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INCLUSIONS AND COINCIDENCES FOR MULTIPLE SUMMING MULTILINEAR MAPPINGS

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Abstract. Using complex interpolation we prove new inclusion and coincidence theorems for multiple (fully) summing multilinear and holomorphic mappings. Among several other results we show that continuous n-linear forms on cotype 2 spaces are multiple $(2; q_k, ..., q_k)$-summing, where $2^{k-1} < n \leq 2^k$, $q_0 = 2$ and $q_{k+1} = \frac{2q_k}{1+q_k}$ for $k \geq 0$.

1. Introduction and notation

The essence of the theory of absolutely summing linear operators can be traced back to Grothendieck's celebrated Resumé [14] and further fundamental works by Pietsch [32] and Lindenstrauss and Pełczyński [19]. For the linear theory of absolutely summing operators the reader is referred to the excellent monograph [11]. In 1983 Pietsch [33] sketched an n-linear approach to the theory of absolutely summing operators and since then a vast number of papers has followed this line (e.g., [1, 5, 6, 8, 10, 12, 13, 15, 18, 21, 22, 27, 28, 30, 26, 29, 31, 35]). In this direction, multiple summing (also called fully summing) multilinear mappings were introduced by Matos [21] and, independently, by Bombal, Pérez-García and Villanueva [5]. This class has proved to be one of the most useful and fruitful multilinear generalizations of the concept of an absolutely summing linear operator. It is worth mentioning that the bilinear case was first treated in 1985 by Ramanujan and Schock [34]. The case of holomorphic mappings is treated in [27]. For the theory of multiple (fully) summing n-linear mappings we refer to [5, 21, 28].

In the following, $\mathbb{N}$ denotes the set of all positive integers, $E, E_1, \ldots, E_n, F$ denote Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By E' we mean the topological dual of E and $B_{E'}$ represents its closed unit ball.

Given $n \in \mathbb{N}$, the space of all continuous n-linear mappings from $E_1 \times \cdots \times E_n$ to F endowed with the sup norm is denoted by $\mathcal{L}(E_1, \ldots, E_n; F)$ ($\mathcal{L}(^nE; F)$ if $E = E_1 = \cdots = E_n$ and $\mathcal{L}(E; F)$ if $n = 1$). The space of all continuous n-homogeneous polynomials with the sup norm will be represented by $\mathcal{P}(^nE; F)$. For $p \geq 1$, the vector space of all sequences $(x_j)_{j=1}^\infty$ in E such that $\|(x_j)_{j=1}^\infty\|_p = (\sum_{j=1}^\infty \|x_j\|^p)^{\frac{1}{p}} < \infty$ is denoted by $\ell_p(E)$. We represent by $\ell_p^w(E)$ the linear space of the sequences

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$(x_j)_{j=1}^{\infty}$ in E such that $(\varphi(x_j))_{j=1}^{\infty} \in \ell_p$ for every $\varphi \in E'$. The expression

$$\|(x_j)_{j=1}^{\infty}\|_{w,p} := \sup_{\varphi \in B_{E'}} \|(\varphi(x_j))_{j=1}^{\infty}\|_p$$

defines a norm on $\ell_p^w(E)$. For the corresponding m-dimensional spaces we write ℓ_p^m and $\ell_{p,w}^m$ instead of ℓ_p and ℓ_p^w respectively.

Given $1 \leq q_j \leq p$, $j = 1, \ldots, n$, an n-linear mapping $T\colon E_1 \times \cdots \times E_n \to F$ is *multiple (or fully)* $(p; q_1, \ldots, q_n)$*-summing* if there exists $C > 0$ such that

$$\left(\sum_{j_1,\ldots,j_n=1}^{m} \|T(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)})\|^p \right)^{1/p} \leq C \prod_{k=1}^{n} \|(x_j^{(k)})_{j=1}^{m}\|_{w,q_k}$$

for every $m \in \mathbb{N}$ and every $x_j^{(k)} \in E_k$, $j = 1, ..., m$ and $k = 1, ..., n$. The space composed by all multiple $(p; q_1, .., q_n)$-summing n-linear mappings from $E_1 \times \cdots \times E_n$ into F is denoted by $\mathcal{L}_{\mathrm{ms}(p;q_1,\ldots,q_n)}(E_1, \ldots, E_n; F)$, and the infimum of the constants C for which the inequality always holds defines a norm $\|\cdot\|_{\mathrm{ms}(p;q_1,\ldots,q_n)}$ on $\mathcal{L}_{\mathrm{ms}(p;q_1,\ldots,q_n)}(E_1, \ldots, E_n; F)$. If $q_1 = \cdots = q_n = q$, we sometimes write $ms(p; q)$ instead of $ms(p; q, ..., q)$ and if $p = q = q_1 = \cdots = q_n$ we simply write ms, p instead of $ms(p; p)$.

An important result due to Bohnenblust and Hille [4] asserts that for each positive integer n, there is a constant c_n so that

$$\left(\sum_{j_1,\ldots,j_n=1}^{\infty} |A(e_{j_1}, ..., e_{j_n})|^{\frac{2n}{n+1}} \right)^{\frac{n+1}{2n}} \leq c_n \|A\| \tag{1.1}$$

for all $A \in \mathcal{L}(^n c_0; \mathbb{K})$. With a simple reformulation of (1.1) one can obtain the "coincidence result"

$$\mathcal{L}(E_1, \ldots, E_n; \mathbb{K}) = \mathcal{L}_{\mathrm{ms}(\frac{2n}{n+1};1,\ldots,1)}(E_1, \ldots, E_n; \mathbb{K}) \tag{1.2}$$

for every $n \geq 2$ and every Banach spaces $E_1, \ldots, E_n$ (this result appears in [28]). As $\frac{2n}{n+1} \longrightarrow 2$, it is natural to wonder if multilinear forms are multiple $(2; q, \ldots, q)$-summing for some $q > 1$. Surprisingly enough we will show that this is true for n-linear forms on cotype 2 spaces but with a q depending on n. More precisely, in Section 2 we will show that

$$\mathcal{L}(E_1, \ldots, E_n; \mathbb{K}) = \mathcal{L}_{\mathrm{ms}(2;q_k,\ldots,q_k)}(E_1, \ldots, E_n; \mathbb{K}),$$

whenever $E_1, \ldots, E_n$ have cotype 2, $2^{k-1} < n \leq 2^k$, $q_0 = 2$ and $q_{k+1} = \frac{2q_k}{1+q_k}$ for $k \geq 0$. Using interpolation techniques, intermediate results are also obtained: if $\theta \in [0, 1]$, then

$$\mathcal{L}(E_1, \ldots, E_n; \mathbb{K}) = \mathcal{L}_{\mathrm{ms}(\frac{2n}{n+\theta}, \frac{q_k}{(q_k-1)\theta+1})}(E_1, \ldots, E_n; \mathbb{K})$$

for $E_1, \ldots, E_n, k, n$ and q_k as above. These results will be firstly proved for complex Banach spaces and the real case will follow by complexification. As far as we know, this interpolation-complexification argument was first applied to multiple summing mappings by Pérez-García [28]. In Section 3 we obtain new inclusions between spaces of multiple summing multilinear, polynomial and holomorphic mappings.

2. Coincidence results for multiple summing forms

Recall that a Banach space E has cotype $q \geq 2$ if there exists a constant $K \geq 0$ such that, no matter how we choose $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in E$,

$$\left(\sum_{j=1}^{k} \|x_j\|^q \right)^{\frac{1}{q}} \leq K \left(\int_0^1 \left\| \sum_{j=1}^{k} r_j(t) x_j \right\|^2 dt \right)^{\frac{1}{2}},$$

where r_j are the Rademacher functions. The infimum of the constants K is denoted by $C_q(E)$.

Since ℓ_2 has cotype 2, a particular case of a result obtained (independently) by Pérez-García [28, Teorema 5.2] and Souza [35, Teorema 1.7.3] gives us the following:

Lemma 2.1. *For any n-tuple $(E_1, \ldots, E_n)$ of Banach spaces we have*

$$\mathcal{L}(E_1, \ldots, E_n; \ell_2) = \mathcal{L}_{\mathrm{ms}(2;1,\ldots,1)}(E_1, \ldots, E_n; \ell_2).$$

Another useful and well-known result that will be useful in the next theorem is the following (the proof is simple, and we omit it):

Lemma 2.2. *If $m \geq 1$, $E_1, \ldots, E_m, F$ are Banach spaces and*

$$\mathcal{L}(E_1, \ldots, E_m; F) = \mathcal{L}_{\mathrm{ms}(p;q)}(E_1, \ldots, E_m; F),$$

then

$$\mathcal{L}(E_1, \ldots, E_n; F) = \mathcal{L}_{\mathrm{ms}(p;q)}(E_1, \ldots, E_n; F),$$

for every $1 \leq n \leq m$.

Henceforth $(q_k)_{k=0}^{\infty}$ will be the sequence of real numbers given by

$$q_0 = 2 \text{ and } q_{k+1} = \frac{2q_k}{1+q_k} \text{ for } k \geq 0.$$

Theorem 2.3. *Let $n \geq 1$ and let $E_1, \ldots, E_n$ be Banach spaces of cotype 2. If k is the natural number such that $2^{k-1} < n \leq 2^k$, then*

$$\mathcal{L}(E_1, \ldots, E_n; \mathbb{K}) = \mathcal{L}_{\mathrm{ms}(2;q_k,\ldots,q_k)}(E_1, \ldots, E_n; \mathbb{K}).$$

Proof. First we prove the complex case $\mathbb{K} = \mathbb{C}$. We can assume that $n = 2^k$ for some natural $k \geq 0$, because otherwise we could choose a natural k such that $n < 2^k$ and extend the n-tuple $(E_1, \ldots, E_n)$ to the 2^k-tuple $(E_1, \ldots, E_n, \mathbb{C}, \ldots, \mathbb{C})$ and use Lemma 2.2. We are going to prove the claim by induction over k. For $k = 0$ there is nothing left to do. Suppose now that the claim is true for $n = 2^k$. Let us consider any $2n$-linear form $T \in \mathcal{L}(E_1, \ldots, E_n, F_1, \ldots, F_n; \mathbb{C})$ with spaces E_i, F_j of cotype 2 ($2^{k+1} = 2n$). In a first step we are going to show that

$$T \in \mathcal{L}_{\mathrm{ms}(2;1,\ldots,1,q_k,\ldots,q_k)}(E_1, \ldots, E_n, F_1, \ldots, F_n; \mathbb{C}). \tag{2.1}$$

For fixed $m \geq 1$ and all $1 \leq r, s \leq n = 2^k$ let any m-tuples

$$(x_{i_r}^{(r)})_{i_r=1}^m \subset E_r \text{ and } (y_{j_s}^{(s)})_{j_s=1}^m \subset F_s$$

be given. For the sake of abbreviation we put

$\mathbf{x_i} = (x_{i_1}^{(1)}, \ldots, x_{i_n}^{(n)})$ for $\mathbf{i} = (i_1, \ldots, i_n)$ and $\mathbf{y_j} = (y_{j_1}^{(1)}, \ldots, y_{j_n}^{(n)})$ for $\mathbf{j} = (j_1, \ldots, j_n)$.

For fixed $\mathbf{y_j}$ we define

$$T_{\mathbf{y_j}} \in \mathcal{L}(E_1, \ldots, E_n; \mathbb{C}) \text{ by } T_{\mathbf{y_j}}(x_1, \ldots, x_n) = T(x_1, \ldots, x_n, \mathbf{y_j}).$$

By the induction assumption we have that $T_{\mathbf{y_j}} \in \mathcal{L}_{\mathrm{ms}(2,q_k,\ldots,q_k)}(E_1,\ldots,E_n;\mathbb{C})$. Now define

$$S \in \mathcal{L}(E_1,\ldots,E_n;\ell_2) \text{ by } S\mathbf{x} = (T(\mathbf{x},\mathbf{y_j})_{\mathbf{j}\in\{1,\ldots,m\}^n}, 0, \ldots) \in \ell_2.$$

Lemma 2.1 gives that $S \in \mathcal{L}_{\mathrm{ms}(2;1,\ldots,1)}(E_1,\ldots,E_n;\ell_2)$, i.e.

$$\left(\sum_{\mathbf{i}} \|S\mathbf{x_i}\|^2\right)^{1/2} \leq c\|S\| \cdot \prod_{r=1}^{n} \|(x_{j_r}^{(r)})_{j_r=1}^m\|_{w,1}. \tag{2.2}$$

Further, from the induction assumption there is c_1 such that $\|T(\mathbf{x},\cdot)\|_{\mathrm{ms}(2;q_k,\ldots,q_k)} \leq c_1\|T(\mathbf{x},\cdot)\|$ for every $\mathbf{x} \in E_1 \times \cdots \times E_n$. So,

$$\begin{aligned}\|S\| &= \sup_{\mathbf{x}\in B_{E_1}\times\cdots\times B_{E_n}} \|S\mathbf{x}\|_2 = \sup_{\mathbf{x}\in B_{E_1}\times\cdots\times B_{E_n}} \left(\sum_{\mathbf{j}} |T(\mathbf{x},\mathbf{y_j})|^2\right)^{1/2} \\ &\leq c_1 \sup_{\mathbf{x}\in B_{E_1}\times\cdots\times B_{E_n}} \|T(\mathbf{x},\cdot)\| \prod_{s=1}^{n} \|(y_{j_s}^{(s)})_{j_s=1}^m\|_{w,q_k} \leq c_1\|T\| \prod_{s=1}^{n} \|(y_{j_s}^{(s)})_{j_s=1}^m\|_{w,q_k}.\end{aligned}$$

Plugging this into (2.2) we end up with

$$\begin{aligned}&\left(\sum_{\mathbf{i}}\sum_{\mathbf{j}} |T(\mathbf{x_i},\mathbf{y_j})|^2\right)^{1/2} \\ &\qquad = \left(\sum_{\mathbf{i}} \|S\mathbf{x_i}\|^2\right)^{1/2} \leq c_2 \cdot \prod_{r=1}^{n} \|(x_{j_r}^{(r)})_{j_r=1}^m\|_{w,1} \cdot \prod_{s=1}^{n} \|(y_{j_s}^{(s)})_{j_s=1}^m\|_{w,q_k},\end{aligned}$$

which proves (2.1). By symmetry we also have

$$T \in \mathcal{L}_{\mathrm{ms}(2;q_k,\ldots,q_k,1,\ldots,1)}(E_1,\ldots,E_n,F_1,\ldots,F_n;\mathbb{C}). \tag{2.3}$$

We proceed by complex interpolation. It follows from (2.1) and (2.3) that the $2n$-linear mappings

$$\begin{aligned}\Psi_T^{(0)} &\colon \ell_{1,w}^m(E_1) \times \cdots \times \ell_{1,w}^m(E_n) \times \ell_{q_k,w}^m(F_1) \times \cdots \times \ell_{q_k,w}^m(F_n) \to \ell_2^{m^{2n}}(\mathbb{C}), \\ \Psi_T^{(1)} &\colon \ell_{q_k,w}^m(E_1) \times \cdots \times \ell_{q_k,w}^m(E_n) \times \ell_{1,w}^m(F_1) \times \cdots \times \ell_{1,w}^m(F_n) \to \ell_2^{m^{2n}}(\mathbb{C})\end{aligned}$$

given by

$$\begin{gathered}\left((x_{i_1}^{(1)})_{i_1},\ldots,(x_{i_n}^{(n)})_{i_n},(y_{j_1}^{(1)})_{j_1},\ldots,(y_{j_n}^{(n)})_{j_n}\right) \mapsto \\ \left(T(x_{i_1}^{(1)},\ldots,x_{i_n}^{(n)},y_{j_1}^{(1)},\ldots,y_{j_n}^{(n)})\right)_{i_1,\ldots,i_n,j_1,\ldots,j_n=1}^m\end{gathered}$$

are bounded independently of m by

$$\left\|\Psi_T^{(0)}\right\| \leq \|T\|_{\mathrm{ms}(2;1,\ldots,1,q_k,\ldots,q_k)} := K_0 \text{ and } \left\|\Psi_T^{(1)}\right\| \leq \|T\|_{\mathrm{ms}(2;q_k,\ldots,q_k,1,\ldots,1)} := K_1,$$

respectively. Remember that for any Banach space G and any $1 \leq s < \infty$ there is a natural linear isometry between $\ell_{s,w}^m(G)$ and the injective tensor product $\ell_s^m \otimes_\varepsilon G$. Therefore, $\Psi_T^{(0)}$ and $\Psi_T^{(1)}$ can also be considered as mappings on the Cartesian product of the associated tensor products with the same operator norm. Using complex multilinear interpolation [2, Theorem 4.4.1] for $\theta = 1/2$, we obtain a $2n$-multilinear operator

$$\Psi_T^{(1/2)}\colon \left[\ell_1^m \otimes_\varepsilon E_1, \ell_{q_k}^m \otimes_\varepsilon E_1\right]_{1/2} \times \cdots \times \left[\ell_{q_k}^m \otimes_\varepsilon F_1, \ell_1^m \otimes_\varepsilon F_1\right]_{1/2} \times \cdots \\ \to \left[\ell_2^{m^{2n}}(\mathbb{C}), \ell_2^{m^{2n}}(\mathbb{C})\right]_{1/2}$$

with $\|\Psi_T^{(1/2)}\| \le K_0^{1/2} K_1^{1/2}$. Now the interpolation result due to Defant and Michels [9] for ε-tensor products comes into play. Since ℓ_q is q-concave for $1 \le q \le 2$, we conclude by [9, Theorem, p. 441] (which is an extension of a classical result due to Kouba [17, Theorem 4.2.11]) that

$$[\ell_1^m \otimes_\varepsilon G, \ell_q^m \otimes_\varepsilon G]_{1/2} = [\ell_1^m, \ell_q^m]_{1/2} \otimes_\varepsilon G = \ell_p^m \otimes_\varepsilon G$$

with isomorphism constants not depending on m, provided that G has cotype 2, $1 \le q \le 2$ and $\frac{1}{p} = \frac{1/2}{1} + \frac{1/2}{q}$. So, $\Psi_T^{(1/2)}$ can also be considered as a map

$$\Psi_T^{(1/2)}\colon \ell_{q_{k+1}}^m \otimes_\varepsilon E_1 \times \cdots \times \ell_{q_{k+1}}^m \otimes_\varepsilon F_n \to \ell_2^{m^{2n}}$$

with $\|\Psi_T^{(1/2)}\| \le c_3 \cdot K_0^{1/2} K_1^{1/2}$ for some constant c_3 not depending on m and $\frac{1}{q_{k+1}} = \frac{1/2}{1} + \frac{1/2}{q_k}$, i.e. $q_{k+1} = \frac{2q_k}{q_k+1}$. In terms of T this means that

$$\left(\sum_{\mathbf{i}} \sum_{\mathbf{j}} |T(\mathbf{x_i}, \mathbf{y_j})|^2\right)^{1/2} \le c_4 \cdot \prod_{r=1}^{n} \|(x_{j_r}^{(r)})_{j_r=1}^m\|_{w,q_{k+1}} \cdot \prod_{s=1}^{n} \|(y_{j_s}^{(s)})_{j_s=1}^m\|_{w,q_{k+1}}$$

with some constant c_4 not depending on m, and so the complex case is done. To prove the real case we proceed by complexification. Given real Banach spaces $E_1, \ldots, E_n$ of cotype 2 and $T \in \mathcal{L}(E_1, \ldots, E_n; \mathbb{R})$, by $\widetilde{E_1}, \ldots, \widetilde{E_n}$ we mean their respective complexifications (see [23, 24]) and by $\tilde{T} \in \mathcal{L}(\widetilde{E_1}, \ldots, \widetilde{E_n}; \mathbb{C})$ the extension of T according to [3, Theorem 3]. By [28, Proposición 4.30(ii)] we know that $\widetilde{E_1}, \ldots, \widetilde{E_n}$ have cotype 2, so the first part of the proof yields that $\tilde{T}$ is multiple $(2; q_k, \ldots, q_k)$-summing. It follows from (an easy adaptation of) [28, Proposición 4.30(i)] that T is multiple $(2; q_k, \ldots, q_k)$-summing as well. □

Now we obtain a scale of coincidences from (1.2) to Theorem 2.3:

Theorem 2.4. *Let $n \ge 1$ and let $E_1, \ldots, E_n$ be Banach spaces of cotype 2. If k is the natural number such that $2^{k-1} < n \le 2^k$, then*

$$\mathcal{L}(E_1, \ldots, E_n; \mathbb{K}) = \mathcal{L}_{\mathrm{ms}(\frac{2n}{n+\theta}; \frac{q_k}{(q_k-1)\theta+1})}(E_1, \ldots, E_n; \mathbb{K})$$

for every $\theta \in [0, 1]$.

Proof. By (1.2) and Theorem 2.3 we know that

$$\mathcal{L}(E_1, \ldots, E_n; \mathbb{C}) = \mathcal{L}_{\mathrm{ms}(\frac{2n}{n+1}; 1, 1, \ldots, 1)}(E_1, \ldots, E_n; \mathbb{C}) \text{ and}$$
$$\mathcal{L}(E_1, \ldots, E_n; \mathbb{C}) = \mathcal{L}_{\mathrm{ms}(2; q_k, \ldots, q_k)}(E_1, \ldots, E_n; \mathbb{C}).$$

Since

$$\frac{1}{\frac{2n}{n+\theta}} = \frac{1-\theta}{2} + \frac{\theta}{\frac{2n}{n+1}} \text{ and } \frac{1}{\frac{q_k}{(q_k-1)\theta+1}} = \frac{1-\theta}{q_k} + \frac{\theta}{1},$$

the same interpolation-complexification argument furnishes the result. □

3. Inclusion results

Given $1 \leq p \leq q < \infty$, it is well known that absolutely p-summing linear operators are absolutely q-summing. For multiple summing mappings, Pérez-García [28, Teorema 4.13] has shown that $\mathcal{L}_{\mathrm{ms},p}(E_1,\ldots,E_n;F) \subseteq \mathcal{L}_{\mathrm{ms},q}(E_1,\ldots,E_n;F)$ whenever $1 \leq p \leq q < 2$ ($1 \leq p \leq q \leq 2$ if $E_1,\ldots,E_n$ have cotype 2). However, in [28, Teorema 4.13] it is also shown that there is no general inclusion theorem for multiple summing multilinear mappings. Some surprising inclusion results for absolutely summing polynomials and holomorphic mappings were recently obtained in [16]. In this section we obtain new inclusions for multiple summing multilinear mappings, polynomials and holomorphic mappings.

A result obtained independently by Pérez-García [28, Teorema 5.2] and Souza [35, Teorema 1.7.3] asserts that if F has finite cotype q, then

$$\mathcal{L}_{\mathrm{ms}(q;1)}({}^nE;F) = \mathcal{L}({}^nE;F) \text{ and } \|\cdot\|_{\mathrm{ms}(q;1)} \leq C_q(F)^n \|\cdot\| . \tag{3.1}$$

Next we will show how (3.1) can be explored in order to obtain surprising inclusion results. For the complexification argument to work we need the following extension of [28, Proposición 4.30(ii)]:

Lemma 3.1. *A real Banach space E has cotype $q > 2$ if and only if its complexification $\widetilde{E}$ has cotype $q > 2$. Also, if E has cotype 2, then $\widetilde{E}$ has cotype 2.*

Proof. The cotype 2 case is done in [28, Proposición 4.30(ii)]. Assume $q > 2$. A celebrated result due to Talagrand [36] asserts that a Banach space E has cotype q if and only if id_E is absolutely $(q;1)$-summing. So, from the linear case of [28, Proposición 4.30(ii)] we have that

$$\begin{aligned} E \text{ has cotype } q &\Leftrightarrow id_E \text{ is } (q;1)\text{-summing} \\ &\Leftrightarrow id_{\widetilde{E}} \text{ is } (q;1)\text{-summing} \Leftrightarrow \widetilde{E} \text{ has cotype } q. \end{aligned}$$

□

Remember that whenever we write $\mathrm{ms}(r;s)$ we are assuming $1 \leq s \leq r$.

Theorem 3.2. *If $E_1,\ldots,E_n$ have cotype 2, F has finite cotype q and $1 \leq s \leq 2$, then*

$$\mathcal{L}_{\mathrm{ms}(r;s)}(E_1,\ldots,E_n;F) \subseteq \mathcal{L}_{\mathrm{ms}(t_1;t_2)}(E_1,\ldots,E_n;F)$$

for every $n \in \mathbb{N}$, $0 \leq \theta \leq 1$ and t_1, t_2 satisfying

$$\frac{1}{t_1} = \frac{1-\theta}{r} + \frac{\theta}{q} \text{ and } \frac{1}{t_2} = \frac{1-\theta}{s} + \theta.$$

Moreover, if $T \in \mathcal{L}_{\mathrm{ms}(r;s)}(E_1,\ldots,E_n;F)$, then

$$\|T\|_{\mathrm{ms}(t_1;t_2)} \leq 16^n \left(C_2(E_1)\cdots C_2(E_n)\right)^{\frac{5}{2}} C_q(F)^{n\theta} \|T\|_{\mathrm{ms}(r;s)}^{(1-\theta)} \|T\|^{\theta} . \tag{3.2}$$

Proof. As before, using [28, Proposición 4.30(ii)] and Lemma 3.1, the real case follows from the complex case. Assume $\mathbb{K} = \mathbb{C}$.

Claim. Under the assumptions of the theorem, if $T \in \mathcal{L}_{\mathrm{ms}(r;s)}(E_1,\ldots,E_n;F) \cap \mathcal{L}_{\mathrm{ms}(p;h)}(E_1,\ldots,E_n;F)$ for some $1 \leq h \leq 2$, then $T \in \mathcal{L}_{\mathrm{ms}(t_1;t_2)}(E_1,\ldots,E_n;F)$ and

$$\|T\|_{\mathrm{ms}(t_1;t_2)} \leq 16^n \left(C_2(E_1)\cdots C_2(E_n)\right)^{\frac{5}{2}} \|T\|_{\mathrm{ms}(r;s)}^{1-\theta} \|T\|_{\mathrm{ms}(p;h)}^{\theta} ,$$

where $\theta \in [0,1]$ and t_1, t_2 satisfy

$$\frac{1}{t_1} = \frac{1-\theta}{r} + \frac{\theta}{p} \text{ and } \frac{1}{t_2} = \frac{1-\theta}{s} + \frac{\theta}{h}. \tag{3.3}$$

Proof of the claim: We proceed by complex interpolation. For each positive integer m the map T induces natural (uniformly) bounded n-linear mappings

$$\Psi_T^{(a)} \colon \ell_{s,w}^m(E_1) \times \cdots \times \ell_{s,w}^m(E_n) \to \ell_r^{m^n}(F) \text{ and}$$
$$\Psi_T^{(b)} \colon \ell_{h,w}^m(E_1) \times \cdots \times \ell_{h,w}^m(E_n) \to \ell_p^{m^n}(F).$$

Applying the complex interpolation method [2, Theorem 4.4.1] to these n-linear operators we get an n-linear mapping

$$\Psi_T^{(\theta)} \colon \left[\ell_{s,w}^m(E_1), \ell_{h,w}^m(E_1)\right]_\theta \times \cdots \times \left[\ell_{s,w}^m(E_n), \ell_{h,w}^m(E_n)\right]_\theta \to \left[\ell_r^{m^n}(F), \ell_p^{m^n}(F)\right]_\theta$$

with $\left\|\Psi_T^{(\theta)}\right\| \leq \left\|\Psi_T^{(a)}\right\|^{1-\theta} \left\|\Psi_T^{(b)}\right\|^{\theta}$. By [2, Theorem 5.1.2] we have the isometry

$$\left[\ell_r^{m^n}(F), \ell_p^{m^n}(F)\right]_\theta = \ell_{t_1}^{m^n}(F),$$

with t_1 as in (3.3). Using the natural isometric identification $\ell_{s,w}^m(E_k) = \ell_s^m \otimes_\varepsilon E_k$, $k = 1, \ldots, n$, as a particular case of [9, Lemma 2 and Proposition 8] (remember that ℓ_s and ℓ_h are 2-concave with constant 1 because $s, h \in [1,2]$), we obtain natural isomorphisms

$$J_k \colon \ell_{t_2,w}^m(E_k) \to \left[\ell_{s,w}^m(E_k), \ell_{h,w}^m(E_k)\right]_\theta$$

with t_2 as in (3.3) and $\|J_k\| \leq 16 C_2(E_k)^{\frac{5}{2}}$. Up to these isomorphisms the mapping $\Psi_T^{(\theta)}$ can be identified with the multilinear mapping

$$\Psi_T \colon \ell_{t_2,w}^m(E_1) \times \cdots \times \ell_{t_2,w}^m(E_n) \to \ell_{t_1}^{m^n}(F),$$
$$\Psi_T\left((x_j^{(1)})_{j=1}^m, \ldots, (x_j^{(n)})_{j=1}^m\right) = \left(T(x_{j_1}^{(1)}, \ldots, x_{j_n}^{(n)})\right)_{j_1,\ldots,j_n=1}^m$$

for all sequences $(x_j^{(k)})_{j=1}^m$ in $\ell_{t_2,w}^m(E_k) = \left[\ell_{s,w}^m(E_k), \ell_{h,w}^m(E_k)\right]_\theta$, $1 \leq k \leq n$. This gives us that $T \in \mathcal{L}_{\mathrm{ms}(t_1;t_2)}(E_1, \ldots, E_n; F)$ and

$$\begin{aligned}\|T\|_{\mathrm{ms}(t_1,t_2)} &\leq \|J_1\| \cdots \|J_n\| \left\|\Psi_T^{(\theta)}\right\| \leq 16^n \left(C_2(E_1) \cdots C_2(E_n)\right)^{\frac{5}{2}} \left\|\Psi_T^{(a)}\right\|^{1-\theta} \left\|\Psi_T^{(b)}\right\|^{\theta} \\ &\leq 16^n \left(C_2(E_1) \cdots C_2(E_n)\right)^{\frac{5}{2}} \|T\|_{\mathrm{ms}(r;s)}^{1-\theta} \|T\|_{\mathrm{ms}(p;h)}^{\theta},\end{aligned}$$

which proves the claim.

To get the result just make $p = q$ and $h = 1$ in the claim and call on (3.1). $\square$

Remark 3.3. Theorem 3.2 is interesting for $r \geq q$. The case $r < q$ is trivial.

Example 3.4. Under the hypotheses of Theorem 3.2 we have

$$\mathcal{L}_{\mathrm{ms}(r;s)}(^nE; F) \subseteq \mathcal{L}_{\mathrm{ms}\left(\frac{qr}{(1-\theta)q+\theta r}; \frac{s}{(1-\theta)+\theta s}\right)}(^nE; F),$$

which can be regarded as a multilinear version (under certain additional hypotheses) of [11, Theorem 10.4]. For instance, making $r = 4, s = 2, q = 3$, $\theta = 1/2$, $E = \ell_2$ and $F = \ell_3$, we obtain

$$\mathcal{L}_{\mathrm{ms}(4;2)}(^n\ell_2; \ell_3) \subseteq \mathcal{L}_{\mathrm{ms}(\frac{24}{7};\frac{4}{3})}(^n\ell_2; \ell_3)$$

for every positive integer n.

We finish the paper by showing how Theorem 3.2 can be applied to multiple summing homogeneous polynomials and holomorphic mappings.

Recall that an n-homogeneous polynomial $P\colon E \to F$ is multiple (or fully) $(r;s)$-summing (in symbols $P \in \mathcal{P}_{\mathrm{ms}(r;s)}(^nE;F)$) if its associated symmetric n-linear mapping $\overset{\vee}{P}$ is multiple $(r;s)$-summing. A natural norm on $\mathcal{P}_{\mathrm{ms}(r;s)}(^nE;F)$ is given by $\|P\|_{\mathrm{ms}(r;s)} = \|\overset{\vee}{P}\|_{\mathrm{ms}(r;s)}$. It is well known [27, Theorem 4.3] that $(\mathcal{P}_{\mathrm{ms}(p;q)}, \|\cdot\|_{\mathrm{ms}(p;q)})$ is a (global) holomorphy type (for the definition and further details on global holomorphy types the reader is referred to [7]).

From Theorem 3.2 and the estimate $\|\overset{\vee}{P}\| \le e^n \|P\|$ we obtain:

Proposition 3.5. *If E has cotype 2, F has finite cotype q, and $1 \le s \le 2$, then*

$$\mathcal{P}_{\mathrm{ms}(r;s)}(^nE;F) \subseteq \mathcal{P}_{\mathrm{ms}(t_1;t_2)}(^nE;F) \tag{3.4}$$

for every $0 \le \theta \le 1$ and t_1, t_2 satisfying

$$\frac{1}{t_1} = \frac{1-\theta}{r} + \frac{\theta}{q} \text{ and } \frac{1}{t_2} = \frac{1-\theta}{s} + \theta.$$

Moreover, if $P \in \mathcal{P}_{\mathrm{ms}(r,s)}(^nE;F)$, then

$$\|P\|_{\mathrm{ms}(t_1,t_2)} \le (16e^\theta)^n C_2(E)^{\frac{5n}{2}} C_q(F)^{n\theta} \|P\|_{\mathrm{ms}(r,s)}^{(1-\theta)} \|P\|^\theta . \tag{3.5}$$

Definition 3.6. An entire mapping $f\colon E \to F$ is said to be of ms$(p;q)$-holomorphy type at $a \in E$ (in the sense of Nachbin [25]) if

(a) $\frac{1}{n!}\widehat{d}^n f(a) \in \mathcal{P}_{\mathrm{ms}(p;q)}(^nE;F)$ and

(b) there exist $C_1 \ge 0$ and $c_1 \ge 0$ such that

$$\left\| \frac{1}{n!}\widehat{d}^n f(a) \right\|_{\mathrm{ms}(p;q)} \le C_1 c_1^n$$

for every positive integer n. If f is of ms$(p;q)$-holomorphy type at every $a \in E$, we say that f is of ms$(p;q)$-holomorphy type and we write $f \in \mathcal{H}_{\mathrm{ms}(p;q)}(E;F)$. The following inclusion follows immediately from [28, Teorema 4.13]:

Proposition 3.7. *If $1 \le p < q < 2$ and E, F are complex Banach spaces, then $\mathcal{H}_{\mathrm{ms},p}(E;F) \subseteq \mathcal{H}_{\mathrm{ms},q}(E;F)$.*

To holomorphic mappings of ms$(p;q)$-holomorphy type we have the following extension of Proposition 3.5:

Proposition 3.8. *Let E, F be complex Banach spaces. If E has cotype 2, F has finite cotype q, and $1 \le s \le 2$, then*

$$\mathcal{H}_{\mathrm{ms}(r;s)}(E;F) \subseteq \mathcal{H}_{\mathrm{ms}(t_1;t_2)}(E;F)$$

for every $0 \le \theta \le 1$ and t_1, t_2 satisfying

$$\frac{1}{t_1} = \frac{1-\theta}{r} + \frac{\theta}{q} \text{ and } \frac{1}{t_2} = \frac{1-\theta}{s} + \theta.$$

Proof. Let $f \in \mathcal{H}_{\mathrm{ms}(r;s)}(E;F)$ and $a \in E$. Then $\frac{1}{n!}\widehat{d}^n f(a) \in \mathcal{P}_{\mathrm{ms}(r;s)}(^nE;F)$ and there are positive constants C_1, c_1, C and c so that

$$\left\| \frac{1}{n!}\widehat{d}^n f(a) \right\| \le C_1 c_1^n \text{ and } \left\| \frac{1}{n!}\widehat{d}^n f(a) \right\|_{\mathrm{ms}(r;s)} \le C c^n$$

for all n. From (3.4) we conclude that $\frac{1}{n!}\widehat{d}^n f(a) \in \mathcal{P}_{\mathrm{ms}(t_1;t_2)}({}^nE;F)$, and from (3.5) we get

$$\begin{aligned}\left\|\frac{1}{n!}\widehat{d}^n f(a)\right\|_{\mathrm{ms}(t_1;t_2)} &\leq (16e^\theta)^n C_2(E)^{\frac{5n}{2}} C_q(F)^{n\theta} \left\|\frac{1}{n!}\widehat{d}^n f(a)\right\|_{\mathrm{ms}(r;s)}^{(1-\theta)} \left\|\frac{1}{n!}\widehat{d}^n f(a)\right\|^\theta \\ &\leq \left(16e^\theta C_2(E)^{\frac{5}{2}} C_q(F)^\theta\right)^n (Cc^n)^{1-\theta} (C_1c_1^n)^\theta \\ &= C^{1-\theta}C_1^\theta \left(16e^\theta C_2(E)^{\frac{5}{2}} C_q(F)^\theta c_1^\theta c^{1-\theta}\right)^n,\end{aligned}$$

which shows that $f \in \mathcal{H}_{\mathrm{ms}(t_1;t_2)}(E;F)$. □

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