PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 3, March 2009, Pages 1161–1168
S 0002-9939(08)09666-4
Article electronically published on October 20, 2008

ON THE SAME N-TYPE CONJECTURE FOR THE SUSPENSION OF THE INFINITE COMPLEX PROJECTIVE SPACE

DAE-WOONG LEE

(Communicated by Paul Goerss)

ABSTRACT. Let $[\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]]$ be an iterated commutator of self-maps φ_{i_j} on the suspension of the infinite complex projective space. In this paper, we produce useful self-maps of the form $I + [\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]]$, where $+$ means the addition of maps on the suspension structure of $\Sigma\mathbb{C}P^\infty$. We then give the answer to the conjecture saying that the set of all the same homotopy n-types of the suspension of the infinite complex projective space is the one element set consisting of a single homotopy type.

1. Introduction

Let $X^{(n)}$ be the nth Postnikov approximation of a space X. We recall that two CW-spaces X and Y are said to have the *same n-type* if the nth Postnikov approximations $X^{(n)}$ and $Y^{(n)}$ are homotopy equivalent. In the early years of algebraic topology, there was an important question posed by J. H. C. Whitehead: if X and Y are two spaces whose Postnikov approximations, $X^{(n)}$ and $Y^{(n)}$, are homotopy equivalent for each integer n, then does it follow that X and Y have the same homotopy type? It is well known that if X is either finite dimensional or if X has only a finite number of non-zero homotopy groups, then the answer to the question is yes! However, in general, there are examples, composed by Adams [1] and Gray [4], demonstrating that the answer to this question is no!

As usual, let Σ and Ω be the suspension and loop functors in the (pointed) homotopy category respectively. Let $\mathrm{Aut}(X)$ be a group of self-homotopy equivalences of X, and let $SNT(X)$ be the set of all homotopy types $[Y]$ such that $X^{(n)}$ and $Y^{(n)}$ are homotopy equivalent for each integer n. We can find valuable results regarding the same homotopy n-type and those kinds of notions (see [5], [9], [12]). In particular, Wilkerson [15, Theorem I] proved that for a connected CW-complex X, there is a bijection of pointed sets $SNT(X) \approx \lim^1\{\mathrm{Aut}(X^{(n)})\}$, where $\lim^1(-)$ is the first derived limit of groups in the sense of Bousfield-Kan [2, p. 251].

The most interesting problems of the same homotopy n-type are the suspensions of the Eilenberg-Mac Lane spaces or the localization of these spaces at any set of primes J [6]. In the case of odd integers $2d+1, d \geq 0$, it is well known that

Received by the editors February 28, 2008, and, in revised form, April 28, 2008.
2000 *Mathematics Subject Classification.* Primary 55P15; Secondary 55S37, 55P40.
Key words and phrases. Same n-type, Aut, commutator, Samelson (Whitehead) product.
This paper was (partially) supported by the Chonbuk National University funds for overseas research, 2008.

$SNT(\Sigma^k K(\mathbb{Z}, 2d+1)) = *$ for $k \geq 0$. The proof of this depends on the fact that the space $\Sigma^k K(\mathbb{Z}, 2d+1)$ has the rational homotopy type of a single sphere in dimension $d = k + 2d + 1$. What will happen in the case of even integers? As the reader can see, this is too complicated and intractable, because the space $\Sigma^k K(\mathbb{Z}, 2d)$ has the rational homotopy type of a bouquet of spheres in dimensions $d = k + 2d, k + 4d, k + 6d, \cdots$. From this point of view, there is a conjecture [11, p. 287] posed by C. A. McGibbon and J. M. Møller:

Conjecture. $SNT(\Sigma\mathbb{C}P^\infty) = *$.

The main purpose of this paper is to present the positive answer to the above conjecture. In Section 2, we describe self-maps on the suspension and loop structures and create useful self-maps of commutators on the suspension structures. In Section 3, we give the answer to the conjecture.

2. Construction of self-maps

McGibbon and Møller [11, Theorem 1] proved

Theorem 1. *Let X be a 1-connected space with finite type over some subring of the rationals. Assume that X has the rational homotopy type of a bouquet of spheres. Then the following three conditions are equivalent:*

(1) $SNT(X) = *$,

(2) *the map* $Aut(X) \xrightarrow{f \mapsto f^{(n)}} Aut(X^{(n)})$ *has a finite cokernel for each n,*

(3) *the map* $Aut(X) \xrightarrow{f \mapsto f_\sharp} Aut(\pi_{\leq n}(X))$ *has a finite cokernel for each n.*

Here, the group $Aut(\pi_{\leq n}(X))$ denotes the group of automorphisms of the graded $\mathbb{Z}$-module, $\pi_{\leq n}(X)$, preserving the Whitehead product pairings.

Actually, we will use this theorem in order to provide the answer to the conjecture. Theorem 1 is the Eckmann-Hilton dual of the corresponding result [10, Theorem 3] concerning the 1-connected H_0-space with finite type over Z_J for some set of primes J along with cohomology.

Let $\hat{\varphi}_1 : \mathbb{C}P^\infty \to \Omega\Sigma\mathbb{C}P^\infty$ and $x_1 : S^2 \to \Omega\Sigma\mathbb{C}P^\infty$ be the inclusions. Morisugi [13] considered the maps inductively:

$$\hat{\varphi}_{n+1} : \mathbb{C}P^\infty \xrightarrow{\bar{\Delta}} \mathbb{C}P^\infty \wedge \mathbb{C}P^\infty \xrightarrow{\hat{\varphi}_1 \wedge \hat{\varphi}_n} \Omega\Sigma\mathbb{C}P^\infty \wedge \Omega\Sigma\mathbb{C}P^\infty \xrightarrow{\sharp} \Omega\Sigma\mathbb{C}P^\infty$$

and

$$x_{n+1} : S^{2n+2} = S^2 \wedge S^{2n} \xrightarrow{x_1 \wedge x_n} \Omega\Sigma\mathbb{C}P^\infty \wedge \Omega\Sigma\mathbb{C}P^\infty \xrightarrow{\sharp} \Omega\Sigma\mathbb{C}P^\infty,$$

where $\bar{\Delta}$ is the reduced diagonal map and $\sharp$ means an extension of the adjoint of the Hopf construction of $\mathbb{C}P^\infty$. Now we take a self-map $\varphi_n : \Sigma\mathbb{C}P^\infty \to \Sigma\mathbb{C}P^\infty$ by the adjoint of $\hat{\varphi}_n : \mathbb{C}P^\infty \to \Omega\Sigma\mathbb{C}P^\infty$ for $n = 1, 2, \cdots$. On the other hand, one can find a self-map $\psi^q : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ which corresponds to $q \in \mathbb{Z} \cong H^2(\mathbb{C}P^\infty; \mathbb{Z}) \cong [\mathbb{C}P^\infty, \mathbb{C}P^\infty]$. The suspension of this map gives a self-map $\Sigma\psi^q$ of $\Sigma\mathbb{C}P^\infty$. McGibbon [8, Theorem 1] showed that every self-map of $\Sigma\mathbb{C}P^\infty$ is a linear combination of these maps, $\Sigma\psi^1, \Sigma\psi^2, \cdots, \Sigma\psi^n$, up to homology. Indeed, we can construct self-maps of $\Sigma\mathbb{C}P^\infty$ by using the suspension structure, up to homology, as follows: $\varphi_1 = \Sigma\psi^1, \varphi_2 = \Sigma\psi^2 - 2\Sigma\psi^1, \varphi_3 = \Sigma\psi^3 - 3\Sigma\psi^2 + 3\Sigma\psi^1$, and in general $\varphi_n = \sum_{r=0}^{n-1}(-1)^r\binom{n}{r}\Sigma\psi^{n-r}$, where $\binom{n}{r}$ is a binomial coefficient. We note that these self-maps are exactly the same as the above self-maps (see [13, Theorem 1.7]) and x_n is the rationally non-trivial indecomposable generator of $\pi_{2n}(\Omega\Sigma\mathbb{C}P^\infty)$ for each n. By Theorem 1.4, *ibid.*, we have

Theorem 2. *$\{\hat{\varphi}_n\}$ and $\{x_n\}$ have the following properties:*

(1) *$\hat{\varphi}_n : \mathbb{C}P^\infty \to \Omega\Sigma\mathbb{C}P^\infty$ factors as $\mathbb{C}P^\infty \xrightarrow{p} \mathbb{C}P^\infty/\mathbb{C}P^{n-1} \xrightarrow{g_n} \Omega\Sigma\mathbb{C}P^\infty$ (where p is the projection) such that the restriction to the bottom sphere of the map g_n coincides with the map $x_n : S^{2n} \to \Omega\Sigma\mathbb{C}P^\infty$.*

(2) *Let $[\hat{\varphi}_m, \hat{\varphi}_n]$ be a commutator in the group $[\mathbb{C}P^\infty, \Omega\Sigma\mathbb{C}P^\infty]$. Then $i^*[\hat{\varphi}_m, \hat{\varphi}_n] = q^* < x_m, x_n >$. Here $i : \mathbb{C}P^{m+n} \to \mathbb{C}P^\infty$ is the inclusion, $q : \mathbb{C}P^{m+n} \to S^{2m+2n}$ is the projection and $< x_m, x_n >$ is the Samelson product in $\pi_*(\Omega\Sigma\mathbb{C}P^\infty)$.*

Similarly, let $[\varphi_m, \varphi_n]$ be the commutator of φ_m and φ_n in the group $[\Sigma\mathbb{C}P^\infty, \Sigma\mathbb{C}P^\infty]$. That is, $[\varphi_m, \varphi_n] = \varphi_m + \varphi_n - \varphi_m - \varphi_n$, where $+$ and $-$ mean the operations of maps induced by the suspension structure on $\Sigma\mathbb{C}P^\infty$. We note that the restriction $[\varphi_m, \varphi_n]|_{(\Sigma\mathbb{C}P^\infty)_{2m+2n}}$ to the skeleton is inessential (see [7]).

Remark 3. In order to produce enough self-maps in $\mathrm{Aut}(\Sigma\mathbb{C}P^\infty)$ so that the map $\mathrm{Aut}(\Sigma\mathbb{C}P^\infty) \to \mathrm{Aut}(\pi_{\leq n}(\Sigma\mathbb{C}P^\infty))$ has a finite cokernel for each n, we now construct the self-homotopy equivalences of $\Sigma\mathbb{C}P^\infty$ by the form $I + [\varphi_{i_n}, [\varphi_{i_{n-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]]$, where I is the identity map on $\Sigma\mathbb{C}P^\infty$ and $[\varphi_{i_n}, [\varphi_{i_{n-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]]$ is the iterated commutator of self-maps φ_{i_j}.

We recall that the Samelson product gives $\pi_*(\Omega X), * \geq 1$, the structure of graded Lie algebra ([14], and [3], p. 141); that is, $< x, y >= -(-1)^{|x||y|} < y, x >$, and $< x, < y, z >>=<< x, y >, z > +(-1)^{|x||y|} < y, < x, z >>$. For another notion in order to understand the given conjecture, we require a steady calculation in $\pi_*(\Omega\Sigma\mathbb{C}P^\infty)$ (or $\pi_*(\Sigma\mathbb{C}P^\infty)$). Let L and $L_{\leq n}$ denote the Lie algebras defined by the Samelson products on the quotients $\pi_*(\Omega\Sigma\mathbb{C}P^\infty)/\text{torsion}$ and $\pi_{\leq 2n}(\Omega\Sigma\mathbb{C}P^\infty)/\text{torsion}$, respectively. That is,

$$L = \pi_*(\Omega\Sigma\mathbb{C}P^\infty)/\text{torsion } (= L < x_1, x_2, \cdots, x_n, \cdots >, \ \dim x_n = 2n)$$

and

$$L_{\leq n} = \pi_{\leq 2n}(\Omega\Sigma\mathbb{C}P^\infty)/\text{torsion} = \pi_*(\Omega\Sigma\mathbb{C}P^\infty)^{(2n)}/\text{torsion},$$

where $(\Omega\Sigma\mathbb{C}P^\infty)^{(2n)}$ means the Postnikov approximation and $x_n : S^{2n} \to \Omega\Sigma\mathbb{C}P^\infty$ is the rationally non-trivial indecomposable generator in $\pi_{2n}(\Omega\Sigma\mathbb{C}P^\infty), n = 1, 2, 3, \cdots$, as mentioned earlier. Equivalently, under the same letter,

$$L = \pi_*(\Sigma\mathbb{C}P^\infty)/\text{torsion } (= L < x_1, x_2, \cdots, x_n, \cdots >),$$

where x_n is also the rationally non-trivial indecomposable generator in $\pi_{2n+1}(\Sigma\mathbb{C}P^\infty)$, and the operation in this case is the Whitehead product on $\pi_*(\Sigma\mathbb{C}P^\infty)/\text{torsion}$. We note that the torsion generators can be ignored in the SNT-computations. Throughout this paper we write the rationally non-trivial indecomposable generators in $\pi_{\leq 2n}(\Omega\Sigma\mathbb{C}P^\infty)/\text{torsion}$ by the same letter $(x_1, x_2, \cdots, x_n, \cdots)$ in $\pi_{\leq 2n+1}(\Sigma\mathbb{C}P^\infty)/\text{torsion}$ for each n.

We recall that $\tilde{H}_*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}\{b_1, b_2, \cdots, b_n, \cdots\}$ as a $\mathbb{Z}$-module, where $b_n \in H_{2n}(\mathbb{C}P^\infty; \mathbb{Z})$ is the standard generator. The rational homology of $\Omega\Sigma\mathbb{C}P^\infty$ is a tensor algebra $T < b_1, b_2, \cdots, b_n, \cdots >$ generated by $\{b_1, b_2, \cdots, b_n, \cdots\}$, where b_n is a generator of $H_{2n}(\Omega\Sigma\mathbb{C}P^\infty; \mathbb{Q})$ with diagonal $\Delta(b_k) = \sum_{i+j=k} b_i \otimes b_j$ and $b_n = E_*(b_n)$. Here, E is the inclusion map $E : \mathbb{C}P^\infty \to \Omega\Sigma\mathbb{C}P^\infty$ defined by $E(x)(t) = (x, t) \in \Sigma\mathbb{C}P^\infty$, and b_0 means $1 \in H_0(\Omega\Sigma\mathbb{C}P^\infty)$.

Let $h : \pi_*(\Omega\Sigma\mathbb{C}P^\infty) \to H_*(\Omega\Sigma\mathbb{C}P^\infty)$ be the Hurewicz homomorphism and let $\chi : \Omega\Sigma\mathbb{C}P^\infty \to \Omega\Sigma\mathbb{C}P^\infty$ be the map of loop inverse. Then, in [13] we have

$$h(x_n) = \begin{cases} b_1 & \text{if } n = 1, \\ (n-1)! \sum_{i=1}^n \chi_*(b_{n-i})(ib_i - b_1 b_{i-1}) = n! b_n + \text{decomposables} & \text{if } n \geq 2, \end{cases}$$

where the product in the above equations is the one in the tensor algebra.

3. Proof of the conjecture

From now on, let P denote the infinite complex projective space $\mathbb{C}P^\infty$ for brevity.

Proof. Let $A_n = \mathrm{Aut}(\pi_{\leq 2n+1}(\Sigma P)/\mathrm{torsion})$. Then we can verify that $A_1 = \mathrm{Aut}(\mathbb{Z}) \cong \{\pm 1\} \cong \mathbb{Z}_2$, and $A_2 \cong \{\pm 1\} \oplus \{\pm 1\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and that $A_n (n \geq 3)$ has a different group structure which is infinite (see [7]). Note that if $\gamma_0 \in \pi_m(\Omega X)$ and $\gamma_1 \in \pi_n(\Omega X)$, then $[h(\gamma_0), h(\gamma_1)] = h(< \gamma_0, \gamma_1 >)$. Let $[\hat{\varphi}_m, \hat{\varphi}_n] = \widehat{[\varphi_m, \varphi_n]} : P \to \Omega\Sigma P$ be the adjoint of $[\varphi_m, \varphi_n] : \Sigma P \to \Sigma P$. Then we have

Proposition 3.1. *$[\hat{\varphi_m}, \hat{\varphi_n}]_*(b_{m+n}) = [h(x_m), h(x_n)]$, where b_{m+n} is a generator of $H_{2m+2n}(P)$, $[h(x_m), h(x_n)] = h(x_m)h(x_n) - (-1)^{|h(x_m)||h(x_n)|} h(x_n)h(x_m)$, and x_m and x_n are rationally non-trivial indecomposable generators.*

In the particular case, we are able to see that $[\hat{\varphi_1}, \hat{\varphi_2}]_*(b_3) = 2(b_1 b_2 - b_2 b_1) = [b_1, 2b_2 - b_1^2] = [h(x_1), h(x_2)]$. The following is the general case:

Proof. Let $q : \mathbb{C}P^{m+n} \to S^{2m+2n}$ be the projection map and let $< , >$ be the Samelson product in $\pi_*(\Omega\Sigma P)$. Theorem 2 says that the following diagram is commutative:

$$\begin{array}{ccccc} \mathbb{C}P^{m+n} & \xrightarrow{i} & P & \xrightarrow{[\hat{\varphi_m}, \hat{\varphi_n}]} & \Omega\Sigma P \\ \big\downarrow q & & & & \big\downarrow = \\ S^{2m+2n} & \xrightarrow{=} & S^{2m+2n} & \xrightarrow{s} & \Omega\Sigma P \end{array}$$

where $s =< x_m, x_n >$. Considering $\mathbb{C}P^{m+n-1} \hookrightarrow \mathbb{C}P^{m+n} \xrightarrow{q} \mathbb{C}P^{m+n}/\mathbb{C}P^{m+n-1} = S^{2m+2n}$, and applying the homology to the above diagram, we have

$$\begin{aligned} [\hat{\varphi_m}, \hat{\varphi_n}]_*(b_{m+n}) &= [\hat{\varphi_m}, \hat{\varphi_n}]_* i_*(b_{m+n}) = s_* q_*(b_{m+n}) \\ &= s_*(b'_{m+n}) = h(< x_m, x_n >) \text{ (Hurewicz homomorphism)} \\ &= [h(x_m), h(x_n)]. \end{aligned}$$

Here, b_{m+n} is also used as a generator of $H_{2m+2n}(\mathbb{C}P^{m+n}) (\cong H_{2m+2n}(P), i_*(b_{m+n}) = b_{m+n})$, $q_* : H_{2m+2n}(\mathbb{C}P^{m+n}) \to H_{2m+2n}(S^{2m+2n})$ is an isomorphism (by using a homology sequence) sending the generator b_{m+n} to the fundamental homology class b'_{m+n}, and x_m and x_n are rationally non-trivial indecomposable generators. □

What will happen in the homomorphism induced by the self-maps $I + [\varphi_m, \varphi_n]$ in the homotopy groups? The answer to this query can be found as follows:

Lemma 3.2. *Let $p = 2m+2n+1$, and let x_{m+n} be the indecomposable generator in $\pi_p(\Sigma P)/torsion$. Then $(I + [\varphi_m, \varphi_n])_\sharp(x_{m+n}) = x_{m+n} + [\varphi_m, \varphi_n]_\sharp(x_{m+n})$, where the first $+$ is the addition of maps induced by the suspension structure on ΣP, while the second $+$ refers to the one of homotopy groups, and $f_\sharp$ denotes an induced homomorphism in homotopy groups.*

Proof. We must show that the diagram

$$\begin{tikzcd} S^p \arrow[r, "x_{m+n}"] \arrow[d, "\nabla"'] & \Sigma P \arrow[r, "\nabla"] & \Sigma P \vee \Sigma P \arrow[d, "{(I,[\varphi_m,\varphi_n])}"] \\ S^p \vee S^p \arrow[r, "x_{m+n} \vee x_{m+n}"] & \Sigma P \vee \Sigma P \arrow[r, "{(I,[\varphi_m,\varphi_n])}"] & \Sigma P \end{tikzcd}$$

is commutative. Consider the following diagram:

$$\begin{tikzcd} S^p \arrow[r, "x_{m+n}"] \arrow[d, "\nabla"'] & \Sigma P \arrow[d, "\nabla"'] \\ S^p \vee S^p \arrow[r, "x_{m+n} \vee x_{m+n}"] & \Sigma P \vee \Sigma P \end{tikzcd}$$

Note that the diagram is not commutative, because x_{m+n} is not a suspension. However, if we consider a quotient map $q : \Sigma P \to \Sigma P/(\Sigma P)_{p-1}$, then the following two diagrams are commutative:

$$\begin{tikzcd} \Sigma P \arrow[r, "{[\varphi_m,\varphi_n]}"] \arrow[d, "q"'] & \Sigma P \arrow[d, "="'] \\ \Sigma P/(\Sigma P)_{p-1} \arrow[r, "{[\varphi_m,\varphi_n]'}"] & \Sigma P \end{tikzcd}$$

and

$$\begin{tikzcd} S^p \arrow[r, "x_{m+n}"] \arrow[d, "\Delta"'] & \Sigma P \arrow[d, "\Delta'"'] \\ S^p \times S^p \arrow[r, "x_{m+n} \times x'_{m+n}"] & \Sigma P \times \Sigma P/(\Sigma P)_{p-1} \end{tikzcd}$$

where $[\varphi_m, \varphi_n]'$ is a map induced by the commutator $[\varphi_m, \varphi_n]$ making the above diagram commutative (this map could be guaranteed since the restriction $[\varphi_m, \varphi_n]|_{(\Sigma P)_{2m+2n}}$ to the skeleton is inessential), and $\Delta(x) = (x, x)$, $\Delta'(x) = (x, q(x))$ and $x'_{m+n} = q(x_{m+n})$. Also note that since $\Sigma P/(\Sigma P)_{p-1}$ is $p-1$ connected,

$$[S^p, \Sigma P \vee \Sigma P/(\Sigma P)_{p-1}] \cong [S^p, \Sigma P \times \Sigma P/(\Sigma P)_{p-1}].$$

This isomorphism takes $(x_{m+n} \vee x'_{m+n}) \circ \nabla \mapsto (x_{m+n} \times x'_{m+n}) \circ \Delta$, and it also takes $\Delta'(x_{m+n})$ to $\nabla'(x_{m+n})$. Therefore the diagram

$$\begin{tikzcd} S^p \arrow[r, "x_{m+n}"] \arrow[d, "\nabla"'] & \Sigma P \arrow[d, "\nabla'"'] & \\ S^p \vee S^p \arrow[r, "x_{m+n} \vee x'_{m+n}"] & \Sigma P \vee \Sigma P/(\Sigma P)_{p-1} \arrow[r, "{(I,[\varphi_m,\varphi_n]')}"] & \Sigma P \end{tikzcd}$$

commutes. Thus we complete the proof. □

It is not difficult to show that the above lemma still holds for the iterated commutators from the fact that $[\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]]|_{(\Sigma P)_{2(i_1+i_2+\cdots+i_k)}} \simeq *$. Moreover, it can be seen that there are many rationally non-trivial indecomposable and decomposable generators on $\pi_{2n+1}(\Sigma P) \otimes \mathbb{Q}$ (or $\pi_{2n}(\Omega\Sigma P) \otimes \mathbb{Q}$). For example, $\{x_1\}$ in dimension 3, $\{x_2\}$ in dimension 5, $\{x_3, [x_1, x_2]\}$ in dimension 7, $\{x_4, [x_1, x_3], [x_1, [x_1, x_2]]\}$ in dimension 9, $\{x_5, [x_1, x_4], [x_1, [x_1, x_3]], [x_1, [x_1[x_1, x_2]]], [x_2, x_3], [x_2, [x_1, x_2]]\}$ in dimension 11, and so on. Indeed, we can show that the above (iterated) Whitehead products are rationally non-trivial by using the cohomology cup products.

Lemma 3.3. *For each Whitehead product* $[x_{i_k}, [x_{i_{k-1}}, \cdots, [x_{i_1}, x_{i_2}], \cdots]]$ *in* $\pi_*(\Sigma P)$, *there exists an iterated commutator* $[\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]]$ *in the group* $[\Sigma P, \Sigma P]$ *such that*

$$(I + [\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]])_\sharp(x_n) = x_n + n![x_{i_k}, [x_{i_{k-1}}, \cdots, [x_{i_1}, x_{i_2}], \cdots]],$$

where x_n *and* x_{i_j} *are rationally non-trivial indecomposable generators and* $n = i_1 + i_2 + \cdots + i_k$.

Remark 3.4. We note that the phenomena of the iterated commutators *completely* depend on those of iterated Whitehead products. What is even more interesting is that the map $[\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]]_\sharp$ sends the indecomposable generators to the (iterated) Whitehead products which are rationally non-trivial decomposable. This fact is just what we need in the proof of the conjecture!

Proof of Lemma 3.3. In the case of the one-fold Whitehead product, we need to show that $[\varphi_{i_1}, \varphi_{i_2}]_\sharp(x_n) = n![x_{i_1}, x_{i_2}]$ $(n = i_1 + i_2)$. The proof follows by chasing the diagram in Proposition 3.1 (by adjointness) and the following commutative diagram:

$$\begin{array}{ccccccc}
S^p & \xrightarrow{x_{i_1+i_2}} & (\Sigma P)_p & \longrightarrow & \Sigma P & \xrightarrow{[\varphi_{i_1}, \varphi_{i_2}]} & \Sigma P \\
{\scriptstyle =}\big\downarrow & & {\scriptstyle q}\big\downarrow & & {\scriptstyle q}\big\downarrow & & {\scriptstyle =}\big\downarrow \\
S^p & \xrightarrow{(i_1+i_2)!} & S^p & \longrightarrow & \Sigma P/(\Sigma P)_{p-1} & \xrightarrow{[\varphi_{i_1}, \varphi_{i_2}]'} & \Sigma P
\end{array}$$

where $p = 2n+1, n = i_1 + i_2$, and the q's are the projections.

We suppose that for the iterated Whitehead products $[x_{i_{k-1}}, \cdots, [x_{i_1}, x_{i_2}], \cdots]$ of length greater than 2, there exist iterated commutators $[\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]$ such that

$$(I + [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots])_\sharp(x_m) = x_m + m![x_{i_{k-1}}, \cdots, [x_{i_1}, x_{i_2}], \cdots],$$

where x_m and x_{i_j} are rationally non-trivial indecomposable generators, and $m = i_1 + i_2 + \cdots + i_{k-1}$. We recall that the restriction $[\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]|_{(\Sigma P)_{2m}}$ to the skeleton is trivial in homotopy and the map $[x_{i_{k-1}}, \cdots, [x_{i_1}, x_{i_2}], \cdots] : S^{2m+1} \to \Sigma P$ is a rationally non-trivial decomposable generator (when rationalized). By applying the first statement and the above results, we can find an iterated commutator $[\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, x_{i_2}], \cdots]]$ such that the desired formula is still guaranteed. $\square$

As previously described, L and $L_{\leq n}$ denote the Lie algebras defined by the Whitehead products on $\pi_*(\Sigma P)/$ torsion and $\pi_{\leq 2n+1}(\Sigma P)/$ torsion (or equivalently the Samelson products on the quotients $\pi_*(\Omega\Sigma P)/$ torsion and $\pi_{\leq 2n}(\Omega\Sigma P)/$ torsion) respectively. We note that ΣP has the rational homotopy type of the bouquet of spheres, i.e., $\Sigma P \simeq_0 S^3 \vee S^5 \vee S^7 \vee \cdots$, where $\simeq_0$ means a rational homotopy. Following McGibbon and Møller [11], we can construct a short exact sequence

$$0 \to \mathrm{Hom}(Q_n L, D_n L) \to \mathrm{Aut}(L_{\leq n}) \to \mathrm{Aut}(L_{<n}) \oplus \mathrm{Aut}(Q_n L) \to 0.$$

Here, $Q_n L$ and $D_n L$ are indecomposables and decomposables respectively, and the maps are given in the same method (see below). Since $\mathrm{Aut}(T)$ is finite, where T is torsion in $\pi_{\leq 2n+1}(\Sigma P)$, in order to prove the given conjecture it suffices to show that the map $\mathrm{Aut}(\Sigma P) \to \mathrm{Aut}(L_{\leq n})$ has a finite cokernel for each n. Using the previous results of this paper, we prove the conjecture by the induction step on n. Since $A_1 = \mathrm{Aut}(L_{\leq 1}) \cong \mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$, the initial step is trivial. Suppose that the map $\mathrm{Aut}(\Sigma P) \to \mathrm{Aut}(L_{<n})$ has a finite cokernel. Let $\mathcal{D}_n$ be the set of (iterated) Whitehead products in $\pi_{2n+1}(\Sigma P)/$ torsion which are, after rationalized, decomposable generators in $\pi_{2n+1}(\Sigma P) \otimes \mathbb{Q}$ for each $n \geq 3$ (say, $\mathcal{D}_3 = \{[x_1, x_2]\}$, $\mathcal{D}_4 = \{[x_1, x_3], [x_1, [x_1, x_2]]\}$, $\mathcal{D}_5 = \{[x_1, x_4], [x_1, [x_1, x_3]], [x_1, [x_1[x_1, x_2]]], [x_2, x_3], [x_2, [x_1, x_2]]\}$, and so on). Then, for each (iterated) Whitehead product $[x_{i_k}, [x_{i_{k-1}}, \cdots, [x_{i_1}, x_{i_2}], \cdots]] \in \mathcal{D}_n$ $(n = i_1 + i_2 + \cdots + i_k)$, by Lemmas 3.2 and 3.3, we can produce a self-map $I + [\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]] \in \mathrm{Aut}(\Sigma P)$ such that the restriction $(I + [\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]])_\sharp|_{L_{<n}}$ is the identity, and

$$(I + [\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]])_\sharp(x_n) = x_n + n![x_{i_k}, [x_{i_{k-1}}, \cdots, [x_{i_1}, x_{i_2}], \cdots]],$$

where x_n is a rationally non-trivial indecomposable generator in $\pi_{2n+1}(\Sigma P)/$torsion. We now consider the diagram

$$\begin{array}{ccccc} & & \mathrm{Aut}(\Sigma P) & \longrightarrow & \mathrm{Aut}(L_{<n}) \\ & & \downarrow & & \uparrow \\ \mathrm{Hom}(Q_nL, D_nL) & \longrightarrow & \mathrm{Aut}(L_{\leq n}) & \longrightarrow & \mathrm{Aut}(L_{<n}) \oplus \mathrm{Aut}(Q_nL) \end{array}$$

and the composition

$$L_{\leq n} \xrightarrow{q} Q_nL \xrightarrow{[\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]]_\sharp} D_nL \xrightarrow{j} L_{\leq n}$$

of maps. Here,

(1) q is the projection and j is the inclusion;
(2) the map $\mathrm{Hom}(Q_nL, D_nL) \to \mathrm{Aut}(L_{\leq n})$ sends $[\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]]_\sharp$ to $I + j \circ [\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]]_\sharp \circ q$, and the map out of $\mathrm{Aut}(L_{\leq n})$ is given by restriction and projection; and
(3) the right vertical arrow is the projection.

Note that the (iterated) commutators $[\varphi_{i_k}, [\varphi_{i_{k-1}}, \cdots, [\varphi_{i_1}, \varphi_{i_2}], \cdots]]$ completely depend on the types of (iterated) Whitehead products $[x_{i_k}, [x_{i_{k-1}}, \cdots, [x_{i_1}, x_{i_2}], \cdots]]$ as in Lemma 3.3. We also note that the map $\mathrm{Aut}(L_{\leq n}) \to \mathrm{Aut}(L_{<n}) \oplus \mathrm{Aut}(Q_nL)$ is an epimorphism and that $\mathrm{Aut}(Q_nL) \cong \mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$ because Q_nL has the only one indecomposable generator, up to sign, for each n. Those facts and the induction hypothesis force the map $\mathrm{Aut}(\Sigma P) \to \mathrm{Aut}(L_{\leq n})$ to have a finite cokernel as required. □

Acknowledgements

I would like to express my gratitude to Professor C. A. McGibbon for his encouragement and fruitful discussion on this paper while I was staying at Wayne State University. The author is also grateful to the referee(s) for offering other suggestions that improved the quality of the paper.

References

1. J. F. Adams, *An example in homotopy theory*, Proc. Camb. Phil. Soc. **53** (1957), 922-923. MR0091477 (19:975d)
2. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math. **304**, Springer-Verlag, Berlin-New York, 1972. MR0365573 (51:1825)

3. F. R. Cohen, J. C. Moore and J. A. Neisendorfer, *Torsion in homotopy groups*, Ann. of Math. (2) **109** (1979), 121-168. MR519355 (80e:55024)
4. B. I. Gray, *Spaces of the same n-type, for all n*, Topology **5** (1966), 241-243. MR0196743 (33:4929)
5. J. R. Harper and J. Roitberg, *Phantom maps and spaces of the same n-type for all n*, J. Pure Appl. Algebra **80** (1992), 123-137. MR1172722 (93g:55010)
6. P. Hilton, G. Mislin and J. Roitberg, *Homotopical localization*, Proc. London Math. Soc. **26** (1973), 693-706. MR0326720 (48:5063)
7. D. Lee, *On self-homotopy equivalences of* $\Sigma\mathbb{C}P^\infty$, preprint.
8. C. A. McGibbon, *Self-maps of projective spaces*, Trans. Amer. Math. Soc. **271** (1982), 325-346. MR648096 (83h:55007)
9. C. A. McGibbon and J. M. Møller, *How can you tell two spaces apart when they have the same n-type for all n?*, Adams Memorial Symposium on Algebraic Topology, N. Ray and G. Walker, eds., London Math. Soc. Lecture Note Series **175**, Cambridge Univ. Press, Cambridge, 1992, 131-143. MR1170575 (93i:55010)
10. C. A. McGibbon and J. M. Møller, *On spaces with the same n-type for all n*, Topology **31** (1992), 177-201. MR1153244 (92m:55008)
11. C. A. McGibbon and J. M. Møller, *On infinite dimensional spaces that are rationally equivalent to a bouquet of spheres*, Proceedings of the 1990 Barcelona Conference on Algebraic Topology, Lecture Notes in Math. **1509**, Springer, Berlin, 1992, 285-293. MR1185978 (93h:55012)
12. C. A. McGibbon and J. Roitberg, *Phantom maps and rational equivalences*, Amer. J. Math. **116** (1994), 1365-1379. MR1305869 (95j:55026)
13. K. Morisugi, *Projective elements in K-theory and self-maps of* $\Sigma\mathbb{C}P^\infty$, J. Math. Kyoto Univ. **38** (1998), 151-165. MR1628087 (99g:55008)
14. G. W. Whitehead, *Elements of homotopy theory*, GTM 61, Springer-Verlag, New York-Heidelberg-Berlin, 1978. MR516508 (80b:55001)
15. C. W. Wilkerson, *Classification of spaces of the same n-type for all n*, Proc. Amer. Math. Soc. **60** (1976), 279-285. MR0474283 (57:13930)

Department of Mathematics, and Institute of Pure and Applied Mathematics, Chonbuk National University, Jeonju, Jeonbuk 561-756, Republic of Korea

E-mail address: dwlee@math.chonbuk.ac.kr