THE AUSLANDER-REITEN CONJECTURE FOR GORENSTEIN RINGS

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ABSTRACT. The Nakayama conjecture is one of the most important conjectures in ring theory. The Auslander-Reiten conjecture is closely related to it. The purpose of this paper is to show that if the Auslander-Reiten conjecture holds in codimension one for a commutative Gorenstein ring R , then it holds for R .

1. INTRODUCTION

Over thirty years ago, Auslander and Reiten [4] conjectured the following, which is called *the generalized Nakayama conjecture*.

Conjecture 1. *Let Λ be an Artin algebra. Then any indecomposable injective Λ -module appears as a direct summand in the minimal injective resolution of Λ .*

They showed that the above conjecture holds for all Artin algebras if and only if the following conjecture holds for all Artin algebras.

Conjecture 2. *Let Λ be an Artin algebra and M a finitely generated Λ -module. If $\text{Ext}_{\Lambda}^i(M, M \oplus \Lambda) = 0$ for all $i > 0$, then M is projective.*

This long-standing conjecture is called *the Auslander-Reiten conjecture*. Auslander and Reiten [4] proved it for several classes of rings including rings of finite representation type and rings of radical square zero. About two decades later, Auslander, Ding and Solberg [3] investigated it for commutative rings. They studied the following condition on a ring, not necessarily an Artinian ring.

(ARC) Let R be a commutative Noetherian ring and M a finitely generated R -module. If $\text{Ext}_R^i(M, M \oplus R) = 0$ for all $i > 0$, then M is projective.

They proved that any complete intersection local ring R satisfies this condition. As a refinement of this result, Araya and Yoshino [1] showed that every local ring R satisfies the condition (ARC) for R -modules M of finite complete intersection dimension.

Conjecture 2 is also called *Tachikawa's conjecture* in the case where the ring Λ is selfinjective, and it is itself an important problem. The purpose of this paper is to explore Conjecture 2 for Gorenstein rings, which are a common generalization

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of a complete intersection ring and a commutative selfinjective ring. To be precise, a commutative Noetherian ring R is called *Gorenstein* if $R_{\mathfrak{p}}$ has finite injective dimension as an $R_{\mathfrak{p}}$ -module for all prime ideals $\mathfrak{p}$ of R . The following theorem is the main result of this paper.

Theorem 3. *Let R be a Gorenstein ring. If $R_{\mathfrak{p}}$ satisfies (ARC) for all prime ideals $\mathfrak{p}$ of height at most one, then $R_{\mathfrak{p}}$ satisfies (ARC) for all prime ideals $\mathfrak{p}$.*

Gorenstein normal rings, which are complete intersections in codimension one, are classical and important rings in commutative algebra and algebraic geometry. One can find the details in [6], for example. Since any normal ring satisfies Serre's condition (R_1) and any regular ring satisfies the condition (ARC), we get the following result as a corollary.

Corollary 4. *Every Gorenstein normal ring satisfies the condition (ARC).*

C. Huneke and G. J. Leuschke proved the following theorem [7, Theorem 0.1].

Theorem 5 (Huneke-Leuschke). *Let R be a Cohen-Macaulay ring which is a quotient of a locally excellent ring S of dimension d by a locally regular sequence. Assume that S is locally a complete intersection ring in codimension one, and further assume either that S is Gorenstein or that S contains the field of rational numbers. Then R satisfies (ARC).*

Since complete intersections satisfy (ARC), Theorem 3 actually shows that every Gorenstein ring which is a complete intersection in codimension one satisfies (ARC), and this does recover the special case of $R = S$ in Theorem 5, and without having to assume that S is locally excellent.

In the next section we shall prove Theorem 3. Our main instruments in the proof are stable cohomology modules.

2. PROOF OF THEOREM 3

To prove Theorem 3, notice that it is enough to show the local case. Throughout the rest of this note, let R be a commutative Gorenstein local ring of Krull dimension d with maximal ideal $\mathfrak{m}$. We begin with recalling the definition of a stable cohomology module.

Definition 6. Let M and N be maximal Cohen-Macaulay R -modules. We set $\underline{\mathrm{Hom}}_R(M, N) = \mathrm{Hom}_R(M, N)/\mathfrak{P}_R(M, N)$, where $\mathfrak{P}_R(M, N)$ is the R -submodule of $\mathrm{Hom}_R(M, N)$ consisting of all homomorphisms from M to N factoring through some finitely generated free R -module. For each integer i , we define the i -th stable cohomology module by $\widehat{\mathrm{Ext}}_R^i(M, N) = \underline{\mathrm{Hom}}_R(\Omega^i M, N)$.

We remark that, since R is Gorenstein, every maximal Cohen-Macaulay R -module M admits a *complete resolution*; namely, there exists an exact sequence $(F_{\bullet}, d_{\bullet})$ of finitely generated free R -modules with $M = \mathrm{Im} d_0$:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{d_0} & F_{-1} \xrightarrow{d_{-1}} F_{-2} \xrightarrow{d_{-2}} \cdots \\ & & & & \searrow & \nearrow & \\ & & & & & M & \end{array}$$

Then $\Omega^i M$ is defined as the image of the map d_i for every integer i . For the details of stable cohomology modules, see [5] for example.

We make a list of several basic properties of $\underline{\text{Hom}}$ and $\widehat{\text{Ext}}$. The details can be found, for instance, in [8, §7].

Proposition 7. *Let M, N be maximal Cohen-Macaulay R -modules.*

- (1) *One has $\widehat{\text{Ext}}_R^0(M, N) = \underline{\text{Hom}}_R(M, N)$.*
- (2) *For integers i, m and n , one has $\widehat{\text{Ext}}_R^i(M, N) \cong \widehat{\text{Ext}}_R^{i-m+n}(\Omega^m M, \Omega^n N)$. In particular, $\widehat{\text{Ext}}_R^i(M, N) \cong \text{Ext}_R^i(M, N)$ for all positive integers i .*
- (3) *One has $\underline{\text{Hom}}_R(M, M) = 0$ if and only if M is free.*

We say that a finitely generated R -module M is a *vector bundle* if $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module for all nonmaximal prime ideals $\mathfrak{p}$.

Theorem 8. *Let M be a maximal Cohen-Macaulay R -module which is a vector bundle. If $\widehat{\text{Ext}}_R^{d-1}(M, M) = 0$, then M is free.*

The lemma below, which is a kind of Auslander-Reiten-Serre duality theorem, is necessary to prove this theorem. The Auslander-Reiten-Serre duality was essentially given by Auslander [2, I.8.8, III.1.8].

Lemma 9. *Let M, N be maximal Cohen-Macaulay R -modules which are vector bundles. Then we have an isomorphism*

$$\text{Ext}_R^d(\underline{\text{Hom}}_R(N, M), R) \cong \widehat{\text{Ext}}_R^{d-1}(M, N).$$

Proof. This follows from [9, Lemma 3.10]. □

Proof of Theorem 8. Since M is a vector bundle, we have $\underline{\text{Hom}}_R(M, M)_{\mathfrak{p}} \cong \underline{\text{Hom}}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$ for every nonmaximal prime ideals $\mathfrak{p}$. Hence the R -module $\underline{\text{Hom}}_R(M, M)$ has finite length. The local duality and Lemma 9 yield isomorphisms $\underline{\text{Hom}}_R(M, M) \cong \text{Ext}_R^d(\text{Ext}_R^d(\underline{\text{Hom}}_R(M, M), R), R) \cong \text{Ext}_R^d(\widehat{\text{Ext}}_R^{d-1}(M, M), R)$. Therefore if $\widehat{\text{Ext}}_R^{d-1}(M, M) = 0$, then M is free. □

As an immediate consequence of Theorem 8, we obtain the following result.

Corollary 10. *Assume $d \geq 2$. Let M be a maximal Cohen-Macaulay R -module which is a vector bundle. If $\text{Ext}_R^{d-1}(M, M) = 0$, then M is free.*

The following example says that in Theorem 8 and Corollary 10 the assumption that M is a vector bundle cannot be removed.

Example 11. Let k be a field. Set $R = k[[x, y, z]]/(xy)$ and $M = R/(x)$. Then R is a Gorenstein local ring of Krull dimension 2 and $\text{Ext}_R^{2-1}(M, M) = 0$, but M is not free.

Now let us prove our main theorem.

Proof of Theorem 3. Let Φ be the set of prime ideals $\mathfrak{p}$ of R such that $R_{\mathfrak{p}}$ does not satisfy the condition (ARC). Assume that Φ is nonempty, and take a minimal element $\mathfrak{q}$ of Φ . Replacing R with $R_{\mathfrak{q}}$, we may assume that R is a Gorenstein local ring of dimension $d \geq 2$ which does not satisfy (ARC) such that $R_{\mathfrak{p}}$ satisfies (ARC) for all nonmaximal prime ideals $\mathfrak{p}$. Then there exists a nonfree finitely generated R -module M such that $\text{Ext}_R^i(M, M \oplus R) = 0$ for all $i > 0$. As $\text{Ext}_R^i(M, R) = 0$ for $i > 0$, M is maximal Cohen-Macaulay. For any nonmaximal prime ideal $\mathfrak{p}$, we have $\text{Ext}_{R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, M_{\mathfrak{p}} \oplus R_{\mathfrak{p}}) = 0$ for every $i > 0$ and $R_{\mathfrak{p}}$ satisfies (ARC), so the $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is free. Hence M is a vector bundle. Since $d - 1 > 0$, we have

$\text{Ext}_R^{d-1}(M, M) = 0$. Corollary 10 implies that M is a free R -module. Thus we get a contradiction, and the set Φ must be empty. $\square$

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