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$(\mathcal{C}_p, \alpha)$-HYPONORMAL OPERATORS AND TRACE-CLASS SELF-COMMUTATORS WITH TRACE ZERO

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(Communicated by Nigel J. Kalton)

This paper is dedicated to the memory of my grandparents.

ABSTRACT. We define the class of $(\mathcal{C}_p, \alpha)$-hyponormal operators and study the inclusion between such classes under various hypotheses for p and α, and then obtain some sufficient conditions for the self-commutator of the Aluthge transform $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ of $(\mathcal{C}_p, \alpha)$-hyponormal operators to be in the trace-class and have trace zero.

1. INTRODUCTION

In this section we define some classes of operators. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For $\alpha > 0$ and $T \in \mathcal{L}(\mathcal{H})$, we call $(T^*T)^\alpha - (TT^*)^\alpha$ the *α-self-commutator* of T and denote it by D_T^α. Also, we will write $|T|$ for $(T^*T)^{1/2}$, $\sigma(T)$, $\sigma_e(T)$, and $\sigma_w(T)$ for the spectrum, essential spectrum, and Weyl spectrum of T, respectively. For a selfadjoint operator A in $\mathcal{L}(\mathcal{H})$ we write A_+, A_- for the positive and negative parts of A, that is, $(|A|+A)/2$, and $(|A|-A)/2$, respectively. We denote by $\mathbb{K}$ the ideal of all compact operators in $\mathcal{L}(\mathcal{H})$, and by $\mathcal{C}_p(\mathcal{H})$, $1 \le p < +\infty$, the ideal of operators in the Schatten p-class (cf. [9]). Although for $0 < p < 1$, the usual definition of $||\cdot||_p$ does not satisfy the triangle inequality, nevertheless $(\mathcal{C}_p, ||\cdot||_p)$ is closed and $||TK||_p \le ||T|| \cdot ||K||_p$, when $T \in \mathcal{L}(\mathcal{H})$ and $K \in \mathcal{C}_p(\mathcal{H})$. Recall that $\mathcal{C}_1(\mathcal{H})$ is the trace-class and that $\mathcal{C}_2(\mathcal{H})$ is the Hilbert-Schmidt class. We write $\operatorname{tr}(T)$ for the canonical scalar-valued trace of an operator T in $\mathcal{C}_1(\mathcal{H})$. We denote by π the natural surjection from $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra, $\mathcal{L}(\mathcal{H})/\mathbb{K}$, and by μ the planar Lebesgue measure. We say that an operator T in $\mathcal{L}(\mathcal{H})$ is *$(\mathcal{C}_p, \alpha)$-normal* if $D_T^\alpha \in \mathcal{C}_p(\mathcal{H})$, and denote the class of $(\mathcal{C}_p, \alpha)$-normal operators by $\mathcal{N}_p^\alpha(\mathcal{H})$. Moreover, an operator T in $\mathcal{L}(\mathcal{H})$ will be called *$(\mathcal{C}_p, \alpha)$-hyponormal* if $D_T^\alpha = P + K$, where P is a positive semidefinite operator ($P \ge 0$) and $K \in \mathcal{C}_p(\mathcal{H})$. The class of $(\mathcal{C}_p, \alpha)$-hyponormal operators will be denoted by $\mathcal{H}_p^\alpha(\mathcal{H})$. In particular, an operator T in $\mathcal{H}_1^1(\mathcal{H})$ will be called *almost hyponormal*. Furthermore, an operator $T \in \mathcal{L}(\mathcal{H})$ whose D_T^α is positive semidefinite is called *α-hyponormal* (notation: $T \in \mathcal{H}_0^\alpha(\mathcal{H})$).

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With only minor changes to the proof of Proposition 1.1 from [7], one can easily prove the following.

Proposition 1. *Let $A \in \mathcal{L}(\mathcal{H})$ be a selfadjoint operator and let $p > 0$. Then A can be written as $P + K$ with $P \geq 0$ and $K \in \mathcal{C}_p(\mathcal{H})$ if and only if $A_- \in \mathcal{C}_p(\mathcal{H})$.*

Consequently, an operator T belongs to $\mathcal{H}_p^\alpha(\mathcal{H})$ if and only if $(D_T^\alpha)_- \in \mathcal{C}_p(\mathcal{H})$.

2. Some inclusions

We will examine various inclusions between these classes of operators. According to Lowner's inequality ($A, B \in \mathcal{L}(\mathcal{H}),\ \ 0 \leq A \leq B,\ \ 0 < r \leq 1 \Rightarrow A^r \leq B^r$), we have the following inclusion, $\mathcal{H}_0^\alpha(\mathcal{H}) \supseteq \mathcal{H}_0^\beta(\mathcal{H})$, when $\alpha \leq \beta$. In the case of $(\mathcal{C}_p, \alpha)$-normal operators, and moreover, for $(\mathcal{C}_p, \alpha)$-hyponormal operators, the similar inclusion for such classes is less obvious. We will give some sufficient conditions when such an inclusion holds for $(\mathcal{C}_p, \alpha)$-normal operators and then for $(\mathcal{C}_p, \alpha)$-hyponormal operators. We will make use of the following.

Lemma 2. *Let $\alpha \geq 1$, $p \geq 1$, and $A, B \in \mathcal{L}(\mathcal{H})$ be positive semidefinite operators such that $A - B \in \mathcal{C}_p(\mathcal{H})$. Then $A^\alpha - B^\alpha \in \mathcal{C}_p(\mathcal{H})$.*

The proof of Lemma 2 uses the following general fact.

Lemma 3. *Let $p \geq 1$, $T \in \mathcal{L}(\mathcal{H})$ and $T_n \in \mathcal{C}_p(\mathcal{H})$, for all $n \in \mathbb{N}$, such that $T_n \xrightarrow{wo} T$ (i.e., weak operator topology) and such that $\|T_n\|_p \leq C < \infty$, for all $n \in \mathbb{N}$ and for some non-negative constant C. Then T belongs to $\mathcal{C}_p(\mathcal{H})$ and $\|T\|_p \leq C$.*

Proof of Lemma 3. We will prove that T belongs to $\mathcal{C}_p(\mathcal{H})$ by proving that

$$\sup\{|\operatorname{tr}(TK)| : \operatorname{rank}(K) < \infty \text{ and } \|K\|_q \leq 1\} < \infty, \tag{1}$$

and the above sup equals $\|T\|_p$, where q is the index conjugate to p (cf. [8], p. 90).

To each T_n one can associate $f_n \in \mathcal{C}_q(\mathcal{H})^*$ defined by $f_n(K) = \operatorname{tr}(T_n K)$. Since $\|f_n\| = \|T_n\|_p \leq C < \infty$, according to Alaoglu's theorem, there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \xrightarrow{w^*} f$, where $f \in \mathcal{C}_q(\mathcal{H})^*$. Therefore $\operatorname{tr}(T_{n_k} K) = f_{n_k}(K) \longrightarrow f(K)$, for all $K \in \mathcal{C}_q(\mathcal{H})$, and $|f(K)| \leq M\|K\|_q$, for some positive constant M. On the other hand, since $T_n \xrightarrow{wo} T$, $\operatorname{tr}(T_n K) \longrightarrow \operatorname{tr}(TK)$ for all operators K of finite rank. The statement follows easily from (1). □

Proof of Lemma 2. Let α, p, A, and B be as in the hypotheses of Lemma 2. For purposes of proving that $A^\alpha - B^\alpha \in \mathcal{C}_p(\mathcal{H})$, we may assume that $\|A\|$ and $\|B\|$ are less than 1, since otherwise we may divide the norm of each operator by a sufficiently large constant. Put $A_n = A + \frac{1}{n}I$, $B_n = B + \frac{1}{n}I$, for $n \geq n_0$, where n_0 is sufficiently large so that $\|A_{n_0}\|$ and $\|B_{n_0}\| < 1$. Put $T_n = A_n^\alpha - B_n^\alpha$ and $T = A^\alpha - B^\alpha$ and observe that $T_n \to T$ in norm. We will prove that $T_n \in \mathcal{C}_p(\mathcal{H})$, and $\|T_n\|_p \leq C$, for $n \geq n_0$, for some $C < \infty$. For an operator $X = X^* \in \mathcal{L}(\mathcal{H})$ with $\sigma(X) \subseteq (0, 1)$, we may write

$$X^\alpha = I + \sum_{k=1}^{\infty} \binom{\alpha}{k} (X - I)^k,$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha - 1)\dots(\alpha - k + 1)}{k!},$$

and the series above converges in the operator norm. Using this representation with $X = A_n$ and $X = B_n$ and then subtracting one from the other, we obtain

$$T_n = \sum_{k=1}^{\infty} \binom{\alpha}{k} [(A_n - I)^k - (B_n - I)^k].$$

Expressing $(A_n - I)^k - (B_n - I)^k$ as a telescopic sum and using the inequality

$$||RS||_p \leq ||R|| \cdot ||S||_p, \text{ for } R \in \mathcal{L}(\mathcal{H}),\ S \in \mathcal{C}_p(\mathcal{H}),$$

we obtain that $(A_n - I)^k - (B_n - I)^k \in \mathcal{C}_p(\mathcal{H})$ and

$$||(A_n - I)^k - (B_n - I)^k||_p \leq ||A - B||_p \, k \, q_n^{k-1},$$

where $q_n = \max\{||A_n - I||, ||B_n - I||\} < 1$. Thus

$$\begin{aligned}
||T_n||_p &\leq \sum_{k=1}^{\infty} |\tbinom{\alpha}{k}| \, ||A - B||_p \, k \, q_n^{k-1} \\
&= \alpha \, ||A - B||_p \big[1 + |\alpha - 1| \, q_n + \dots + \frac{|(\alpha - 1) \dots (\alpha - k + 1)|}{(k-1)!} \, q_n^{k-1} + \dots\big] \\
&= \alpha \, ||A - B||_p \big[1 + \sum_{k=1}^{[\alpha]} \frac{(\alpha - 1) \dots (\alpha - k)}{k!} \, q_n^k\big] \\
&+ \alpha \, ||A - B||_p \big[\sum_{k=[\alpha]+1}^{\infty} \frac{(\alpha - 1) \dots (\alpha - [\alpha]) |(\alpha - [\alpha] - 1) \dots (\alpha - k)|}{k!} \, q_n^k\big].
\end{aligned}$$

If the integer part of α, $[\alpha]$, is an even number written as $2k_0$, then the above sums, ignoring the factor $\alpha \, ||A - B||_p$, can be written as

$$\begin{aligned}
&\big[1 + \sum_{k=1}^{\infty} \frac{(\alpha - 1) \dots (\alpha - k)}{k!} \, (-q_n)^k + 2 \sum_{k=0}^{k_0} \frac{(\alpha - 1) \dots (\alpha - 2k - 1)}{(2k+1)!} \, q_n^{2k+1}\big] \\
&= \big[(1 - q_n)^{\alpha - 1} + 2 \sum_{k=0}^{k_0} \frac{(\alpha - 1) \dots (\alpha - 2k - 1)}{(2k+1)!} \, q_n^{2k+1}\big].
\end{aligned}$$

Since $q_n \in (0, 1)$, we can conclude that

$$||T_n||_p \leq \alpha ||A - B||_p \, \big[1 + \sum_{k=0}^{k_0} \frac{(\alpha - 1) \dots (\alpha - 2k - 1)}{(2k+1)!}\big],$$

when $[\alpha] = 2k_0$. The case when $[\alpha]$ is an odd number can be easily derived from the above case. Applying Lemma 3, the proof of Lemma 2 is complete. □

With only minor adaptations of the proof of Lemma 2, one can prove the following.

Corollary 4. *Let* $\alpha \in \mathbb{R}$, $p \geq 1$, *and* A, $B \in \mathcal{L}(\mathcal{H})$ *be invertible positive definite operators such that* $A - B \in \mathcal{C}_p(\mathcal{H})$. *Then* $A^\alpha - B^\alpha \in \mathcal{C}_p(\mathcal{H})$.

In the following proposition we study how the class of $(\mathcal{C}_p, \alpha)$-normal operators varies when α changes.

Proposition 5. *Let $\alpha > 0$, $p \geq 1$, and let T be in $\mathcal{N}_p^{\alpha}(\mathcal{H})$.*
(a) *If $\beta \geq \alpha$, then T belongs to $\mathcal{N}_p^{\beta}(\mathcal{H})$, and therefore $\mathcal{N}_p^{\alpha}(\mathcal{H}) \subseteq \mathcal{N}_p^{\beta}(\mathcal{H})$.*
(b) *If either T^*T or TT^* is a semi-Fredholm operator and $0 < \gamma \leq \alpha$, then T belongs to $\mathcal{N}_p^{\gamma}(\mathcal{H})$.*

Denote by $Q_0(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) \mid T^*T \text{ or } TT^* \text{ is semi-Fredholm}\}$. An alternative characterization of $Q_0(\mathcal{H})$ is $Q_0(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) \mid 0 \in \rho_{le}(T) \cup \rho_{re}(T)\}$, where $\rho_{le}(T)$, $\rho_{re}(T)$ are the left essential and right essential resolvents of the operator $T \in \mathcal{L}(\mathcal{H})$, respectively.

Corollary 6. *Let $p \geq 1$ and $\alpha, \beta > 0$. Then $\mathcal{N}_p^{\alpha}(\mathcal{H}) \cap Q_0(\mathcal{H}) = \mathcal{N}_p^{\beta}(\mathcal{H}) \cap Q_0(\mathcal{H})$.*

Proof of Proposition 5. Let α, p, and T be as in the hypotheses and let $T = U|T|$ be the polar decomposition of T, and set $S := U|T|^{\alpha}$. Then obviously, $S^*S = |T|^{2\alpha} = (T^*T)^{\alpha}$ and $SS^* = U|T|^{2\alpha}U^* = (TT^*)^{\alpha}$, and therefore, $[S^*, S] = D_T^{\alpha} = K \in \mathcal{C}_p(\mathcal{H})$. On the other hand,

$$D_S^r = (S^*S)^r - (SS^*)^r = (T^*T)^{\alpha r} - (TT^*)^{\alpha r} = D_T^{\alpha r}.$$

Let $\beta \geq \alpha$ and put $r = \frac{\beta}{\alpha} \geq 1$ and apply Lemma 2 to conclude (a). To prove (b), we assume that T^*T is semi-Fredholm; the proof when TT^* is semi-Fredholm is similar. Indeed, when T^*T is semi-Fredholm, obviously T^*T is Fredholm; i.e., $\pi(T^*T)$ is invertible in the Calkin algebra. Let $\pi(X)$ be the inverse of $\pi(T^*T)$ in the Calkin algebra; then $\pi(X)$ is a positive element and $\pi(X)^s\pi(T^*T)^s = \pi(T^*T)^s\pi(X)^s = I_{Calkin}$, for any $s \geq 0$. In particular, for $s = 2\alpha$, $\pi(T^*T)^{2\alpha} = \pi(S^*S)$ is invertible in the Calkin algebra. Thus, for any $r \geq 0$, there exist some operators A_r and B_r in $\mathcal{L}(\mathcal{H})$ so that

$$(S^*S)^r \cdot A_r = I + K_r^1 \text{ and } B_r \cdot (S^*S)^r = I + K_r^2, \tag{2}$$

with K_r^1, K_r^2 of finite rank, thus in $\mathcal{C}_1(\mathcal{H})$. Since $[S^*, S] = K \in \mathcal{C}_p(\mathcal{H})$, according to the argument used above, $D_S^q \in \mathcal{C}_p(\mathcal{H})$, for any $q \geq 1$. We prove that $D_S^{\{q\}}$ belongs to $\mathcal{C}_p(\mathcal{H})$, for any $q \geq 1$, where $q = [q] + \{q\}$ is the decomposition of q into its integer and fractional part. Indeed,

$$\begin{aligned} D_S^q &= (S^*S)^q - (SS^*)^{[q]+\{q\}} \\ &= (S^*S)^q - (S^*S - K)^{[q]}(SS^*)^{\{q\}} \\ &= (S^*S)^q - [(S^*S)^{[q]} + K'](SS^*)^{\{q\}} \\ &= (S^*S)^{[q]}D_S^{\{q\}} + K'', \end{aligned}$$

where K, K', K'' are in $\mathcal{C}_p(\mathcal{H})$. Multiplying the equality

$$D_S^q = (S^*S)^{[q]}D_S^{\{q\}} + K''$$

by $B_{[q]}$ and using the fact that $D_S^q \in \mathcal{C}_p(\mathcal{H})$, we obtain according to (2) that $D_S^{\{q\}}$ belongs to $\mathcal{C}_p(\mathcal{H})$, for any $q \geq 1$; therefore, D_S^r belongs to $\mathcal{C}_p(\mathcal{H})$, for any $0 \leq r \leq 1$. Therefore, for $r = \frac{\gamma}{\alpha}$, we have $D_S^r = D_T^{\alpha r} = D_T^{\gamma} \in \mathcal{C}_p(\mathcal{H})$, and (b) is established. □

Next we study the class of $(\mathcal{C}_p, \alpha)$-hyponormal operators. Since the class $\mathcal{H}_0^{\alpha}(\mathcal{H})$ is monotone decreasing (as a subset) in terms of α, we can only expect that the class $\mathcal{H}_p^{\alpha}(\mathcal{H})$ will be monotone decreasing.

Proposition 7. *Let $\alpha > 0,\ p \geq 1$, and let $T \in \mathcal{H}_p^\alpha(\mathcal{H})$ with $D_T^\alpha = P + K,\ \ P \geq 0$, $K \in \mathcal{C}_p(\mathcal{H})$. If $0 < \beta \leq \alpha$ and if one of the following is satisfied:*
(a) *either T^*T or TT^* is a semi-Fredholm operator or*
(b) *both $(T^*T)^\alpha$ and $(TT^*)^\alpha + P$ are invertible,*
then T belongs to $\mathcal{H}_p^\beta(\mathcal{H})$.

Proof. Let $\alpha,\ p$, and T be as in the hypotheses and let $T = U|T|$ be the polar decomposition of T, and put $S = U|T|^\alpha$. The calculations used in the proof of Proposition 5 show that S belongs to $\mathcal{H}_p^1(\mathcal{H})$, and according to Proposition 1, $S^*S - (SS^* + P) = K$, with $P \geq 0$ and $K \in \mathcal{C}_p(\mathcal{H})$. If either T^*T or TT^* is a semi-Fredholm operator, then using the same circle of ideas as in the proof of Proposition 5, one can conclude that

$$(S^*S)^{\frac{\beta}{\alpha}} - (SS^* + P)^{\frac{\beta}{\alpha}} = K',$$

with $K' \in \mathcal{C}_p(\mathcal{H})$. On the other hand, using Lowner's inequality, one can write

$$(SS^* + P)^{\frac{\beta}{\alpha}} = (SS^*)^{\frac{\beta}{\alpha}} + P',$$

with $P' \geq 0$. These two equalities can be written in terms of operators T and T^* as

$$(T^*T)^\beta - (TT^*)^\beta = P' + K',$$

which, according to Proposition 1, implies that $T \in \mathcal{H}_p^\beta(\mathcal{H})$. This ends the proof under assumption (a). The proof with assumption (b) makes use of Corollary 4 and is left for the reader. □

Corollary 8. *Let $\alpha \geq \beta > 0,\ p \geq 1$. Then $\mathcal{H}_p^\alpha(\mathcal{H}) \cap Q_0(\mathcal{H}) \subseteq \mathcal{H}_p^\beta(\mathcal{H}) \cap Q_0(\mathcal{H})$.*

Proof. Apply part (a) of Proposition 7. □

In section 3 we will use the lemmas below, one of them being a consequence of the following corollary. This corollary is a consequence of Theorem 3.4 of [5].

Corollary 9. *Let A, $B \in \mathcal{L}(\mathcal{H})$ be positive semidefinite operators. If $\alpha \in (0,1]$ and $1 \leq p < \infty$, then*

$$||B^\alpha - A^\alpha||_p \leq ||\,|B - A|^\alpha\,||_p.$$

Lemma 10. *Let $A \in \mathcal{L}(\mathcal{H}),\ A \geq 0,\ \alpha \in (0,1],\ p \geq \alpha,\ K \in \mathcal{C}_p(\mathcal{H})$, such that $A + K \geq 0$. Then $(A+K)^\alpha = A^\alpha + K_1$, where $K_1 \in \mathcal{C}_{\frac{p}{\alpha}}(\mathcal{H})$. If in addition $K \geq 0$, then $K_1 \geq 0$.*

Proof. Set $K_1 := (A+K)^\alpha - A^\alpha$. From Corollary 9 one obtains

$$||K_1||_{\frac{p}{\alpha}} \leq ||\,|K|^\alpha||_{\frac{p}{\alpha}} = ||K||_p^\alpha < \infty,$$

which implies $K_1 \in \mathcal{C}_{\frac{p}{\alpha}}(\mathcal{H})$.

If $K \geq 0$, then we can apply Lowner's inequality to $A + K$ and A and obtain $(A+K)^\alpha \geq A^\alpha$. Therefore $K_1 \geq 0$.

□

Lemma 11. *Let $A \in \mathcal{L}(\mathcal{H}),\ A \geq 0,\ p \geq 1,\ K \in \mathcal{C}_p(\mathcal{H})$, such that $A + K \geq 0$, and let $\alpha \in [1, +\infty)$. Then $(A+K)^\alpha = A^\alpha + K_1$, where $K_1 \in \mathcal{C}_p(\mathcal{H})$.*

Proof. Apply Lemma 2. □

3. Application

In [4] the following sufficient condition for an almost hyponormal operator to have trace-class self-commutator with trace zero was obtained.

Theorem A ([4])**.** *If* $T \in \mathcal{H}_1^1(\mathcal{H})$ *and* $\mu(\sigma_w(T)) = 0$, *then* $T \in \mathcal{N}_1^1(\mathcal{H})$ *and*

$$\operatorname{tr}(D_T^1) = 0.$$

In [1] the following was obtained.

Theorem B ([1])**.** *If* $T \in \mathcal{H}_0^\alpha(\mathcal{H})$ *for some* $\alpha \in (0,1]$, *then*

$$||D_T^\alpha|| \le \frac{\alpha}{\pi} \iint_{\sigma_w(T)} r^{2\alpha-1}\, dr\, d\theta.$$

An obvious consequence of Theorem B is the following.

Corollary 12. *If* $T \in \mathcal{H}_0^\alpha(\mathcal{H})$ *for some* $\alpha > 0$ *and* $\mu(\sigma_w(T)) = 0$, *then* T *is normal.*

The above results naturally lead to the following.

Question. Let T be in $\mathcal{H}_p^\alpha(\mathcal{H})$ for some $\alpha > 0,\ p > 0$, and such that $\mu(\sigma_w(T)) = 0$. Does this imply that T or some transform of T, say $\phi(T)$, belongs to $\mathcal{N}_1^\beta(\mathcal{H})$, for some β, and $\operatorname{tr}(D_{\phi(T)}^\beta) = 0$?

This question is also justified by Theorem C below. For a subset E of $\mathbb{C}$, let

$$\omega_p(E) = \frac{p}{2} \int\int_E \rho^{p-1}\, d\rho\, d\theta,$$

and for $T \in \mathcal{L}(\mathcal{H})$, let $m(T)$ be the rational cyclicity of T, that is, the least cardinal number of a set $\mathcal{M} \subseteq \mathcal{H}$ such that $\bigvee\{r(T)x :\ r \in \operatorname{Rat}(\sigma(T)),\ x \in \mathcal{M}\} = \mathcal{H}$.

Theorem C ([2])**.** *Let* $T \in \mathcal{L}(\mathcal{H})$ *and* $\frac{1}{2} \le \alpha < \infty$.

(a) *If* $\frac{1}{2} \le \alpha \le 1$ *and* $T \in \mathcal{H}_1^\alpha(\mathcal{H})$, *and* $K \in \mathcal{C}_{2\alpha}(\mathcal{H})$, *then*

$$\operatorname{tr}(D_T^\alpha) \le \frac{1}{\pi}\, m(T+K)\, \omega_{2\alpha}(\sigma(T+K)).$$

(b) *If* $1 \le \alpha < \infty$ *and* $T \in \mathcal{H}_0^\alpha(\mathcal{H})$, *then*

$$\operatorname{tr}(D_T^\alpha) \le \frac{1}{\pi}\, m(T)\, \omega_{2\alpha}(\sigma(T)).$$

Part (b) of Theorem C with the additional hypotheses that $\mu(\sigma_w(T)) = 0$ and $m(T) < \infty$ holds the same conclusion as Corollary 12. Indeed, if $T \in \mathcal{H}_0^\alpha(\mathcal{H})$ for some $\alpha \ge 1$, then T is a hyponormal operator. It is now well known that for some class of operators, including the hyponormal ones, Weyl's theorem holds; that is,

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\pi_{00}(T)$ is the set of isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. Therefore, $\mu(\sigma_w(T)) = 0$ implies that $\mu(\sigma(T)) = 0$, and thus $\operatorname{tr}(D_T^1) = 0$, that is, $D_T^1 = 0$.

On the other hand, concerning part (a) of Theorem C, J. Stampfli in [10] proved that for $T \in \mathcal{L}(\mathcal{H})$, there exists a compact operator K such that $\sigma(T+K) \setminus \sigma_w(T)$ consists of a countable set. In fact, the proof that was provided in [10] says more.

Lemma D ([10]). *Let $T \in \mathcal{L}(\mathcal{H})$ and $p \geq 1$. Then for any $\varepsilon > 0$, there exists $K \in \mathcal{C}_p(\mathcal{H})$ such that $||K||_p < \varepsilon$ and $\sigma(T+K) \setminus \sigma_w(T)$ consists of a countable set which clusters only on $\sigma_w(T)$.*

Consequently, for an operator $T \in \mathcal{H}_1^\alpha(\mathcal{H})$, for some $\alpha \in [\frac{1}{2}, 1]$, and the operator K of Lemma D, we have $\omega_{2\alpha}(\sigma(T+K)) = 0$, provided that $\mu(\sigma_w(T)) = 0$. If in addition $m(T+K) < \infty$, then according to part (a) of Theorem C, $T \in \mathcal{N}_1^1(\mathcal{H})$ and $\operatorname{tr}(D_T^\alpha) = 0$.

We make a modest contribution towards answering the above question. Let T belong to $\mathcal{H}_p^\alpha(\mathcal{H})$, for some $\alpha > 0$, $p > 0$, such that $D_T^\alpha = P + K$ with $P \geq 0$ and $K \in \mathcal{C}_p(\mathcal{H})$. Since $K = K^* = K_+ - K_-$ and $K_+,\ K_- \geq 0$ are $\mathcal{C}_p$-class operators, in what follows we will assume that $D_T^\alpha = P - K$ with $P \geq 0$ and $K \geq 0,\ K \in \mathcal{C}_p(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $T = U|T|$ be the polar decomposition of T and write $\tilde{T}$ for the Aluthge transform of T, that is, $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.

Theorem 13. *Let $p > 0$, $\alpha \in [\frac{1}{2}, 1]$, and $T \in \mathcal{H}_p^\alpha(\mathcal{H})$ such that $D_T^\alpha = P - K$ with $P,\ K \geq 0$, $K \in \mathcal{C}_p(\mathcal{H})$, and let $\varepsilon \in (0, \frac{1}{2}]$ such that $\alpha + \varepsilon \leq 1$. Then $\tilde{T} \in \mathcal{H}_{(\frac{4\alpha p}{\varepsilon+\alpha})}^{(\alpha+\varepsilon)}(\mathcal{H})$.*

In proving Theorem 13 we will make use of an elementary lemma (Lemma 14, whose proof is omitted) and of Furuta's inequalities [3].

Lemma 14. *For $T \in \mathcal{L}(\mathcal{H})$ there exists a Hilbert space $\mathcal{K}$ that includes $\mathcal{H}$ and an operator $A \in \mathcal{L}(\mathcal{K})$ such that $D_T^\alpha \oplus 0_{\mathcal{K}\ominus\mathcal{H}} = D_A^\alpha$, for any $\alpha > 0$, and $\sigma(T) \setminus \{0\} = \sigma(A) \setminus \{0\}$, where $A = U|A|$ with U unitary.*

Theorem E ([3]). *For operators $E \geq F \geq 0$ in $\mathcal{L}(\mathcal{H})$ and $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p + 2r$, we have*

(F_1) $\quad (F^r E^p F^r)^{\frac{1}{q}} \geq (F^{p+2r})^{\frac{1}{q}}$,

(F_2) $\quad (E^{p+2r})^{\frac{1}{q}} \geq (E^r F^p E^r)^{\frac{1}{q}}$.

Proof of Theorem 13. Let T be as in the hypotheses of Theorem 13. According to Lemma 14, we may assume that $T = U|T|$ with U unitary. The equality $D_T^\alpha = P - K$ with $P,\ K \geq 0$ implies $|T|^{2\alpha} + K \geq U|T|^{2\alpha}U^*$. Multiplying this inequality by U^* to the left and by U to the right, one obtains

$$A := U^*|T|^{2\alpha}U + U^*KU \geq |T|^{2\alpha} =: B.$$

According to Lemma 10,

$$A^{\frac{1}{2\alpha}} = [U^*(|T|^{2\alpha} + K)U]^{\frac{1}{2\alpha}} = U^*(|T|^{2\alpha} + K)^{\frac{1}{2\alpha}}U = U^*(|T| + K_1)U,$$

with $K_1 \in \mathcal{C}_{2\alpha p}^+(\mathcal{H})$. Setting $K_2 := |T|^{\frac{1}{2}}U^*K_1U|T|^{\frac{1}{2}}$, we have

$$\begin{aligned}
\left(\tilde{T}^*\tilde{T} + K_2\right)^{\alpha+\varepsilon} &= \left\{|T|^{\frac{1}{2}}\left[U^*(|T| + K_1)U\right]|T|^{\frac{1}{2}}\right\}^{\alpha+\varepsilon} \\
&= \left\{|T|^{\frac{1}{2}}\left[U^*(|T|^{2\alpha} + K)U\right]^{\frac{1}{2\alpha}}|T|^{\frac{1}{2}}\right\}^{\alpha+\varepsilon} \\
&= \left(B^{\frac{1}{4\alpha}}A^{\frac{1}{2\alpha}}B^{\frac{1}{4\alpha}}\right)^{\alpha+\varepsilon} \\
&\overset{(F_1)}{\geq} (B^{\frac{1}{\alpha}})^{\alpha+\varepsilon} = |T|^{2(\alpha+\varepsilon)}.
\end{aligned}$$

On the other hand, according to Lemma 10,

$$(\tilde{T}^* \tilde{T} + K_2)^{\alpha+\varepsilon} = (\tilde{T}^* \tilde{T})^{\alpha+\varepsilon} + K_3,$$

with $K_3 \in \mathcal{C}^+_{\frac{2\alpha p}{\alpha+\varepsilon}}(\mathcal{H})$ since $K_2 \in \mathcal{C}^+_{2\alpha p}(\mathcal{H})$. Thus we have obtained the inequality

$$(*) \qquad (\tilde{T}^* \tilde{T})^{\alpha+\varepsilon} + K_3 \geq |T|^{2(\alpha+\varepsilon)}, \qquad K_3 \in \mathcal{C}^+_{\frac{2\alpha p}{\alpha+\varepsilon}}(\mathcal{H}).$$

On the other hand, the inequality

$$D := U|T|^{2\alpha}U^* \leq |T|^{2\alpha} + K =: C$$

can be used in conjunction with (F_2) to obtain a similar inequality to (*). Indeed, we have

$$(C^{\frac{1}{4\alpha}} D^{\frac{1}{2\alpha}} C^{\frac{1}{4\alpha}})^{\alpha+\varepsilon} \overset{(F_2)}{\leq} (C^{\frac{1}{\alpha}})^{\alpha+\varepsilon}.$$

Next, we compute each side of the above inequality. Again, according to Lemma 10,

$$C^{\frac{1}{4\alpha}} = (|T|^{2\alpha} + K)^{\frac{1}{4\alpha}} = |T|^{\frac{1}{2}} + K_4,$$

with $K_4 \in \mathcal{C}^+_{4\alpha p}(\mathcal{H})$. Obviously, $D^{\frac{1}{2\alpha}} = U|T|U^*$. Therefore, the left-hand side of the above inequality becomes

$$\begin{aligned}
(C^{\frac{1}{4\alpha}} D^{\frac{1}{2\alpha}} C^{\frac{1}{4\alpha}})^{\alpha+\varepsilon} &= \Big[(|T|^{\frac{1}{2}} + K_4)(U|T|U^*)(|T|^{\frac{1}{2}} + K_4)\Big]^{\alpha+\varepsilon} \\
&= \Big(|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}} + K_5\Big)^{\alpha+\varepsilon}, \quad K_5 \in \mathcal{C}_{4\alpha p}(\mathcal{H}) \\
&= \Big(\tilde{T}\tilde{T}^* + K_5\Big)^{\alpha+\varepsilon} \\
&= (\tilde{T}\tilde{T}^*)^{\alpha+\varepsilon} + K_6, \quad K_6 \in \mathcal{C}_{\frac{4\alpha p}{\alpha+\varepsilon}}(\mathcal{H}).
\end{aligned}$$

The right-hand side of the above inequality can be handled with Lemmas 10 and 11 as follows:

$$(C^{\frac{1}{\alpha}})^{\alpha+\varepsilon} \overset{L11}{=} (|T|^2 + K_7)^{\alpha+\varepsilon} \overset{L10}{=} |T|^{2(\alpha+\varepsilon)} + K_8,$$

with $K_7 \in \mathcal{C}_p(\mathcal{H})$ and $K_8 \in \mathcal{C}_{\frac{p}{\alpha+\varepsilon}}(\mathcal{H})$. Thus

$$|T|^{2(\alpha+\varepsilon)} + K_8 \geq (\tilde{T}\tilde{T}^*)^{\alpha+\varepsilon} + K_6,$$

where $K_6 \in \mathcal{C}_{\frac{4\alpha p}{\alpha+\varepsilon}}(\mathcal{H})$ and $K_8 \in \mathcal{C}_{\frac{p}{\alpha+\varepsilon}}(\mathcal{H})$, which implies

$$(**) \qquad |T|^{2(\alpha+\varepsilon)} \geq (\tilde{T}\tilde{T}^*)^{\alpha+\varepsilon} + K_9, \quad K_9 = K_6 - K_8 \in \mathcal{C}_{\frac{4\alpha p}{\alpha+\varepsilon}}(\mathcal{H}).$$

Combining (*) and (**) we obtain

$$(\tilde{T}^* \tilde{T})^{\alpha+\varepsilon} - (\tilde{T}\tilde{T}^*)^{\alpha+\varepsilon} \geq K_{10},$$

where $K_{10} = K_9 - K_3 \in \mathcal{C}_{\frac{4\alpha p}{\alpha+\varepsilon}}(\mathcal{H})$, and the proof is finished. □

Corollary 15. *Let* $T \in \mathcal{H}^{(1/2)}_{(1/2)}(\mathcal{H})$ *such that* $D_T^{\frac{1}{2}} = P - K$ *with* $P, K \geq 0$, $K \in \mathcal{C}_{\frac{1}{2}}(\mathcal{H})$. *Then* $\tilde{T} \in \mathcal{H}^1_1(\mathcal{H})$.

Theorem 16. *Let* $T \in \mathcal{H}^{(1/2)}_{(1/2)}(\mathcal{H})$ *such that* $D_T^{\frac{1}{2}} = P - K$ *with* $P, K \geq 0$, $K \in \mathcal{C}_{\frac{1}{2}}(\mathcal{H})$. *If* $\mu(\sigma_w(T)) = 0$, *then* $\tilde{T} \in \mathcal{N}^1_1(\mathcal{H})$ *and* $\operatorname{tr}(D^1_{\tilde{T}}) = 0$.

Proof. Let T be as in the hypotheses. According to Corollary 15, the operator $\tilde{T}$ is in $\mathcal{N}_1^1(\mathcal{H})$. Furthermore, according to Theorem 1.8 of [6], we obtain that $\mu(\sigma_w(\tilde{T})) = 0$. Then apply Theorem A to finish the proof. $\square$

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