \hat{\tau}_n(b) = \tau_n(a_t) \cdot \tau_n(b_t) + \hat{\tau}_n(a'_t) \cdot \hat{\tau}_n(b_t) + \hat{\tau}_n(a_t) \cdot \hat{\tau}_n(b'_t) + \hat{\tau}_n(a'_t) \cdot \hat{\tau}_n(b'_t).$$

Note that $\tau_n(a_t \cdot b_t) = \tau_n(a_t) \cdot \tau_n(b_t)$. It follows from subsection 2.6 (iii) that

$$\hat{\tau}_n(ab) - \hat{\tau}_n(a) \cdot \hat{\tau}_n(b) \in \mathfrak{m}_{n-1}^{t+1}.$$

Let $t \to \infty$. Then it follows from the completeness of S_{n-1} that

$$\hat{\tau}_n(ab) = \hat{\tau}_n(a) \cdot \hat{\tau}_n(b).$$

Using the same argument (replacing $\hat{\tau}_n$ with $\hat{\delta}_n$), we can get

$$\hat{\delta}_n(ab) = \hat{\delta}_n(a)b + \hat{\tau}_n(a)\hat{\delta}_n(b).$$

Therefore $(\hat{\tau}_n, \hat{\delta}_n)$ is a skew derivation on S_{n-1}.

In view of the assumptions in setup 2.5 and (2.1), we see that

$$\hat{\tau}_n(\mathfrak{m}_{n-1}) \subseteq \mathfrak{m}_{n-1}, \quad \hat{\delta}_n(S_{n-1}) \subseteq \mathfrak{m}_{n-1}, \quad \text{and} \quad \hat{\delta}_n(\mathfrak{m}_{n-1}) \subseteq \mathfrak{m}_{n-1}^2.$$

Following subsection 2.3 (i), (ii), the skew power series ring $S_n = S_{n-1}[[y_n; \tau_n, \delta_n]]$ is well defined, and S_n is a complete local ring with maximal ideal $\mathfrak{m}_n = \mathfrak{m} + \langle y_1, \ldots, y_n\rangle$. This completes the inductive step. The theorem is proved by induction. □

The following is a consequence of subsections 2.3, 2.4 and Theorem 2.8.

2.9. **Corollary.** (i) *The power series extension S_n in Theorem 2.8 is the completion of R_n with respect to the ideal $\mathfrak{m}_n = \mathfrak{m} + \langle y_1, \ldots, y_n\rangle$. Any power series in S_n is a unit (in S_n) if and only if its constant term is a unit in C.*

(ii) *The associated graded ring $\operatorname{gr} S_n$ is isomorphic to an iterated skew polynomial ring $(\operatorname{gr} C)[y_1; \bar{\tau}_1] \ldots [y_n; \bar{\tau}_n]$.*

(iii) *Assume further that $\bar{\tau}_1, \ldots, \bar{\tau}_n$ are automorphisms. If $\operatorname{gr} C$ is a domain, S_n is a domain. If $\operatorname{gr} C$ is right (respectively left) noetherian, so is S_n. If $\operatorname{gr} C$ is Auslander regular, then S_n is also Auslander regular.*

(iv) *Suppose that $\operatorname{gr} C$ is right noetherian and that $\bar{\tau}_1, \ldots, \bar{\tau}_n$ are automorphisms. Then it follows that $\operatorname{rKdim} S_n \leq \operatorname{rKdim} \operatorname{gr} C + n$ and $\operatorname{rgl} S_n \leq \operatorname{rgl} \operatorname{gr} C + n$.*

3. Examples

Throughout, let $\mathbf{k}$ be a field.

3.1. **Quantum matrices.** Let $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\mathbf{k}))$ be the multiparameter quantum coordinate ring of $n\times n$ matrices over $\mathbf{k}$, as studied in [2] (cf., e.g., [3]). Here $\mathbf{p}=(p_{ij})$ is a multiplicatively antisymmetric $n\times n$ matrix over $\mathbf{k}$, and λ is a nonzero element of $\mathbf{k}$ not equal to 1. Further information about this algebra can be found in [3]. As shown in [2], $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\mathbf{k}))$ can be presented as a skew polynomial ring

$$\mathbf{k}[y_{11}]\,[y_{1\,2};\tau_{1\,2}]\,\cdots\,[y_{lm};\,\tau_{lm},\delta_{lm}]\,\cdots\,[y_{nn};\tau_{nn},\delta_{nn}].$$

Each (τ_{lm},δ_{lm}) is a skew derivation as follows:

$$\tau_{lm}(y_{ij})=\begin{cases} p_{li}p_{jm}y_{ij}, & \text{when } l\geq i \text{ and } m>j,\\ \lambda p_{li}p_{jm}y_{ij}, & \text{when } l>i \text{ and } m\leq j,\end{cases}$$

$$\delta_{lm}(y_{ij})=\begin{cases} (\lambda-1)p_{li}y_{im}y_{lj}, & \text{when } l>i \text{ and } m>j,\\ 0, & \text{otherwise.}\end{cases}$$

It is not hard to see that these skew derivations satisfy the assumptions in setup 2.5. Hence, by Theorem 2.8, the power series extension of $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\mathbf{k}))$ is the iterated skew power series ring

$$\mathbf{k}[[y_{11}]]\,[[y_{1\,2};\hat{\tau}_{1\,2}]]\,\cdots\,[[y_{lm};\,\hat{\tau}_{lm},\hat{\delta}_{lm}]]\,\cdots\,[[y_{nn};\hat{\tau}_{nn},\hat{\delta}_{nn}]],$$

where each extended skew derivation is defined as in (2.1). Also note that each τ_{lm} acts by nonzero scalar multiplication on the generators, and so each $\bar{\tau}_{lm}$ is an automorphism. It now follows from Corollary 2.9 that the preceding power series completion is a local, noetherian, Auslander regular domain.

3.2. **Quantized k-algebras K_n.** There are other well-known quantum coordinate rings, for example coordinate rings of quantum symplectic space and quantum Euclidean $2n$-space (see, e.g., [3]). Horton introduced a class of algebras, denoted $K^{P,Q}_{n,\Gamma}(\mathbf{k})$ or more briefly K_n, that includes coordinate rings of both quantum symplectic space and quantum Euclidean $2n$-space; see [7]. To describe this class of algebras, let $P,\,Q\in(\mathbf{k}^\times)^n$ such that $P=(p_1,\ldots,p_n)$ and $Q=(q_1,\ldots,q_n)$ where $p_i\neq q_i$ for each $i\in\{1,\ldots,n\}$. Further, let $\Gamma=(\gamma_{i,j})\in M_n(\mathbf{k}^\times)$ with $\gamma_{j,i}=\gamma_{i,j}^{-1}$ and $\gamma_{i,i}=1$ for all $i,\,j$. Then, as in [7], $K^{P,Q}_{n,\Gamma}(\mathbf{k})$ is generated by $x_1,\,y_1,\,\ldots,x_n,\,y_n$ satisfying certain relations determined by P, Q and Γ. This algebra can be presented as an iterated skew polynomial ring,

$$\mathbf{k}[x_1][y_1;\,\tau_1][x_2;\,\sigma_2][y_2;\,\tau_2,\,\delta_2]\cdots[x_n;\,\sigma_n][y_n;\,\tau_n,\,\delta_n];$$

see [7, Proposition 3.5]. Automorphisms σ_i, τ_i and τ_i-derivations δ_i are defined as follows:

$$\begin{array}{ll}
\sigma_i(x_j)=q_j^{-1}p_i\gamma_{i,j}x_j & 1\leq j\leq i-1,\\
\sigma_i(y_j)=q_j\gamma_{j,i}y_j & 1\leq j\leq i-1,\\
\tau_i(x_j)=p_i^{-1}\gamma_{j,i}x_j & 1\leq j\leq i-1,\\
\tau_i(y_j)=\gamma_{i,j}y_j & 1\leq j\leq i-1,\\
\tau_i(x_i)=q_i^{-1}x_i, & \\
\delta_i(x_j)=0 & 1\leq j\leq i-1,\\
\delta_i(y_j)=0 & 1\leq j\leq i-1,\\
\delta_i(x_i)=-q_i^{-1}\sum_{l<i}(q_l-p_l)y_lx_l. &
\end{array}$$

Note that these automorphisms and derivations give quadratic relations, and so, by Theorem 2.8, K_n has the power series extension

$$\mathbf{k}[[x_1]][[y_1;\,\hat{\tau}_1]][[x_2;\,\hat{\sigma}_2]][[y_2;\,\hat{\tau}_2,\,\hat{\delta}_2]]\cdots[[x_l;\,\hat{\sigma}_l]][[y_l;\,\hat{\tau}_l,\,\hat{\delta}_l]]\cdots[[x_n;\,\hat{\sigma}_n]][[y_n;\,\hat{\tau}_n,\,\hat{\delta}_n]],$$

where the extended skew derivations are defined as in (2.1). Again, it follows from Corollary 2.9 that this completion is a local, noetherian, Auslander regular domain.

Moreover, comparing with Corollary 2.9 (iv), the dimensions of the power series completions in subsection 3.2 can be more precisely determined as follows:

3.3. Let E be an algebra in the class K_n, and $\hat{E}$ be the power series completion of E with respect to the ideal $\langle x_1, y_1, \ldots, x_n, y_n\rangle$. From subsection 3.2, we see that, among the defining commutation relations, nonzero derivations only occur in the following cases:

$$y_i x_i = \tau_i(x_i) y_i + \delta_i(x_i), \quad \text{for } i = 2, \ldots, n.$$

Also note that $\delta_i(x_i) \in I_{i-1} = \langle x_1, y_1, \ldots, x_{i-1}, y_{i-1}\rangle$. Hence, the set of generators $\{x_1, y_1, \ldots, x_n, y_n\}$ forms a *regular normalizing set* (see [15, Definition 1.1]). Since $J(\hat{E}) = \langle x_1, y_1, \ldots, x_n, y_n\rangle$, it now follows from [15, Theorem 2.7] that the Krull dimension, classical Krull dimension and global dimension of $\hat{E}$ are all equal to $2n$.

3.4. *Remark.* For the quantum coordinate rings and quantum algebras in subsections 3.1 and 3.2, it is well known that the derivations δ_{lm} and δ_l are locally nilpotent. In [4], using this fact (and other assumptions), Cauchon constructed the "Derivation-Elimination Algorithm". But, for power series completions of these examples, the extended derivations $\hat{\delta}_{lm}$ and $\hat{\delta}_l$ are not locally nilpotent.

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References

[1] V. I. Arnautov, S. T. Glavatsky and A.V. Mikhalev, *Introduction to the Theory of Topological Rings and Modules*, Pure and Applied Mathematics 197, Marcel Dekker, Inc., New York, 1996. MR1368852 (97g:16057)

[2] M. Artin, W. Schelter, and J. Tate, *Quantum deformations of GL_n*, *Comm. Pure Appl. Math.*, 44 (1991), 879-895. MR1127037 (92i:17014)

[3] K. Brown and K. Goodearl, *Lectures on Algebraic Quantum Groups*, Basel: Birkhäuser, 2002. MR1898492 (2003f:16067)

[4] G. Cauchon, Effacement des dérivations et spectres premiers des algèbres quantiques, *J. Alg.*, 260 (2003), 476-518. MR1967309 (2004g:16044)

[5] P. M. Cohn, *Skew Fields: Theory of General Division Rings*, Encyclopedia of Mathematics and Its Applications 57, Cambridge University Press, Cambridge, 1995. MR1349108 (97d:12003)

[6] K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings, Second Edition*, London Mathematical Society Student Texts 61, Cambridge University Press, Cambridge, 2004. MR2080008 (2005b:16001)

[7] K. L. Horton, The prime and primitive spectra of multiparameter quantum symplectic and Euclidean spaces, *Comm. Alg.*, (10) 31 (2003), 4713-4743. MR1998025 (2004f:16073)

[8] R. M. Kashaev, The Heisenberg double and the pentagon relation, *Algebra i Analiz*, 8 (1996), no. 4, 63-74. MR1418255 (97j:16055)

[9] T. H. Koornwinder, Special functions and q-commuting variables, *Special Functions, q-Series and Related Topics (Toronto, Ontario, 1995)*, Fields Institute Communications 14, Amer. Math. Soc., Providence, RI, 1997, 131–166. MR1448685 (99e:33024)

[10] H. Li and F. Van Oystaeyen, *Zariskian Filtrations*, Kluwer Academic Publishers, Dordrecht, 1996. MR1420862 (97m:16083)

[11] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics, vol. 30, American Mathematical Society, Providence, RI, 2000. MR1811901 (2001i:16039)

[12] C. Năstaşescu and F. Van Oystaeyen, *Graded Ring Theory*, North-Holland, Amsterdam, 1982. MR676974 (84i:16002)

[13] P. Schneider and O. Venjakob, On the codimension of modules over skew power series rings with applications to Iwasawa algebras, *J. Pure Appl. Algebra*, 204 (2006), 349–367. MR2184816 (2006i:16067)

[14] O. Venjakob, A non-commutative Weierstrass preparation theorem and applications to Iwasawa theory (with an appendix by Denis Vogel), *J. Reine Angew. Math.*, 559 (2003), 153–191. MR1989649 (2004e:11123)

[15] R. Walker, Local rings and normalizing sets of elements, *Proc. London Math. Soc.*, (3) 24 (1972), 27-45. MR0294399 (45:3469)

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