

SIMPLE ALGEBRAS OF GELFAND-KIRILLOV DIMENSION TWO

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ABSTRACT. Let k be a field. We show that a finitely generated simple Goldie k -algebra of quadratic growth is noetherian and has Krull dimension 1. Thus a simple algebra of quadratic growth is left noetherian if and only if it is right noetherian. As a special case, we see that if A is a finitely generated simple domain of quadratic growth, then A is noetherian and by a result of Stafford every right and left ideal is generated by at most two elements. We conclude by posing questions and giving examples in which we consider what happens when the hypotheses are relaxed.

1. INTRODUCTION

We study finitely generated algebras of quadratic growth. Given a field k and a finitely generated k -algebra A , a k -subspace V of A is called a *frame* of A if V is finite dimensional, $1 \in V$, and V generates A as a k -algebra. We say that A has *quadratic growth* if there exist a frame V of A and constants $C_1, C_2 > 0$ such that

$$C_1 n^2 \leq \dim_k(V^n) \leq C_2 n^2 \quad \text{for all } n \geq 1.$$

We note that an algebra of quadratic growth has GK dimension 2. More generally, the Gelfand-Kirillov dimension (GK dimension, for short) of a finitely generated k -algebra A is defined to be

$$\text{GKdim}(A) = \limsup_{n \rightarrow \infty} \frac{\log(\dim(V^n))}{\log n},$$

where V is a frame of A . While algebras of quadratic growth have GK dimension 2, it is not the case that an algebra of GK dimension 2 necessarily has quadratic growth.

GK dimension can be viewed as a noncommutative analogue of Krull dimension in the following sense: if A is a finitely generated commutative k -algebra, then the Krull dimension of A and the GK dimension of A coincide. Krull dimension can also be extended to the noncommutative setting in a natural way by considering the so-called *deviation* of the poset of left ideals of a ring (see McConnell and Robson [5, Chapter 6] for the actual definition of Krull dimension). Unlike GK dimension,

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however, Krull dimension does not always exist. This is one weakness of Krull dimension, but it always exists for noetherian rings. It is in this more general sense that we use the term Krull dimension in this paper. For more information on GK dimension we refer the reader to Krause and Lenagan [4].

We study simple Goldie algebras of quadratic growth. A ring R is *right Goldie* if it satisfies the following two conditions:

- (1) R does not contain an infinite direct sum of nonzero right ideals;
- (2) R satisfies the ascending chain condition on right annihilators.

Left Goldie is defined analogously. A ring that is both left and right Goldie is called Goldie. A result of Irving and Small [3] shows that right Goldie and left Goldie are equivalent for semiprime algebras of finite GK dimension.

The Goldie condition is named after Alfred Goldie, who defined this property in his study of noncommutative localization. Just as a commutative domain has a field of fractions, a semiprime Goldie ring R has a semisimple Artinian ring of quotients $Q(R)$. We note that noetherian¹ algebras are always Goldie, but there exist non-noetherian Goldie algebras; for example, a polynomial ring in infinitely many variables over a field is Goldie but not noetherian.

Our main result is the following theorem, which shows that the Goldie and noetherian properties are equivalent for simple algebras of quadratic growth.

Theorem 1.1. *Let k be a field and let A be a finitely generated simple Goldie k -algebra of quadratic growth. Then A is a noetherian algebra of Krull dimension 1.*

A result of Jategaonkar (cf. Krause and Lenagan [4, Proposition 4.13]) shows that a domain of finite GK dimension is Goldie. Hence we get the following corollary.

Corollary 1.2. *Let k be a field and let A be a finitely generated simple k -algebra of quadratic growth that is a domain. Then A is a noetherian algebra of Krull dimension 1. In particular, every left and right ideal can be generated by two elements.*

The fact that every right and left ideal of a simple noetherian algebra of Krull dimension one is generated by at most two elements follows from a result of Stafford [8]. Another interesting corollary we obtain is the following.

Corollary 1.3. *Let A be a finitely generated simple algebra of quadratic growth. Then A is left noetherian if and only if A is right noetherian.*

The outline of the paper is as follows. In section 2, we give proofs of Theorem 1.1 and Corollary 1.2. In section 3, we give some examples in which we consider what happens when various hypotheses are relaxed and give some concluding remarks.

2. PROOFS

In this section we give the proofs of Theorem 1.1 and Corollary 1.2. We begin with a few estimates for algebras of quadratic growth.

Lemma 2.1. *Let K be a field, let A be a finitely generated K -algebra with quadratic growth, and let m be a positive integer. If V is a finite-dimensional K -subspace of*

¹We take noetherian to mean satisfying the ascending chain condition on *both* left and right ideals.

A that generates A as a K -algebra, then there is a positive constant $C = C(m, V)$ such that

$$\dim(V^n) - \dim(V^{n-m}) \leq Cn$$

for infinitely many $n \geq 1$.

Proof. By assumption, A has quadratic growth and hence there is a positive constant C_0 such that

$$(2.1) \quad \dim(V^n) \leq C_0 n^2 \quad \text{for } n \text{ sufficiently large.}$$

We take

$$(2.2) \quad C = 4C_0 m.$$

Suppose that there is a number N such that

$$\dim(V^n) - \dim(V^{n-m}) > Cn$$

for all $n > N$. Then

$$\begin{aligned} \dim(V^{nm+N}) &\geq \dim(V^{nm+N}) - \dim(V^N) \\ &= \sum_{j=0}^{n-1} (\dim(V^{N+jm+m}) - \dim(V^{N+jm})) \\ &\geq \sum_{j=0}^{n-1} C(N+jm+m) \\ &= CnN + Cm(n+1)n/2 \\ &\geq 2C_0 n^2 m^2 \\ &> C_0 (nm+N)^2 \quad (\text{for } n \text{ sufficiently large}). \end{aligned}$$

This is a contradiction. The result follows. \square

Our next lemma gives an estimate for the growth of A/I when I contains a regular element a . We recall that an element a is *regular* if its right and left annihilators are both (0) .

Lemma 2.2. *Let k be a field and let A be a prime finitely generated k -algebra with quadratic growth. Let V be a finite-dimensional generating subspace of A that contains 1. If I is a right ideal of A and I contains a regular element a , then there is a positive constant C such that*

$$\dim(V^n/V^n \cap I) \leq Cn$$

for infinitely many n .

Proof. By assumption A has quadratic growth and hence there is a positive constant C_0 such that

$$\dim(V^n) \leq C_0 n^2 \quad \text{for } n \geq 1.$$

Let I be a nonzero left ideal of A and suppose there is a regular element $a \in I$. Then $a \in V^m$ for some natural number m . Then

$$\begin{aligned} \dim(V^n/(V^n \cap I)) &= \dim(V^n) - \dim(V^n \cap I) \\ &\leq \dim(V^n) - \dim(V^n \cap aA) \\ &\leq \dim(V^n) - \dim(aV^{n-m}) \\ &= \dim(V^n) - \dim(V^{n-m}). \end{aligned}$$

By the preceding lemma there is a positive constant C such that

$$\dim(V^n) - \dim(V^{n-m}) \leq Cn$$

for infinitely many n . The result follows. \square

Lemma 2.3. *Let K be a field and let A be an infinite simple affine K -algebra. Let V be a finite-dimensional generating subspace of A which contains 1. If $I \subseteq J$ are two right ideals in A and $I \neq J$, then there is $m > 0$ such that*

$$\dim(V^n \cap J/V^n \cap I) \geq n - m$$

for all $n \geq 0$.

Proof. Pick $a \in J \setminus I$. Without loss of generality, we may assume $J = I + aA$. Note that if $I + aV^m \supseteq I + aV^{m+1}$, then $I + aA \subseteq I + aV^m$, and hence J/I is finite dimensional as a K -vector space. But this says that A has a finite-dimensional nonzero right A -module, which is impossible since A is infinite and simple. It follows that

$$\dim(I + aV^n/I) \geq n$$

for all $n \geq 0$. Since V is a generating subspace, there is some m such that $a \in V^m$. Hence

$$\dim(J \cap V^{n+m}/I \cap V^{n+m}) \geq \dim((I + aV^n)/I) \geq n.$$

The result follows. \square

Proof of Theorem 1.1. Let V be a finite-dimensional K -subspace of A that generates A as a K -algebra. By assumption A has quadratic growth and hence there is a positive constant C such that

$$\dim(V^n) \leq Cn^2 \quad \text{for } n \geq 1.$$

Let I be a nonzero right ideal. By Lemma 2.2 there is a positive constant $C = C(I, V)$ such that

$$\dim(V^n/(V^n \cap I)) \leq Cn$$

for infinitely many n . Let m denote the *uniform dimension* of A (see McConnell and Robson [5, 2.2.10]). We claim that if

$$I = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_e$$

is an ascending chain of ideals in which each containment is proper, then $e \leq mC$. If not, there must exist i and j such that $d := j - i > C$ and the uniform dimension of A/I_i is the same as the uniform dimension of A/I_j . Thus there exists a right ideal J such that $I_\ell \cap J = (0)$ for $i \leq \ell \leq j$ and $I_\ell + J$ is essential (that is, it contains a regular element) for $i \leq \ell \leq j$. Let

$$J_\ell = I_{i+\ell} + J$$

for $0 \leq \ell \leq d$.

By Lemma 2.3 there are positive integers m_1, m_2, \dots, m_d such that

$$\dim(V^n \cap J_j/V^n \cap J_{j-1}) \geq n - m_j$$

for all n . Hence

$$\begin{aligned} \dim(V^n) &\geq \dim(V^n \cap J_d) \\ &= \dim(V^n \cap J_0) + \sum_{i=1}^d \dim(V^n \cap J_i / V^n \cap J_{i-1}) \\ &\geq \dim(V^n \cap J_0) + \sum_{i=1}^d (n - m_i) \\ &\geq \dim(V^n \cap J_0) + dn - (m_1 + \cdots + m_d). \end{aligned}$$

Thus

$$\dim(V^n) - \dim(V^n \cap J_0) \geq dn - (m_1 + \cdots + m_d)$$

for all n . But J_0 contains a regular element and hence for infinitely many n we have

$$\dim(V^n) - \dim(V^n \cap J_0) \leq Cn$$

by Lemma 2.2. Thus $d \leq C$, since m_1, \dots, m_d are fixed. It follows that A is right noetherian. By looking at the opposite ring, we see A is also left noetherian. To see that A has Krull dimension 1, note that if $J \supset I$, and I is nonzero, then any chain of submodules of J/I in which containments are proper must have length at most $C = mC(I, V)$. In particular, J/I is artinian. Hence A has Krull dimension 1. By a result of Stafford, every right and left ideal of A is generated by at most two elements. \square

Proof of Corollary 1.2. This follows immediately from Theorem 1.1 and Proposition 4.13 of Krause and Lenagan [4]. \square

Proof of Corollary 1.3. By a result of Irving and Small [3], a simple algebra of quadratic growth is right Goldie if and only if it is left Goldie. If such an algebra is either right or left noetherian, then it is Goldie and so it is noetherian on both sides. \square

3. CONCLUDING REMARKS

We note that the finitely generated hypothesis is necessary in Theorem 1.1 to ensure we obtain noetherianity. Let

$$B = k[t, t^{1/2}, t^{1/4}, \dots],$$

and let $A = B[x; \delta]$, where δ is differentiation with respect to t . Then A is a free B -module and since B is not noetherian, neither is A . We note that A is a simple domain and any finitely generated subalgebra of A has quadratic growth.

We note that prime Goldie algebras of quadratic growth need not be noetherian in general. To see this, let

$$A = \begin{pmatrix} k[x, y, y^{-1}] & k[x, y, y^{-1}]x \\ k[x, y, y^{-1}] & k[y] + k[x, y, y^{-1}]x \end{pmatrix}.$$

We note that A is a finitely generated non-noetherian prime Goldie k -algebra (cf. McConnell and Robson [5, 13.10.2]). It is straightforward to check that A has quadratic growth. A non-PI example of a non-noetherian finitely generated simple Goldie algebra of quadratic growth can be constructed over any field k that is not algebraic over a finite field by using $k_q[x, y, y^{-1}]$ in A instead of $k[x, y, y^{-1}]$, where $k_q[x, y, y^{-1}]$ is a localization of the *quantum plane*; that is, $xy = qyx$, with q nonzero

and not a root of unity. This type of non-PI construction cannot be made over a finite field since every nonzero element of a finite field is a root of unity and so the resulting algebra is PI. Smoktunowicz [6] has shown that in many cases, a finitely generated prime algebra over a finite field satisfies a polynomial identity. We thus make the following conjecture.

Conjecture 3.1. *Let k be a finite field and let A be a finitely generated k -algebra of quadratic growth. Then A is not simple.*

Artin and Stafford showed in the course of their description of graded domains of GK dimension two that all non-PI graded domains of quadratic growth that are generated in degree 1 are noetherian. It is an interesting question as to whether a similar description of the quotient division algebras of simple domains of quadratic growth can be given. More specifically, can it be shown that a simple domain of quadratic growth is birationally isomorphic to a graded domain of quadratic growth or to the ring of differential operators on a curve? Artin and Stafford also made the conjecture that a “gap” should exist in the possible GK dimensions that a graded domain can attain. Smoktunowicz [7] proved this conjecture. It would be interesting if a similar gap result could be attained for simple domains; that is, can it be shown that there are no finitely generated simple domains of GK dimension strictly between 2 and 3?

We also note that the techniques used in the proof of Theorem 1.1 do not extend to higher GK dimension. Nevertheless, we are unaware of a counter-example for algebras of GK dimension 3. We thus pose the following questions.

Question 3.2. Does there exist a non-noetherian finitely generated simple domain of GK dimension 3?

Question 3.3. Is the Krull dimension of a finitely generated simple noetherian algebra of GK dimension 3 at most 2?

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