PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 3, March 2009, Pages 1139–1148
S 0002-9939(08)09705-0
Article electronically published on September 29, 2008

THE DISTORTION OF A KNOTTED CURVE

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(Communicated by Daniel Ruberman)

Abstract. The distortion of a curve measures the maximum arc/chord length ratio. Gromov showed that any closed curve has distortion at least $\pi/2$ and asked about the distortion of knots. Here, we prove that any nontrivial tame knot has distortion at least $5\pi/3$; examples show that distortion under 7.16 suffices to build a trefoil knot. Our argument uses the existence of a shortest essential secant and a characterization of borderline-essential arcs.

Gromov introduced the notion of distortion for curves as the supremal ratio of arclength to chord length. (See [Gro78], [Gro83, p. 114] and [GLP81, pp. 6–9].) He showed that any closed curve has distortion $\delta \geq {}^{\pi}/_{2}$, with equality only for a circle. He then asked whether every knot type can be built with, say, $\delta \leq 100$.

As Gromov knew, there are infinite families with such a uniform bound. For instance, an open trefoil (a long knot with straight ends) can be built with $\delta < 10.7$, as follows from an explicit computation for a simple shape. Then connect sums of arbitrarily many trefoils—even infinitely many, as in Figure 1—can be built with this same distortion. (O'Hara [O'H92] exhibited a similar family of prime knots.)

Despite such examples, many people expect a negative answer to Gromov's question. We provide a first step in this direction, namely a lower bound depending on knottedness: we prove that any nontrivial tame knot has $\delta \geq {}^{5\pi}/_{3}$, more than three times the minimum for an unknot.

To make further progress on the original question, one should try to bound distortion in terms of some measure of knot complexity. Examples such as Figure 1 show that crossing number and even bridge number are too strong: distortion can stay bounded as they go to infinity. Perhaps it is worth investigating hull number [CKKS03, Izm06].

Our bound $\delta \geq {}^{5\pi}/_{3}$ arises from considering *essential secants* of the knot, a notion introduced by Kuperberg [Kup94] and developed further in [DDS06]. There, we used the essential alternating quadrisecants of [Den04] to give a good lower bound for the ropelength [GM99, CKS02] of nontrivial knots.

The main tool in [DDS06] was a geometric characterization of a borderline-essential arc (quoted here as Theorem 1.1), showing that its endpoints are part of an essential trisecant. This result captures the intuition that in order for an arc to become essential, it must wrap around some other point of the knot; but it also demonstrates that secants to that other point are themselves essential. This theorem will be important for our distortion bounds as well.

Received by the editors February 28, 2008.
2000 *Mathematics Subject Classification.* Primary 57M25; Secondary 49Q10, 53A04, 53C42.

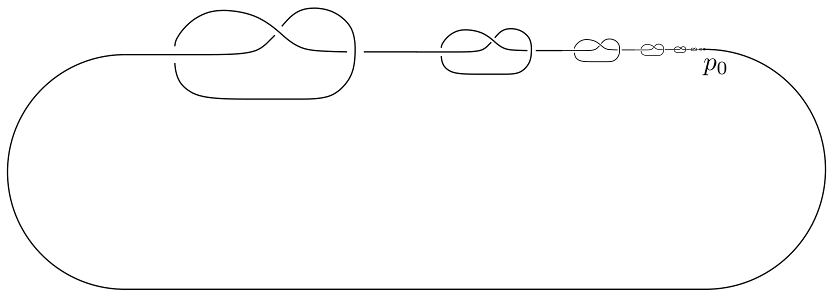

FIGURE 1. A wild knot, the connect sum of infinitely many trefoils, can be built with distortion less than 10.7 by repeating scaled copies of a low-distortion open trefoil. To ensure that the distortion will be realized within one trefoil, we merely need to make the copies sufficiently small compared to the overall loop of the knot and sufficiently distant from each other. This knot is smooth except at the one point p_0.

Our main new technical tool is Corollary 2.6, which guarantees the existence of a shortest essential secant. While this is obvious for smooth or polygonal knots, we are not free to make any such assumption about the geometry of knots of low distortion. Thus it is important that Corollary 2.6 holds merely under the topological assumption of tameness.

With these two tools, the intuition behind our main result is clear. Let $\overline{ab}$ be a shortest essential secant for a nontrivial tame knot and scale the knot so that $|a-b|=1$. We prove the knot has distortion $\delta \geq {}^{5\pi}/_{3}$ by showing that each of the two arcs γ between a and b has length at least ${}^{5\pi}/_{3}$. Indeed, for γ to become essential, by Theorem 1.1 it must wrap around some other point $\mathbf{0}$ of the knot. If $\overline{x\mathbf{0}}$ is essential for all points $x \in \gamma$, then γ stays outside the unit ball around $\mathbf{0}$, and thus must wrap around ${}^{5}/_{6}$ of a circle, as in Figure 7 (left). The other cases, where some $\overline{x\mathbf{0}}$ is inessential, take longer to analyze but turn out to need even more length.

Note that a wild knot, even if its distortion is low, can have arbitrarily short essential arcs, as in the example of Figure 1. For this technical reason, our main theorem applies only to tame knots, even though we expect wild knots must have even greater distortion. Every wild knot has infinite total curvature, and thus infinite bridge number and infinite crossing number. Thus it is initially surprising how many wild knots can be built with finite distortion. Even some standard examples with uncountably many wild points (on a Cantor set) can be constructed with finite distortion. An interesting question is whether there is some (necessarily wild) knot type which requires infinite distortion. (A knot requiring infinite length would be an example.) Perhaps a knot with no tame points would have this property, or perhaps even the knot described by J.W. Alexander [Ale24] (and later by G.Ya. Zuev, see [Sos02, p. 12]), whose wild set is Antoine's necklace.

Our bound $\delta \geq {}^{5\pi}/_{3}$ is of course not sharp, but numerical simulations [Mul06] have found a trefoil knot with distortion less than 7.16, so we are not too far off. We expect the true minimum distortion for a trefoil is closer to that upper bound than to our lower bound. A sharp bound (characterizing that minimum value) would presumably require a criticality theory for distortion minimizers. Perhaps this could be developed along the lines of the balance criterion for (Gehring) ropelength [CFK+06], but the technical difficulties seem formidable.

On the other hand, it is easy to see how to slightly improve our bound $\delta \geq {}^{5\pi}\!/_{3}$. Indeed, the circular arc shown in Figure 7 (left) must actually spiral out in the middle (to avoid greater distortion between c and $\mathbf{0}$). In the first version [DS04] of this paper, our bounds considered a shortest essential arc; we used logarithmic spirals to improve an initial bound $\delta \geq \pi$ to $\delta > 3.99$. Bereznyak and Svetlov [BS06] then obtained $\delta > 4.76$ by focusing on a shortest borderline-essential secant and making further use of spirals. We expect that such spirals could improve our bound here by only a few percent, at the cost of tripling the length of this paper; thus we have not pursued this idea.

There are easy upper bounds for distortion in terms of other geometric quantities for space curves. For instance, an arc of total curvature $\alpha < \pi$ has distortion at most $\sec {}^{\alpha}\!/_{2}$. (See [Sul08, §7].) Similarly, a closed curve of ropelength R has distortion at most ${}^{R}\!/_{2}$. (This was [LSDR99, Thm. 3] and also follows easily from [DDS06, Lem. 3.1].) But there are no useful bounds the other way: the example in Figure 1 has bounded distortion but infinite total curvature and ropelength, while a steep logarithmic spiral shows that arcs of infinite total curvature can have distortion arbitrarily close to 1.

This means that a lower bound like ours for the distortion of a nontrivial knot cannot be based on the known lower bounds for total curvature [Mil50] or ropelength [DDS06]. Indeed, before our work here, it remained conceivable that the infimal distortion of knotted curves was ${}^{\pi}\!/_{2}$.

An alternative approach might be to consider explicitly the geometry of curves of small distortion. A closed plane curve with distortion close to ${}^{\pi}\!/_{2}$ must be pointwise close to a round circle [DEG+07]; we have modified that argument to apply to space curves [DS04]. Being close to a circle, of course, does not preclude being knotted. We note that the proof looks only at distortion between opposite points on the curve, and, indeed, any knot type can be realized so that this restricted distortion is arbitrarily close to ${}^{\pi}\!/_{2}$.

1. Definitions and background

We deal with oriented, compact, connected curves embedded in $\mathbb{R}^3$. Such a curve is either an *arc* homeomorphic to an interval, or a *knot* (a simple closed curve) homeomorphic to a circle.

Two points p, q along a knot K separate K into two complementary arcs, γ_{pq} (from p to q) and γ_{qp}. We let ℓ_{pq} denote the length of γ_{pq}. Distortion contrasts the shorter arclength distance $d(p,q) := \min(\ell_{pq}, \ell_{qp}) \leq {}^{\ell(K)}\!/_{2}$ with the straight-line (chord) distance $|p-q|$ in $\mathbb{R}^3$. (For an arc γ, if p, q lie in order along γ, then $d(q,p) = d(p,q) := \ell_{pq}$ is the length of the subarc γ_{pq}.)

Definition. The *distortion* between distinct points p and q on a curve γ is

$$\delta(p,q) := \frac{d(p,q)}{|p-q|} \geq 1.$$

The *distortion* of γ is the supremum $\delta(\gamma) := \sup \delta(p,q)$, taken over all pairs of distinct points.

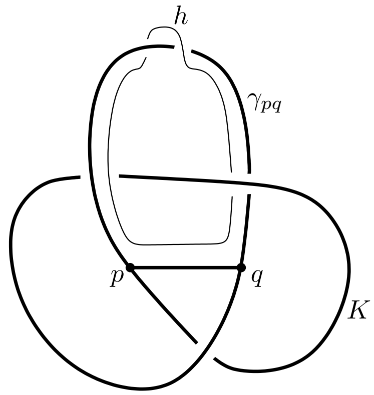

FIGURE 2. The arc γ_{pq} is essential in the knot K: the parallel h, whose linking number with K is zero, is homotopically nontrivial in the knot complement. This shows there is no disk spanning $\gamma_{pq} \cup \overline{pq}$ while avoiding K. In this example, γ_{qp} is also essential, so $\overline{pq}$ is essential.

Our distortion bound for knots uses the notion of essential arcs, introduced in [DDS06] as an extension of ideas of Kuperberg [Kup94]. Note that generically a knot K together with a chord $\overline{pq}$ forms a θ-graph in space; being essential is a topological feature of this knotted graph, as shown in Figure 2.

Definition. Suppose α, β and γ are interior-disjoint arcs from p to q, forming a knotted θ-graph in $\mathbb{R}^3$. We say the ordered triple (α, β, γ) is *essential* if the loop $\alpha \cup \beta$ bounds no (singular) disk whose interior is disjoint from the knot $\alpha \cup \gamma$.

Now suppose K is a knot and $p, q \in K$. If the secant $\overline{pq}$ has no interior intersections with K, we say γ_{pq} is an *essential arc* of K if $(\gamma_{pq}, \overline{pq}, \gamma_{qp})$ is essential. If $\overline{pq}$ does intersect K, we say γ_{pq} is an *essential arc* if for any $\varepsilon > 0$ there is an ε-perturbation S of $\overline{pq}$ such that $(\gamma_{pq}, S, \gamma_{qp})$ is essential. We say $\overline{pq}$ is an *essential secant* if both γ_{pq} and γ_{qp} are essential.

Note that the ε-perturbation ensures that the set of essential arcs is closed within the set $(K \times K) \smallsetminus \Delta$ of all subarcs. We say the arc γ_{pq} is *borderline-essential* if it is in the boundary of the set of essential arcs. That is, γ_{pq} is essential, but there are inessential subarcs of K with endpoints arbitrarily close to p and q.

The following theorem [DDS06, Thm. 7.1] lies at the heart of our distortion bounds. It describes the special geometric configuration, shown in Figure 3, arising from any borderline-essential arc.

Theorem 1.1. *Suppose γ_{pq} is a borderline-essential subarc of a knot K. Then the interior of the segment $\overline{pq}$ must intersect K at some point $x \subset \gamma_{qp}$ for which the secants $\overline{xp}$ and $\overline{xq}$ are both essential.* □

In [DDS06, Lem. 4.3] we showed that the minimum length of an arc $\gamma_{ab} \subset \mathbb{R}^n$ staying outside the unit ball $B_1(\mathbf{0})$ is $m\big(|a|, |b|, \angle a\mathbf{0}b\big)$, where for $r, s \geq 1$ and $\theta \in [0, \pi]$ we set

$$m(r,s,\theta) := \begin{cases} \sqrt{r^2 + s^2 - 2rs\cos\theta} & \text{if } \theta \leq \theta_0, \\ \sqrt{r^2-1} + \sqrt{s^2-1} + \theta - \theta_0 & \text{if } \theta \geq \theta_0, \end{cases}$$

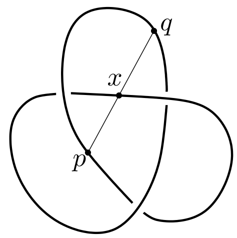

FIGURE 3. If the arc γ_{pq} is borderline-essential in the knot K, Theorem 1.1 gives a point $x \in K \cap \overline{pq}$ for which $\overline{xp}$ and $\overline{xq}$ are essential.

with $\theta_0 = \theta_0(r, s) := \operatorname{arcsec} r + \operatorname{arcsec} s$. (In the case of plane curves, this can be dated back to [Kub23].)

This bound is hard to apply since $m(r, s, \theta)$ is not monotonic in r and s. Thus we are led to define $m_1(s, \theta) := \min_{r \geq 1} m(r, s, \theta)$, from which we calculate

$$m_1(s,\theta) = \begin{cases} s \sin\theta & \text{if } \theta \leq \operatorname{arcsec} s, \\ \sqrt{s^2-1} + \theta - \operatorname{arcsec} s & \text{if } \theta \geq \operatorname{arcsec} s. \end{cases}$$

This function m_1 is continuous, increasing in s and in θ, and concave in θ. We have:

Lemma 1.2. *An arc γ_{ab} staying outside $B_1(\mathbf{0})$ has length at least $m_1\big(|b|, \angle a\mathbf{0}b\big) \geq \angle a\mathbf{0}b$.* $\square$

Remark. For $\theta = \pi$, we are always in the second case in the definition of m_1, and we have $m_1(s, \pi) > \sqrt{s^2+1} + {}^{\pi}/_{2}$, the right-hand side being the length of a curve that follows a quarter-circle from a and then goes straight to b (cutting into the unit ball).

2. Shortest essential arcs and secants

To get our lower bound on ropelength, we showed [DDS06, Lem. 8.1] that in a knot of unit thickness, arcs of length less than π (and secants of length less than 1) are inessential. Here, we show that in *any* tame knot, sufficiently short arcs and secants are inessential. It follows that every nontrivial tame knot has shortest essential arcs and secants.

If K is unknotted, any subarc is inessential. Conversely, Dehn's lemma can be used to show [DDS06, Thm. 5.2] that if both γ_{pq} and γ_{qp} are inessential (for some $p, q \in K$), then K is unknotted. Equivalently, if K is a nontrivial knot, then the complement of any inessential arc is essential.

Lemma 2.1. *If γ_{pq} is a borderline-essential subarc of a knot K, then $\overline{pq}$ is an essential secant.*

Proof. Since γ_{pq} is borderline-essential, there are inessential arcs $\gamma_{p'q'}$ converging to γ_{pq}. Since K (having the essential subarc γ_{pq}) must be nontrivial, the complements $\gamma_{q'p'}$ are essential. Thus their limit γ_{qp} is also essential. $\square$

Corollary 2.2. *Given any point p on a nontrivial knot K, there is some $q \in K$ such that $\overline{pq}$ is essential.*

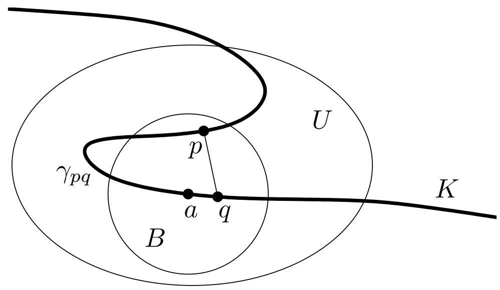

FIGURE 4. Near a locally flat point $a \in K$, short arcs and secants $\overline{pq}$ are inessential.

Proof. Since K is nontrivial, at least some subarcs starting or ending at p are essential. If they all are, then so are all secants from p. Otherwise there is some borderline-essential arc starting or ending at p. By the lemma, this gives us an essential secant $\overline{pq}$. $\square$

Lemma 2.3. *Suppose K is a knot and U is a topological ball such that K intersects U in a single unknotted arc. Suppose p and q are two points in order along this arc, and β is any arc within U from p to q, disjoint from K. Then $(\gamma_{pq}, \beta, \gamma_{qp})$ is inessential.*

Proof. By definition of an unknotted ball/arc pair, after applying an ambient homeomorphism we may assume that U is a round ball and $K \cap U$ a diameter. Pick any homeomorphism between γ_{pq} and β (fixing p and q). Join all pairs of corresponding points by straight segments; these fill out a (singular) disk with boundary $\gamma_{pq} \cup \beta$, which by convexity stays entirely within U. The disk avoids K (except of course for the segment endpoints along γ_{pq}) because β avoids the straight segment $K \cap U$. $\square$

Proposition 2.4. *Given a knot K and any locally flat point $a \in K$, we can find $r > 0$ such that any subarc γ_{pq} or secant $\overline{pq}$ of K which lies in the ball $B_r(a)$ is inessential.*

Proof. Locally flat means, by definition, that a has a neighborhood U in which $K \cap U$ is a single unknotted arc. Choose r such that $B := B_r(a)$ is contained in U, as in Figure 4. For any points $p, q \in K \cap B$, the segment $\overline{pq}$ is contained in B by convexity, hence in U. So any sufficiently small perturbation S of this segment (as in the definition of essential) stays in U. Since $K \cap U$ is a single arc, after switching the labels p and q if necessary, we have $\gamma_{pq} \subset U$. (If we are proving the first claim, of course, we already know $\gamma_{pq} \subset B$.) By Lemma 2.3, $(\gamma_{pq}, S, \gamma_{qp})$ is inessential, implying by definition that the subarc γ_{pq} and the secant $\overline{pq}$ are inessential. $\square$

Theorem 2.5. *Given any tame knot K, there exists $\varepsilon > 0$ such that any subarc γ_{pq} of length $\ell_{pq} < \varepsilon$ is inessential and any secant $\overline{pq}$ of length $|p-q| < \varepsilon$ is inessential.*

Proof. Suppose there were sequences $p_n, q_n \in K$ giving essential arcs or essential secants with length decreasing to zero. By compactness of $K \times K$ we can extract a convergent subsequence $(p_n, q_n) \to (a, a)$. But the tame knot K is by definition locally flat at every point $a \in K$. Choose $r > 0$ as in Proposition 2.4 and choose n large enough so that $\gamma_{p_n q_n} \subset B_r(a)$. Then the proposition says $\gamma_{p_n q_n}$ and $\overline{p_n q_n}$ are inessential, contradicting our choice of p_n, q_n. $\square$

Corollary 2.6. *Any nontrivial tame knot K has a shortest essential secant and a shortest essential subarc.*

Proof. Being nontrivial, K does have essential subarcs and secants by Corollary 2.2. By compactness, a length-minimizing sequence (p_n, q_n) for either case has a subsequence converging to some $(p, q) \in K \times K$, and $p \neq q$ by Theorem 2.5. Since being essential is a closed condition, this limit arc or secant is still essential, with minimum length. □

3. Distortion bounds

The key to our distortion bounds will be to focus on a shortest essential secant, as guaranteed by Corollary 2.6; we usually rescale so this secant has length 1. Then Theorem 1.1 implies that any borderline-essential secant has length at least 2.

Remark. We can immeditately deduce that the distortion of the knot is at least 4. Indeed, if $\overline{ab}$ is a shortest essential secant, scaled so that $|a - b| = 1$, then each arc between a and b (in particular the shorter one) must include a point x at distance 2 from a as well as a point y at distance 2 from b. The shortest possibility has length 4 with $x = y$ as in Figure 5.

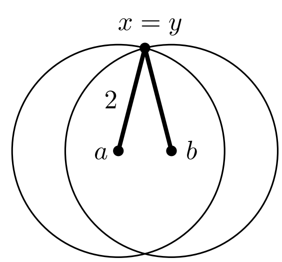

Figure 5. If $\overline{ab}$ is a shortest essential secant with $|a-b| = 1$, then each arc between a and b must leave both circles of radius 2 shown. It thus has length at least 4.

Of course, this crude bound does not account for the fact that, while a borderline-essential secant can have $|p - q| = 2$, a borderline essential arc γ_{pq} must be significantly longer, since it wraps around the point x guaranteed by Theorem 1.1. As with the main theorem, our intuition is guided by the case where all points along γ_{pq} give essential secants to x. Here clearly $\ell_{pq} \geq \pi$; the following proposition confirms that this is indeed the critical case.

Proposition 3.1. *Let K be a nontrivial tame knot, scaled so that a shortest essential secant has length* 1*. Suppose arc γ_{pq} is borderline-essential, and $x \in \overline{pq} \cap K$ is a point as guaranteed by Theorem* 1.1 *with $\overline{xp}$ and $\overline{xq}$ essential. If we set $s := \min(|q - x|, 2) \in [1, 2]$, we have $\ell_{pq} \geq m_1(s, \pi) \geq \pi$.*

Proof. Translate so that x is the origin $\mathbf{0}$. If $\overline{y\mathbf{0}}$ is essential for all $y \in \gamma_{pq}$, then by our scaling, γ_{pq} stays outside $B_1(\mathbf{0})$. Thus by Lemma 1.2 and monotonicity of m_1, we get $\ell_{pq} \geq m_1(|q|, \pi) \geq m_1(s, \pi)$ as desired.

Otherwise, let $y, z \in \gamma_{pq}$ be the first and last points making borderline-essential secants $\overline{\mathbf{0}y}$ and $\overline{\mathbf{0}z}$. By our choice of scaling, $|y|, |z| \geq 2$ and the arcs γ_{py} and γ_{zq} stay outside $B_1(\mathbf{0})$. As in Figure 6, define angles

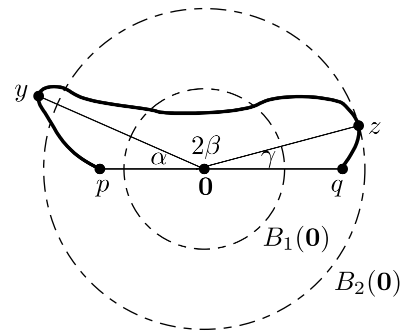

FIGURE 6. In the second case in the proof of Proposition 3.1, since y and z are borderline-essential to $\mathbf{0}$, they are outside $B_2(\mathbf{0})$. Even though γ_{yz} can go inside $B_1(\mathbf{0})$ the total length ℓ_{pq} in this case is at least $m_1(2,\pi)$.

$$\alpha := \angle p\mathbf{0}y, \quad 2\beta := \angle y\mathbf{0}z, \quad \gamma := \angle z\mathbf{0}q,$$

so that $\alpha + 2\beta + \gamma = \pi$. By Lemma 1.2, we have

$$\ell_{pq} = \ell_{py} + \ell_{yz} + \ell_{zq} \geq m_1(2,\alpha) + 4\sin\beta + m_1(2,\gamma).$$

By concavity of m_1, for any given $\alpha + \gamma$, the sum of the first and last terms is minimized for $\gamma = 0$. Thus we get

$$\ell_{pq} \geq m_1(2, \pi - 2\beta) + 4\sin\beta.$$

For $\beta \geq \pi/3$, the first case in the definition of m_1 applies, so

$$\ell_{pq} \geq 2\sin 2\beta + 4\sin\beta = 4\sin\beta\,(1+\cos\beta) \geq 4.$$

For $\beta \leq \pi/3$, the second case applies, so

$$\ell_{pq} \geq \sqrt{3} + 2\pi/3 - 2\beta + 4\sin\beta \geq \sqrt{3} + 2\pi/3 = m_1(2,\pi).$$

Noting that $4 > m_1(2,\pi) \approx 3.826$, we find that in either case,

$$\ell_{pq} \geq m_1(2,\pi) \geq m_1(s,\pi). \qquad \square$$

Theorem 3.2. *Let K be a nontrivial tame knot, scaled so that a shortest essential secant has length* 1. *Suppose $\overline{ab}$ is an essential secant with length $|a-b| \leq 2$. Then*

$$d(a,b) \geq 2\pi - 2\arcsin\frac{|a-b|}{2}.$$

Proof. Switching a and b if necessary, we may assume $d(a,b) = \ell_{ab} \leq \ell_{ba}$. Setting $\varphi := 2\arcsin{}^{|a-b|}/_2 \geq |a-b|$, we wish to show that $\ell_{ab} \geq 2\pi - \varphi$.

Let $\gamma_{ac} \subset \gamma_{ab}$ be the shortest initial subarc that is essential, and translate so that the origin $\mathbf{0} \in \overline{ac} \cap K$ is a point as in Theorem 1.1. By Proposition 3.1, $\ell_{ac} \geq m_1(|c|,\pi) \geq \pi$, so it suffices to show $\ell_{cb} \geq \pi - \varphi$.

For a fixed length $|a-b|$, consider $\angle a\mathbf{0}b$ as a function of $|a|, |b| \geq 1$. It is maximized when $|a| = 1 = |b|$, with $\angle a\mathbf{0}b = \varphi$. Thus $\angle c\mathbf{0}b \geq \pi - \varphi$. If $\overline{\mathbf{0}x}$ is essential for all $x \in \gamma_{cb}$, then γ_{cb} remains outside $B_1(\mathbf{0})$, as in Figure 7 (left), so $\ell_{cb} \geq m_1(1, \angle c\mathbf{0}b) = \angle c\mathbf{0}b$ and we are done.

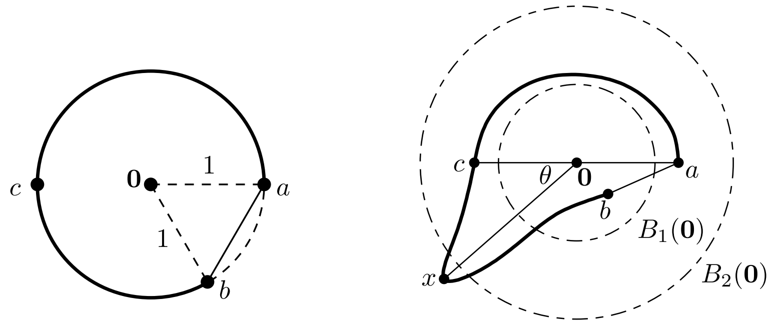

FIGURE 7. (Left) Since a and b are outside $B_1(\mathbf{0})$, we have $\angle a\mathbf{0}b \le 2\arcsin {}^{|a-b|}/_2$. When γ_{bc} stays outside the ball, its length is at least $\pi - \angle a\mathbf{0}b$. This figure shows the case $|a-b| = 1$ used in Corollary 3.3. (Right) If there is a point $x \in \gamma_{cb}$ with $|x| \ge 2$, then γ_{xb} can go inside $B_1(\mathbf{0})$. We use the bound $\ell_{cb} \ge |x-c| + \big(|x-a| - |a-b|\big)$.

Otherwise, let $x \in \gamma_{cb}$ be the first point for which $\overline{\mathbf{0}x}$ is borderline-essential, implying that $|x| \ge 2$. By the triangle inequality, $\ell_{xb} \ge |x-b| \ge |x-a| - |a-b|$, so

$$\ell_{cb} \ge \ell_{cx} + |x-a| - |a-b|.$$

Now set $\theta := \angle c\mathbf{0}x$ as in Figure 7 (right) and consider two cases.

For $\theta \ge {}^{\pi}/_2$, we get $\ell_{cx} \ge m_1(2,\theta) = \sqrt{3} + \theta - {}^{\pi}/_3$, while $|x-a| \ge 2\sin\theta$ since $|x| \ge 2$. The concave function $\theta + 2\sin\theta$ is minimized at the endpoint $\theta = \pi$, so, as desired,

$$\ell_{cb} \ge {}^{2\pi}/_3 + \sqrt{3} - |a-b| > \pi - \varphi.$$

For $\theta \le {}^{\pi}/_2$, we use $\ell_{cx} \ge |x-c|$ and consider fixed values of $|c| \ge 1$ and $|x| \ge 2$. We want to minimize the sum $|x-c| + |x-a|$. Since $|x-a|$ is increasing in $|a|$, we may assume $|a| = 1$. Then since $|c| \ge |a|$ and $\theta \le {}^{\pi}/_2$, we have $\angle cx\mathbf{0} \ge \angle \mathbf{0}xa$. This means that $|x-c| + |x-a|$ is an increasing function of θ, minimized at $\theta = 0$, where we have $|x-c| + |x-a| \ge 2 - |c| + 3$. Thus $\ell_{cb} \ge 5 - |c| - |a-b|$. Using the remark after Lemma 1.2, we have $\ell_{ac} \ge m_1(|c|,\pi) > |c| + {}^{\pi}/_2$. Thus finally, as desired,

$$\ell_{ab} > {}^{\pi}/_2 + 5 - |a-b| > 2\pi - \varphi. \qquad \square$$

Corollary 3.3. *Any nontrivial tame knot has $\delta \ge {}^{5\pi}/_3$.*

Proof. Let $\overline{ab}$ be a shortest essential secant for the knot K, and scale so that $|a-b| = 1$. Applying the theorem, we get $\delta(a,b) = d(a,b) \ge {}^{5\pi}/_3$. $\square$

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