REDUCTION THEOREMS FOR NOETHER'S PROBLEM

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ABSTRACT. Let K be any field, and G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K -automorphisms and $h \cdot x(g) = x(hg)$. Denote by $K(G) = K(x(g) : g \in G)^G$ the fixed field. Noether's problem asks whether $K(G)$ is rational (= purely transcendental) over K . We will give several reduction theorems for solving Noether's problem. For example, let $\tilde{G} = G \times H$ be a direct product of finite groups. Theorem. Assume that $K(H)$ is rational over K . Then $K(\tilde{G})$ is rational over $K(G)$. In particular, if $K(G)$ is rational (resp. retract rational) over K , so is $K(\tilde{G})$ over K .

1. INTRODUCTION AND STATEMENTS

Let K be any field, and G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K -automorphisms and $h \cdot x(g) = x(hg)$. Denote by $K(G) = K(x(g) : g \in G)^G$ the fixed field. Noether's problem asks whether $K(G)$ is rational (= purely transcendental) over K . For a survey of Noether's problem, see Swan's paper [10].

The purpose of this article is to prove several reduction theorems when we try to solve Noether's problem for some group.

Our first result generalizes [6, Proposition 7] (the case $p = 2$).

Theorem 1.1. *Let K be a field with $\text{char } K = p > 0$ and $\tilde{G}$ be a group extension defined by $1 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \tilde{G} \rightarrow G \rightarrow 1$, where G is a finite group. Then $K(\tilde{G})$ is rational over $K(G)$.*

The meaning of the above conclusion is that there is a K -embedding of $K(G)$ into $K(\tilde{G})$, i.e. an injective K -linear homomorphism of fields from $K(G)$ into $K(\tilde{G})$, so that $K(\tilde{G})$ is rational over $K(G)$.

Note that, for any field K , if G and $\tilde{G}$ are finite groups so that $K(\tilde{G})$ is rational over $K(G)$, then $K(\tilde{G})$ is rational (resp. stably rational, retract rational) over K provided that so is $K(G)$. (Recall that “rational” $\Rightarrow$ “stably rational” $\Rightarrow$ “retract rational”. Here, a field L is stably rational over K if there exists a field which is rational over both L and K . For the definition of retract rationality, see [9, Definition 3.2].)

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Successive applications of the above theorem yield the following.

Corollary 1.2. *Let K be a field with $\text{char } K = p > 0$ and $\tilde{G}$ be a group extension defined by $1 \rightarrow H \rightarrow \tilde{G} \rightarrow G \rightarrow 1$, where H and G are finite groups. If either*

- (i) *H is a cyclic p -group or*
- (ii) *H is an abelian p -group lying in the center of $\tilde{G}$ or*
- (iii) *H is a p -group and $\tilde{G} \cong H \times G$,*

then $K(\tilde{G})$ is rational over $K(G)$.

Proof. In each of these cases, there is a subgroup $C \cong \mathbb{Z}/p\mathbb{Z}$ of H such that C is normal in $\tilde{G}$ and the extension $1 \rightarrow H/C \rightarrow \tilde{G}/C \rightarrow G \rightarrow 1$ belongs again to the same type (i), (ii) or (iii). $\square$

Case (iii) of this result may be thought of as a generalization of Kuniyoshi's Theorem: If K is a field with $\text{char } K = p > 0$ and G is a finite p -group, then $K(G)$ is rational over K [5]. Also, by Kuniyoshi's Theorem, one can view case (iii) of Corollary 1.2 as a consequence of our next result.

Theorem 1.3. *Let K be any field, and let H and G be finite groups. If $K(H)$ is rational (resp. stably rational, retract rational) over K , so is $K(H \times G)$ over $K(G)$.*

In particular, if both $K(H)$ and $K(G)$ are rational (resp. stably rational, retract rational) over K , so is $K(H \times G)$ over K .

Remark 1.4. Saltman already proved in [8, Theorem 1.5] that if $K(H)$ and $K(G)$ are retract rational over K , then so is $K(H \times G)$ over K .

Corollary 1.5. *Let $\tilde{G} = H \times G$ be a direct product of finite groups, and let K be a field. Assume that: (i) H is an abelian group with exponent e , i.e. $e = \max\{\text{ord}(h) : h \in H\}$; and (ii) the field K contains the e -th roots of unity. Then $K(\tilde{G})$ is rational over $K(G)$.*

In particular, $K((\mathbb{Z}/2\mathbb{Z}) \times G)$ is rational over $K(G)$ for every field K and every finite group G .

Proof. First, by Corollary 1.2 (iii), we can assume that the characteristic of K does not divide e . Then, in this case, the result follows from Theorem 1.3 and Fischer's Theorem: $K(H)$ is rational over K for every finite abelian group H of exponent e and every field K containing a primitive e -th root of unity [10, Theorem 6.1]. $\square$

Remark 1.6. We don't know whether Theorem 1.3 (or even Corollary 1.5) is valid for $\tilde{G}$, which is a semi-direct product but not a direct product. In fact, we don't know whether there exist distinct prime numbers p and q such that $\tilde{G} = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$ is a non-abelian semi-direct product and $\mathbb{C}(\tilde{G})$ is not rational over $\mathbb{C}$. Note that $\mathbb{C}(\tilde{G})$ is retract rational over $\mathbb{C}$ [8, Theorem 3.5].

However, consider the non-abelian group $\tilde{G} = \mathbb{Z}/17\mathbb{Z} \rtimes \mathbb{Z}/16\mathbb{Z}$, where $\mathbb{Z}/16\mathbb{Z}$ acts faithfully on $\mathbb{Z}/17\mathbb{Z}$. By [8, Theorems 3.1 and 5.11], $\mathbb{Q}(\tilde{G})$ is not even retract rational over $\mathbb{Q}$, while it is known that $\mathbb{C}(\tilde{G})$ is rational over $\mathbb{C}$ [2].

For wreath products, we will prove the following.

Theorem 1.7. *Let K be any field, and let $H \wr G$ be the wreath product of finite groups H and G . If $K(H)$ is rational (resp. stably rational) over K , so is $K(H \wr G)$ over $K(G)$.*

In particular, if both $K(H)$ and $K(G)$ are rational (resp. stably rational) over K , so is $K(H \wr G)$ over K .

Remark 1.8. Assuming that K is an infinite field, Saltman showed in [8, Theorem 3.3] that $K(H \wr G)$ is retract rational over K provided that so are $K(H)$ and $K(G)$.

Obvious combinations of the above theorems produce new rationality results as, for example, the following one.

Corollary 1.9. *Let K be a field, p a prime number, and G a p -Sylow subgroup of some finite symmetric group. If $K(\mathbb{Z}/p\mathbb{Z})$ is rational over K , so is $K(G)$ over K .*

In particular, if K is a field containing the p -th roots of unity, then $K(G)$ is rational over K .

Proof. It is well known (e.g. [7, p. 177]) that G is a direct product of iterated wreath products $\mathbb{Z}/p\mathbb{Z} \wr \cdots \wr \mathbb{Z}/p\mathbb{Z}$. Apply Theorem 1.3 and Theorem 1.7. $\square$

We also have an application to Noether's problem for dihedral groups.

Corollary 1.10. *Let K be any field, n be an odd integer, and D_n be the dihedral group of order $2n$.*

- (a) *$K(D_{2n})$ is rational over $K(D_n)$. In particular, if $K(D_n)$ is rational (resp. retract rational) over K , so is $K(D_{2n})$.*
- (b) *If $K(\mathbb{Z}/n\mathbb{Z})$ is rational over K , then both $K(D_n)$ and $K(D_{2n})$ are stably rational over K .*

Proof. (a) If n is odd, then $D_{2n} = \langle \sigma, \tau : \sigma^{2n} = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ is a direct product of the groups $\langle \sigma^2, \tau \rangle \cong D_n$ and $\langle \sigma^n \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Apply Corollary 1.5.

(b) It is easy to check that, for odd n , the map $(\mathbb{Z}/n\mathbb{Z}) \times D_n \rightarrow (\mathbb{Z}/n\mathbb{Z}) \wr (\mathbb{Z}/2\mathbb{Z})$ given by $(a, \sigma^b \tau^\epsilon) \mapsto ((a+b, a-b), \epsilon)$ is well-defined and is an isomorphism (see Section 4 for the definition of the wreath product). Hence, the stable rationality of $K(D_n)$ over K follows from Theorem 1.3 and Theorem 1.7. Then, by (a), also $K(D_{2n})$ is stably rational over K . $\square$

Remark 1.11. Note that part (a) is implicit in [4].

If n is an odd integer and $K(\mathbb{Z}/n\mathbb{Z})$ is rational over K , the first-named author is able to show that $K(D_n)$ is indeed rational over K , by using other methods.

We will prove Theorem 1.1, Theorem 1.3 and Theorem 1.7 in Section 2, Section 3 and Section 4, respectively.

Standing notation. If G is a finite group, we will write $V = \bigoplus_{g \in G} K \cdot x(g)$ as the regular representation space of G , where G acts on V by $h \cdot x(g) = x(hg)$ for any $g, h \in G$. Recall the definition $K(G) := K(x(g) : g \in G)^G$ given at the beginning of this section.

2. PROOF OF THEOREM 1.1

Before proving Theorem 1.1 we recall two basic facts.

Theorem 2.1 (Hajja and Kang [3, Theorem 1]). *Let G be a finite group acting on $L(x_1, \dots, x_n)$, the rational function field of n variables over a field L . Suppose that*

- (i) *for any $\sigma \in G$, $\sigma(L) \subset L$;*
- (ii) *the restriction of the action of G to L is faithful;*

(iii) for any $\sigma \in G$,

$$\begin{pmatrix} \sigma(x_1) \\ \sigma(x_2) \\ \vdots \\ \sigma(x_n) \end{pmatrix} = A(\sigma) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma),$$

where $A(\sigma) \in GL_n(L)$ and $B(\sigma)$ is an $n \times 1$ matrix over L .

Then there exist $z_1, \dots, z_n \in L(x_1, \dots, x_n)$ so that $L(x_1, \dots, x_n) = L(z_1, \dots, z_n)$ with $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq n$.

Theorem 2.2 (Ahmad, Hajja and Kang [1, Theorem 3.1]). *Let L be any field, $L(x)$ the rational function field of one variable over L , and G a group acting on $L(x)$. Suppose that, for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma \cdot x + b_\sigma$, where $a_\sigma, b_\sigma \in L$ and $a_\sigma \neq 0$. Then $L(x)^G = L^G$ or $L^G(f)$ for some polynomial $f \in L[x]$. In fact, if the integer $m := \min\{\deg g(x) : g(x) \in L[x]^G, g(x) \notin L\}$ does exist, then $L(x)^G = L^G(f(x))$ for any $f(x) \in L[x]^G$ satisfying $\deg f = m$.*

Proof of Theorem 1.1. In this section, K is a field with $\text{char } K = p > 0$ and $1 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \tilde{G} \rightarrow G \rightarrow 1$. Let c be a generator of the normal subgroup $\mathbb{Z}/p\mathbb{Z}$ and $\pi : \tilde{G} \rightarrow G \rightarrow 1$ be the given epimorphism.

Step 1. Let $u : G \rightarrow \tilde{G}$ be a section of π .

As before let $\tilde{V} = \bigoplus_{g \in \tilde{G}} K \cdot x(\tilde{g})$ and $V = \bigoplus_{g \in G} K \cdot x(g)$ be the regular representation spaces of $\tilde{G}$ and G respectively.

Step 2. For each $g \in G$, define

$$\begin{aligned} y(g) &= \sum_{0 \leq i \leq p-1} x(c^i u(g)) \in \tilde{V}, \\ z(g) &= \sum_{0 \leq i \leq p-1} ix(c^i u(g)) \in \tilde{V}, \\ z &= \sum_{g \in G} z(g) \in \tilde{V}, \\ W &= \left(\bigoplus_{g \in G} K \cdot y(g) \right). \end{aligned}$$

Note that $c \cdot y(g) = y(g)$. As G -spaces, W and V are G -equivariant. Hence $K(W)^{\tilde{G}} \simeq K(G)$.

Step 3. We will examine the action of $\tilde{G}$ on $z(g)$ and z .

It is clear that $c \cdot z(g) = z(g) - y(g)$.

For any $h, g \in G$, suppose that $u(h) \cdot u(g) = c^m \cdot u(hg)$ and $u(h) \cdot c \cdot u(h)^{-1} = c^n$. Note that m is an integer depending on g and h , and n is invertible in K . When the element h is fixed, we may write $m = m(g)$ to emphasize the dependence of m on g .

We find that $u(h) \cdot z(g) = \sum_{0 \leq i \leq p-1} ix(u(h)c^i u(g)) = \sum_{0 \leq i \leq p-1} ix(c^{in} u(h)u(g)) = \sum_{0 \leq i \leq p-1} ix(c^{in+m} u(hg)) = c^m \cdot (1/n) \sum_{0 \leq i \leq p-1} ix(c^i u(hg)) = (1/n)z(hg) - (m/n)y(hg)$.

It follows that $u(h) \cdot z = (1/n)z - \sum_{g \in G} (m(g)/n)y(hg)$, where $m(g)$ denotes the integer m depending on g .

Step 4. Define $\widetilde{W} = W \oplus K \cdot z$. Then $\widetilde{W}$ is a faithful $\widetilde{G}$ -subspace of $\widetilde{V}$. By Theorem 2.1, $K(\widetilde{G})$ is rational over $K(\widetilde{W})^{\widetilde{G}}$.

Consider the pair $\widetilde{W}$ and W and apply Theorem 2.2. We find that $K(\widetilde{W})^{\widetilde{G}}$ is rational over $K(W)^{\widetilde{G}}$. Since $K(W)^{\widetilde{G}} = K(W)^G \simeq K(G)$, we are done.

3. PROOF OF THEOREM 1.3

Without loss of generality, we may assume that neither H nor G is the trivial group.

Step 1. Write $\widetilde{G} = H \times G$.

Let $U = \bigoplus_{h \in H} K \cdot x(h)$ and $V = \bigoplus_{g \in G} K \cdot x(g)$ be the regular representation spaces of H and G respectively.

For any element $\tilde{g} \in \widetilde{G}$, any $u \otimes v \in U \otimes_K V$, define $\tilde{g} \cdot (u \otimes v) = (h \cdot u) \otimes (g \cdot v)$ if $\tilde{g} = hg$, where $h \in H$ and $g \in G$. It is easy to see that $U \otimes_K V$ is isomorphic to the regular representation space of $\widetilde{G}$.

Step 2. Define

$$\begin{aligned} u_0 &= \sum_{h \in H} x(h) \in U, \quad v_0 = \sum_{g \in G} x(g) \in V, \\ \widetilde{U} &= \sum_{u \in U} K \cdot u \otimes v_0 \subset U \otimes_K V, \quad \widetilde{V} = \sum_{v \in V} K \cdot u_0 \otimes v \subset U \otimes_K V. \end{aligned}$$

It is easy to see that $\widetilde{U} \oplus \widetilde{V}$ is a faithful $\widetilde{G}$ -subspace of $U \otimes_K V$. Moreover, when restricted to the action of H , the space $\widetilde{U}$ is H -equivariant isomorphic to the space U , and similarly for $\widetilde{V}$ and V as G -spaces.

Step 3. By Theorem 2.1, $K(\widetilde{G}) = K(U \otimes_K V)^{\widetilde{G}}$ is rational over $K(\widetilde{U} \oplus \widetilde{V})^{\widetilde{G}}$.

On the other hand, $K(\widetilde{U} \oplus \widetilde{V})^{\widetilde{G}} = (K(\widetilde{U} \oplus \widetilde{V})^H)^G$, which is K -isomorphic to $K(H) \cdot K(G)$. We conclude that $K(\widetilde{G})$ is rational over $K(H) \cdot K(G)$. (Note that the composite $K(H) \cdot K(G)$ is a free composite; i.e. the transcendence degree of it is the sum of those of $K(H)$ and $K(G)$.)

Step 4. If $K(H)$ is rational (resp. stably rational) over K , it is easy to see that so is $K(H) \cdot K(G)$ over $K(G)$. Thus $K(\widetilde{G})$ is rational (resp. stably rational) over $K(G)$.

As to the retract rationality, from the definition of retract rationality [9, Definition 3.2], it is not difficult to show that (i) if $K(H)$ is retract rational over K , then $K(H) \cdot K(G)$ is retract rational over $K(G)$; and (ii) if both $K(H)$ and $K(G)$ are retract rational, then $K(H) \cdot K(G)$ is retract rational over K . Hence the result.

4. PROOF OF THEOREM 1.7

Step 1. Write $\widetilde{G} = H \wr G$.

Recall the definition of the wreath product $H \wr G$.

Define $N = \bigoplus_{g \in G} H_g$, where each H_g is a copy of H . When we write an element $x = (\cdots, x_g, \cdots) \in N$, it is understood that x_g is the component of x in H_g .

We will define a left action of G on N as follows. If $\sigma \in G$ and $x = (\cdots, x_g, \cdots) \in N$, define ${}^\sigma x = y$ where $y = (\cdots, y_g, \cdots) \in N$ with $y_g = x_{\sigma^{-1}g}$.

The wreath product $H \wr G$ is the semi-direct product $N \rtimes G$. More precisely, if $x, y \in N$ and $\sigma, \tau \in G$, then $(x, \sigma) \cdot (y, \tau) = (x \cdot (\sigma y), \sigma\tau)$. Thus we have

$$(1) \quad (\sigma x)(\tau y) = (\sigma\tau)(\tau^{-1}x \cdot y),$$

where $\sigma, \tau \in G$ and $x, y \in N$.

We will fix our notation for the group $\tilde{G} = H \wr G$, which will be used in subsequent discussions. The groups N and G may be identified (in the usual way) with subgroups of $\tilde{G}$. As above, if $x \in N$ and $\sigma \in G$, then (x, σ) or $x\sigma$ denotes an element (and the same element) in $\tilde{G}$. For any $g \in G$, let H_g be the subgroup of N consisting of elements $x = (\dots, x_{g'}, \dots)$ satisfying the condition that $x_{g'} = 1$ for any $g' \in G \setminus \{g\}$; define a group isomorphism $\phi_g : H \rightarrow H_g$ such that, for any $h \in H$, if $x = \phi_g(h)$ and $x = (\dots, x_{g'}, \dots) \in H_g$, then $x_g = h$.

Define a subgroup $M = \sum_{g \in G \setminus \{1\}} H_g$. Note that the coset decomposition of $\tilde{G}$ with respect to M is given as $\tilde{G} = \bigcup (\sigma \cdot \phi_1(h))M$ where σ and h run over all elements in G and H respectively.

Step 2. Let $V = \bigoplus_{g \in G} K \cdot u(g)$ and $W = \bigoplus_{x \in N} K \cdot v(x)$ be the regular representation spaces of G and N respectively.

Define an action of $\tilde{G}$ on $V \otimes_K W$ by $(gx) \cdot (u(g') \otimes v(y)) = u(gg') \otimes v(g'^{-1}x \cdot y)$ (following Equation (1)), where $g, g' \in G$ and $x, y \in N$.

It follows that $V \otimes_K W$ is isomorphic to the regular representation space of $\tilde{G}$.

Step 3. For each $g \in G$, let $W_g = \bigoplus_{h \in H} K \cdot v(\phi_g(h))$ be the regular representation space of H_g . For any $g \in G \setminus \{1\}$, define

$$w_g = \sum_{h \in H} v(\phi_g(h)) \in W_g.$$

As in the proof of Theorem 1.3, we may regard $\bigotimes_{g \in G \setminus \{1\}} W_g$ as the regular representation space of M , and regard $\bigotimes_{g \in G} W_g$ as the regular representation space of N , i.e. W . Define

$$w' = \bigotimes_{g \in G \setminus \{1\}} w_g \in \bigotimes_{g \in G \setminus \{1\}} W_g.$$

Define

$$w_0 = v(1) \otimes w' \in W, \quad u_0 = u(1) \otimes w_0 \in V \otimes_K W.$$

Note that $x \cdot u_0 = u_0$ for any $x \in M$.

Step 4. For any $g \in G, h \in H$, define

$$u(g; h) = (g \cdot \phi_1(h)) \cdot u_0 = u(g) \otimes (v(\phi_1(h)) \otimes w') \in V \otimes_K W.$$

Note that, for $g, g' \in G$ and $h, h' \in H$, we have $g \cdot u(g'; h) = u(gg'; h)$, $\phi_g(h) \cdot u(g; h') = u(g; hh')$, and $\phi_g(h) \cdot u(g'; h') = u(g'; h')$ if $g \neq g'$.

For each $g \in G$, define

$$U_g = \bigoplus_{h \in H} K \cdot u(g; h) \subset V \otimes_K W,$$

and define

$$\tilde{U} := \bigoplus_{g \in G} U_g \subset V \otimes_K W.$$

It is not difficult to show that $\tilde{U}$ is a faithful $\tilde{G}$ -subspace of $V \otimes_K W$. Note that G permutes the spaces U_g ($g \in G$) regularly; H_g acts regularly on U_g , while H_g acts trivially on $U_{g'}$ if $g \neq g'$.

Step 5. Apply Theorem 2.1. We find that $K(\tilde{G})$ is rational over $K(\tilde{U})^{\tilde{G}}$. It remains to show that $K(\tilde{G})$ is rational (resp. stably rational) over $K(G)$ provided that $K(H)$ is rational (resp. stably rational) over K .

We consider first the situation when $K(H)$ is rational over K . Since G permutes the spaces U_g ($g \in G$) regularly, we may choose a transcendence basis $\{z(g; i) : 1 \leq i \leq d\}$ for $K(U_g)^{H_g}$ (where d is the order of H); i.e. we may write $K(U_g)^{H_g} = K(z(g; i) : 1 \leq i \leq d)$, such that $g \cdot z(g'; i) = z(gg'; i)$ for $1 \leq i \leq d$.

Thus $K(\tilde{U})^{\tilde{G}} = (K(\tilde{U})^N)^G = K(z(g; i) : g \in G, 1 \leq i \leq d)^G$. Apply Theorem 2.1. It is easy to see that $K(z(g; i) : g \in G, 1 \leq i \leq d)^G$ is rational over $K(z(g; 1) : g \in G)^G$, which is isomorphic to $K(G)$.

Step 6. Assume now that $K(H)$ is stably rational over K . More precisely, suppose that $K(H)(t_1, \dots, t_m)$ is rational over K .

Let $\{t(g; j)\}$ denote new indeterminates and define a $\tilde{G}$ -space $\tilde{V}$ by

$$\tilde{V} := \bigoplus_{g \in G, 1 \leq j \leq m} K \cdot t(g; j),$$

where $g \cdot t(g'; j) = t(gg'; j)$ and $x \cdot t(g; j) = t(g; j)$ for any $g, g' \in G$, any $x \in N$, any $1 \leq j \leq m$.

Note that $\tilde{U} \oplus \tilde{V}$ is a faithful $\tilde{G}$ -subspace of $(V \otimes_K W) \oplus \tilde{V}$. By Theorem 2.1, we find that $K((V \otimes_K W) \oplus \tilde{V})^{\tilde{G}}$ is rational over $K(V \otimes_K W)^{\tilde{G}} = K(\tilde{G})$. Again by Theorem 2.1, $K((V \otimes_K W) \oplus \tilde{V})^{\tilde{G}}$ is rational over $K(\tilde{U} \oplus \tilde{V})^{\tilde{G}}$.

Now $K(\tilde{U} \oplus \tilde{V})^N = \prod_{g \in G} K(U_g)^{H_g}(t(g; j) : 1 \leq j \leq m)$, where each $K(U_g)^{H_g}$ is K -isomorphic to $K(H)$ with $g \cdot K(U_{g'})^{H_{g'}} = K(U_{gg'})^{H_{gg'}}$ for any $g, g' \in G$. For each $g \in G$, the field $K(U_g)^{H_g}(t(g; j) : 1 \leq j \leq m)$ is rational over K . As in Step 5, we may choose a transcendence basis $\{z(g; i) : 1 \leq i \leq d + m\}$ for $K(U_g)^{H_g}(t(g; j) : 1 \leq j \leq m)$ so that G acts regularly on each set $\{z(g; i) : g \in G\}$, for every $1 \leq i \leq d + m$. The remaining arguments are quite similar to Step 5 and are omitted.

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