CYCLES ON NASH ALGEBRAIC MODELS OF SMOOTH MANIFOLDS

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ABSTRACT. A Nash algebraic model of a compact smooth manifold M is a pair (X, X_0) where X is a nonsingular real algebraic set and X_0 is the union of some connected components of X such that X_0 is diffeomorphic to M . We study the homology classes on X_0 represented by algebraic subsets of X contained in X_0 for various Nash algebraic models (X, X_0) of M .

1. INTRODUCTION

In 1973 A. Tognoli [25], improving upon an earlier result of J. Nash [20], demonstrated that every compact smooth (of class C^∞) manifold M is diffeomorphic to a nonsingular real algebraic set (in $\mathbb{R}^n$ for some n), henceforth called an *algebraic model* of M . Since then a lot of effort has been devoted to the construction of algebraic models with certain additional desirable properties; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 18]. In the recent papers [13, 14, 15, 19], a relaxed concept of algebraic model is used, which is closer in spirit to Nash's original paper [20]. By a *Nash algebraic model* of M , we mean a pair (X, X_0) , where X is a compact nonsingular real algebraic set, X_0 is the union of some connected components of X , and X_0 is diffeomorphic to M . Note that $X \setminus X_0$ is a smooth manifold, being either empty or a union of connected components of X . We say that the Nash algebraic model (X, X_0) of M is *nice* if $X \setminus X_0$ is diffeomorphic to M , in which case the topology of X is determined by M . All Nash algebraic models constructed in this paper happen to be nice.

In order to state our results we need some preparation. A homology class in $H_k(X, \mathbb{Z}/2)$, where X is a compact nonsingular real algebraic set, is said to be *algebraic* if it can be represented by a k -dimensional algebraic subset of X . The set $H_k^{\text{alg}}(X, \mathbb{Z}/2)$ of all algebraic homology classes in $H_k(X, \mathbb{Z}/2)$ is a subgroup. Early papers dealing with algebraic homology classes provided examples of X with $H_k^{\text{alg}}(X, \mathbb{Z}/2) \neq H_k(X, \mathbb{Z}/2)$ for some k ; see [1, 5, 6, 16, 21, 22]. The groups $H_k^{\text{alg}}(X, \mathbb{Z}/2)$ play an important role in real algebraic geometry. Their properties and applications are surveyed in [10]. If X_0 is the union of some connected components of X , we denote by $H_k^{\text{alg}}(X_0, \mathbb{Z}/2)$ the subgroup of $H_k(X, \mathbb{Z}/2)$ consisting of

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the homology classes represented by k -dimensional algebraic subsets of X contained in X_0 .

The behavior of the groups $H_k^{\text{alg}}(X, \mathbb{Z}/2)$ (or their counterparts in cohomology) as X runs through the class of all algebraic models of a given compact smooth manifold M is investigated in [7, 8, 9, 18].

In the present paper, we study the groups $H_k^{\text{alg}}(X_0, \mathbb{Z}/2)$ as (X, X_0) runs through the class of all Nash algebraic models of M .

Given homology classes α in $H_k(M, \mathbb{Z}/2)$ and β in $H_{m-k}(M, \mathbb{Z}/2)$, $m = \dim M$, we denote by $\alpha \bullet \beta$ their intersection number in $\mathbb{Z}/2$. Recall that

$$\alpha \bullet \beta = \langle D_M^{-1}(\alpha) \cup D_M^{-1}(\beta), [M] \rangle,$$

where $[M]$ is the fundamental class of M in $H_m(M, \mathbb{Z}/2)$ and

$$D_M : H^*(M, \mathbb{Z}/2) \rightarrow H_*(M, \mathbb{Z}/2), \quad D_M(v) = v \cap [M]$$

is the Poincaré duality isomorphism.

Theorem 1.1. *Let M be a compact connected smooth manifold and let G be a subgroup of $H_k(M, \mathbb{Z}/2)$, where $1 \leq k \leq m/2$, $m = \dim M \geq 3$. Assume that the subgroup*

$$\bar{G} = \{\beta \in H_{m-k}(M, \mathbb{Z}/2) \mid \alpha \bullet \beta = 0 \text{ for all } \alpha \in G\}$$

of $H_{m-k}(M, \mathbb{Z}/2)$ is generated by the homology classes represented by some $(m-k)$ -dimensional smooth submanifolds of M with trivial normal bundle. Then there exist a nice Nash algebraic model (X, X_0) of M and a smooth diffeomorphism $\varphi : M \rightarrow X_0$ such that $\varphi_(G) = H_k^{\text{alg}}(X_0, \mathbb{Z}/2)$.*

Theorem 1.1 is nontrivial even when $G = 0$; however, it is then a special case of our next result.

Theorem 1.2. *Any compact smooth manifold M of dimension m has a nice Nash algebraic model (X, X_0) such that $H_k^{\text{alg}}(X_0, \mathbb{Z}/2) = 0$ for each $k \geq 1$, for which the group $H_{m-k}(M, \mathbb{Z}/2)$ is generated by the homology classes represented by $(m-k)$ -dimensional smooth submanifolds of M with trivial normal bundle.*

According to Theorem 1.2, the m -torus $\mathbb{R}^m/\mathbb{Z}^m$ has a nice Nash algebraic model (X, X_0) such that $H_k^{\text{alg}}(X_0, \mathbb{Z}/2) = 0$ for all $k \geq 1$. It is not known whether for odd m greater than 2 there is an algebraic model Y of $\mathbb{R}^m/\mathbb{Z}^m$ with $H_1^{\text{alg}}(Y, \mathbb{Z}/2) = 0$.

Theorems 1.1 and 1.2 are proved in the next section.

2. PROOFS

Let B^k be a nonsingular irreducible real algebraic set with precisely two connected components B_0^k and B_1^k , each diffeomorphic to the unit k -sphere, $k \geq 1$. For example, one can take

$$B^k = \{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} \mid x_0^4 - 4x_0^2 + 1 + x_1^2 + \dots + x_k^2 = 0\}.$$

Let

$$B^k(d) = B^k \times \dots \times B^k \text{ and } B_0^k(d) = B_0^k \times \dots \times B_0^k$$

be the d -fold products, $d \geq 1$.

Lemma 2.1. *Let X be a compact nonsingular real algebraic set and let X_0 be the union of some connected components of X . If $g : X \rightarrow B^k(d)$ is a regular map such that $g(X_0) \subseteq B_0^k(d)$, then*

$$(g|X_0)_*(\beta) = 0 \text{ in } H_k(B^k(d), \mathbb{Z}/2)$$

for every homology class β in $H_k^{\text{alg}}(X_0, \mathbb{Z}/2)$.

Proof. In [17, 18], for every compact nonsingular real algebraic set Y a certain subgroup $\text{Alg}^k(Y)$ of $H^k(Y, \mathbb{Z}/2)$ is defined. If w is in $\text{Alg}^k(Y)$ and γ is in $H_k^{\text{alg}}(Y, \mathbb{Z}/2)$, then

$$(i) \quad \langle w, \gamma \rangle = 0.$$

Indeed, (i) is equivalent to [18, Theorem 4.4] (and to [17, Theorem 2.1]), since

$$\langle w, \gamma \rangle = \langle w, D_Y^{-1}(\gamma) \cap [Y] \rangle = \langle w \cup D_Y^{-1}(\gamma), [Y] \rangle.$$

Moreover, if $\delta : B_0^k(d) \hookrightarrow B^k(d)$ is the inclusion map, then

$$(ii) \quad H^k(B_0^k(d), \mathbb{Z}/2) = \delta^*(\text{Alg}^k(B^k(d)));$$

cf. [18, Example 4.5]. Both (i) and (ii) will be crucial.

Let β be a homology class in $H_k^{\text{alg}}(X_0, \mathbb{Z}/2)$. Then $i_*(\beta)$ is in $H_k^{\text{alg}}(X, \mathbb{Z}/2)$, where $i : X_0 \hookrightarrow X$ is the inclusion map. Hence $g_*(i_*(\beta))$ is in $H_k^{\text{alg}}(B^k(d), \mathbb{Z}/2)$, the map g being regular; see [6, 11]. Denoting by $\bar{g} : X_0 \rightarrow B_0^k(d)$ the restriction of g , we have $\delta \circ \bar{g} = g \circ i$, and hence

$$(iii) \quad \langle \delta^*(v), \bar{g}_*(\beta) \rangle = \langle v, \delta_*(\bar{g}_*(\beta)) \rangle = \langle v, g_*(i_*(\beta)) \rangle = 0$$

for every v in $\text{Alg}^k(B^k(d))$, where the last equality follows from (i). Making use of (ii) and (iii), we get $\bar{g}_*(\beta) = 0$. Since $g|X_0 = \delta \circ \bar{g}$, we obtain $(g|X)_*(\beta) = 0$, as required. $\square$

Let M be a compact smooth manifold. Given an n -dimensional smooth submanifold N of M (always assumed to be a closed subset of M), we set

$$[N]_M := i_*([N]),$$

where $i : N \hookrightarrow M$ is the inclusion map. In other words, $[N]_M$ is the homology class in $H_n(M, \mathbb{Z}/2)$ represented by N .

A cohomology class v in $H^k(M, \mathbb{Z}/2)$, $k \geq 1$, is said to be *spherical* provided $v = f^*(c)$, where $f : M \rightarrow S^k$ is a continuous (or equivalently smooth) map from M into the unit k -sphere S^k and c is the unique generator of $H^k(S^k, \mathbb{Z}/2) \cong \mathbb{Z}/2$. It is well known that v is spherical if and only if the homology class $D_M(v)$ that is Poincaré dual to v can be represented by a smooth submanifold of M with trivial normal bundle; see [24, Théorème II.2].

Proof of Theorem 1.1. Since $k \leq m/2$, according to [24, Théorème II.26] there exist k -dimensional smooth submanifolds $K_1, \dots, K_\ell$ of M such that

$$(a_1) \quad [K_1]_M, \dots, [K_\ell]_M \text{ generate } G.$$

We may assume that each submanifold K_i , $1 \leq i \leq \ell$, is connected and, for $k \geq 2$, nonorientable.

To justify this claim, we argue as follows. Since M is connected and $k \geq 1$, K_i can be replaced by a connected smooth submanifold of M obtained by joining the connected components of K_i with k -dimensional tubes. If $k \geq 2$, then $m - k \geq 2$,

and hence there is a k -dimensional connected nonorientable smooth submanifold P of M which is contained in an open subset of M diffeomorphic to $\mathbb{R}^m$. It follows that for $k \geq 2$, K_i can be replaced by a nonorientable smooth submanifold of M obtained by joining K_i and P with a k -dimensional tube. The claim is proved.

By transversality, the submanifolds $K_1, \dots, K_\ell$ can be chosen in general position.

Let

$$(a_2) \quad \Gamma := \{v \in H^k(M, \mathbb{Z}/2) \mid \langle v, \alpha \rangle = 0 \text{ for all } \alpha \in G\}.$$

Since for all v in $H^k(M, \mathbb{Z}/2)$ and α in $H_k(M, \mathbb{Z}/2)$,

$$\begin{aligned} \langle v, \alpha \rangle &= \langle v, D_M^{-1}(\alpha) \cap [M] \rangle \\ &= \langle v \cup D_M^{-1}(\alpha), [M] \rangle \\ &= \langle D_M^{-1}(\alpha) \cup v, [M] \rangle \\ &= \alpha \bullet D_M(v), \end{aligned}$$

it follows that $D_M(\Gamma) = \bar{G}$. Therefore the assumption about $\bar{G}$ implies that Γ is generated by spherical cohomology classes. Hence, with notation as in Lemma 2.1, there exist a positive integer d and a smooth map

$$f = (f_1, \dots, f_d) : M \rightarrow B^k(d) = B^k \times \dots \times B^k$$

satisfying

$$(a_3) \quad f(M) \subseteq B_0^k(d) \text{ and } \Gamma = f^*(H^k(B^k(d), \mathbb{Z}/2)).$$

By (a_2) ,

$$G = \{\alpha \in H_k(M, \mathbb{Z}/2) \mid \langle v, \alpha \rangle = 0 \text{ for all } v \in \Gamma\}.$$

Since $\langle f^*(w), \alpha \rangle = \langle w, f_*(\alpha) \rangle$, we have that for every w in $H^k(B^k(d), \mathbb{Z}/2)$ and every α in $H_k(M, \mathbb{Z}/2)$, the equality in (a_3) gives

$$(a_4) \quad G = \{\alpha \in H_k(M, \mathbb{Z}/2) \mid f_*(\alpha) = 0\}.$$

Let M' be a smooth manifold, G' be a subgroup of $H_k(M', \mathbb{Z}/2)$, $K'_1, \dots, K'_\ell$ be smooth submanifolds of M' , and $f' : M' \rightarrow B^k(d)$ be a smooth map. We say that $(M', G'; K'_1, \dots, K'_\ell, f')$ is a modification of $(M, G; K_1, \dots, K_\ell, f)$ if there exists a smooth diffeomorphism $h : M \rightarrow M'$ such that $G' = h_*(G)$, $K'_i = h(K_i)$ for $1 \leq i \leq \ell$, and f' is homotopic to $f \circ h^{-1}$. Clearly, in the proof we may replace $(M, G; K_1, \dots, K_\ell, f)$ by a modification.

Assertion. Replacing $(M, G; K_1, \dots, K_\ell, f)$ by a suitable modification if necessary, we may assume that M is a nonsingular real algebraic set, $K_1, \dots, K_\ell$ are nonsingular algebraic subsets of M , and $f|K : K \rightarrow B^k(d)$ is a regular map, where

$$K := K_1 \cup \dots \cup K_\ell.$$

In the proof of the Assertion, we will consider separately the two cases $k \geq 2$ and $k = 1$.

Case 1. Suppose $k \geq 2$. Since $K_1, \dots, K_\ell$ are in general position in M , by [3, Theorem 2.8.9] or [4, Theorem 4] we may assume that M is a nonsingular algebraic subset of $\mathbb{R}^n$ and $K_1, \dots, K_\ell$ are nonsingular algebraic subsets of M .

We will now prove that the map $f_{j0}|K : K \rightarrow B_0^k$ is null homotopic, where $f_{j0} : M \rightarrow B_0^k$ is defined by $f_{j0}(x) = f_j(x)$ for all x in M . Recall that B_0^k is diffeomorphic to S^k . By Hopf's classification theorem, $f_{j0}|K$ is null homotopic if and only if $(f_{j0}|K)^*(b) = 0$ in $H^k(K, \mathbb{Z})$, where b is a generator of $H^k(B_0^k, \mathbb{Z}) \cong \mathbb{Z}$.

By the Mayer-Vietoris exact sequence, the equality $(f_{j0}|K)^*(b) = 0$ in $H^k(K, \mathbb{Z})$ is equivalent to $(f_{j0}|K_i)^*(b) = 0$ in $H^k(K_i, \mathbb{Z})$ for $1 \leq i \leq \ell$. Since K_i is a connected and nonorientable manifold, $(f_{j0}|K_i)^*(b) = 0$ in $H^k(K_i, \mathbb{Z})$ if and only if $(f_{j0}|K_i)_*([K_i]) = 0$ in $H_k(B_0^k, \mathbb{Z}/2)$. We have

$$(f_{j0}|K_i)_*([K_i]) = (f_{j0})_*([K_i]_M) = 0,$$

where the last equality follows from (a_1) and (a_4) . Hence the map $f|K : K \rightarrow B^k(d)$ is null homotopic.

In order to complete the proof of the Assertion for $k \geq 2$, it suffices to show that $f : M \rightarrow B^k(d)$ is homotopic to a smooth map which is constant on K . This can be done as follows. Choose a compact subset A of M such that (M, A) is a polyhedral pair, K is contained in the interior of A , and K is a deformation retract of A . Then $f|A : A \rightarrow B^k(d)$ is null homotopic and, by the homotopy extension theorem [23, p. 118, Corollary 5], there exists a continuous map $f' : M \rightarrow B^k(d)$ which is homotopic to f and such that $f'|A : A \rightarrow B^k(d)$ is a constant map. Thus one can find a smooth map $f'' : M \rightarrow B^k(d)$ that is homotopic to f' and equal to f' on K . Since f'' is homotopic to f and is constant on K , the proof of the Assertion for $k \geq 2$ is complete.

Case 2. Suppose now that $k = 1$. We have

$$(f|K_i)_*([K_i]) = f_*([K_i]_M) = 0,$$

where the last equality follows from (a_1) and (a_4) . Consequently, the map $f|K_i : K_i \rightarrow B^k(d)$ represents 0 in the unoriented bordism group $\mathcal{N}_*(B^k(d))$ (this is elementary, but see [12, (17.3)] if desired). By [3, Theorem 2.8.4], there exist a nonsingular real algebraic set K'_i , a regular map $r_i : K'_i \rightarrow B^k(d)$, and a smooth diffeomorphism $h_i : K_i \rightarrow K'_i$ such that r_i is homotopic to $(f|K_i) \circ h_i^{-1}$. We may assume that $K'_1, \dots, K'_\ell$ are pairwise disjoint nonsingular algebraic subsets of $\mathbb{R}^n$, $n \geq 2m + 1$. Hence one can find a smooth embedding $e : M \rightarrow \mathbb{R}^n$ such that $e(x) = h_i(x)$ for all x in K_i , $1 \leq i \leq \ell$ (note that $K_1, \dots, K_\ell$ are pairwise disjoint, as they are in general position in M , $\dim M = m \geq 3$). Let $M' := e(M)$ and let $\bar{e} : M \rightarrow M'$ be a smooth diffeomorphism defined by $\bar{e}(x) = e(x)$ for all x in M . Thus M' is a smooth submanifold of $\mathbb{R}^n$ containing $K'_1 \cup \dots \cup K'_\ell$, and the map $(f \circ \bar{e}^{-1})|K'_i$ is homotopic to r_i for $1 \leq i \leq \ell$. By the homotopy extension theorem (see the argument in Case 1 above), there exists a smooth map $r : M' \rightarrow B^k(d)$ homotopic to $f \circ \bar{e}^{-1}$ and with $r|K'_i = r_i$ for $1 \leq i \leq \ell$.

Therefore, without loss of generality, we may assume that M is a smooth submanifold of $\mathbb{R}^n$, the original $K_1, \dots, K_\ell$ are nonsingular algebraic subsets of $\mathbb{R}^n$, and $f|K : K \rightarrow B^k(d)$ is a regular map. Since $\dim K = 1$, every topological real vector bundle on K is isomorphic to an algebraic (strongly algebraic, in the terminology used in [5]) vector bundle; see [7, Theorem 12.5.1]. By [5, Theorem 3.2], there exist a nonsingular algebraic subset M'' of $\mathbb{R}^n$ and a smooth diffeomorphism $\theta : M \rightarrow M''$ such that $K \subseteq M''$ and $\theta(x) = x$ for all x in K . This completes the proof of the Assertion for $k = 1$.

Henceforth we assume that $M, K_1, \dots, K_\ell, f$ are as in the Assertion. In addition, we assume that M is a nonsingular algebraic subset of $\mathbb{R}^n$. For any nonnegative integer p , we regard $\mathbb{R}^n$ as a subset of $\mathbb{R}^{n+p}$, identifying $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^n \times \mathbb{R}^p = \mathbb{R}^{n+p}$. Let $t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a translation such that $M \cap t(M) = \emptyset$, and let $\bar{M} := M \cup t(M)$. Thus $\bar{M}$ is the disjoint union of two copies

of M . Define

$$\bar{f} : \bar{M} \rightarrow B^k(d)$$

by $\bar{f}(x) = \bar{f}(t(x)) = f(x)$ for all x in M . Clearly, the map $\bar{f} : \bar{M} \rightarrow B^k(d)$ represents 0 in the unoriented bordism group $\mathcal{N}_*(B^k(d))$. Moreover, $\bar{f}|_K = f|_K$ is a regular map. By [3, Theorem 2.8.4], there exist a nonnegative integer p , a nonsingular algebraic subset X of $\mathbb{R}^{n+p}$, a regular map $g : X \rightarrow B^k(d)$, and a smooth diffeomorphism $\psi : \bar{M} \rightarrow X$ such that $K \subseteq X$, $\psi(x) = x$ for all x in K , and g is homotopic to $\bar{f} \circ \psi^{-1}$. Set $X_0 := \psi(M)$ and let $\varphi : M \rightarrow X_0$ be the restriction of ψ . Then (X, X_0) is a Nash algebraic model of M , and φ is a smooth diffeomorphism.

It remains to prove that $\varphi_*(G) = H_k^{\text{alg}}(X_0, \mathbb{Z}/2)$. This can be done as follows. By construction, $\varphi_*([K_i]_M) = [K_i]_{X_0}$ for $1 \leq i \leq \ell$, and hence in view of (a_1) ,

$$(a_5) \quad \varphi_*(G) \subseteq H_k^{\text{alg}}(X_0, \mathbb{Z}/2).$$

Note that $g|_{X_0}$ is homotopic to $(f \circ \psi^{-1})|_{X_0} = f \circ \varphi^{-1}$, and hence (a_4) implies

$$\varphi_*(G) = \{\beta \in H_k(X_0, \mathbb{Z}/2) \mid (g|_{X_0})_*(\beta) = 0\}.$$

If β is in $H_k^{\text{alg}}(X_0, \mathbb{Z}/2)$, then according to Lemma 2.1, $(g|_{X_0})_*(\beta) = 0$, which in turn means that β is in $\varphi_*(G)$. In view of (a_5) , we obtain

$$\varphi_*(G) = H_k^{\text{alg}}(X_0, \mathbb{Z}/2).$$

The proof is complete. $\square$

Proof of Theorem 1.2. Let T be the set of all integers k such that $1 \leq k \leq m-1$ and the group $H_{m-k}(M, \mathbb{Z}/2)$ is generated by $(m-k)$ -dimensional smooth submanifolds of M with trivial normal bundle.

Suppose T is empty. If $m = 0$, there is nothing to prove. If $m \geq 1$, then according to [26], there exists an irreducible algebraic model X of the disjoint union of two copies of M . Let X_0 be the union of some connected components of X such that X_0 is diffeomorphic to M . Then (X, X_0) is a nice Nash algebraic model of M . Since X is irreducible, we have $H_k^{\text{alg}}(X_0, \mathbb{Z}/2) = 0$ for $k \geq m$, as required.

Henceforth, we assume T is nonempty, with

$$T = \{k_1, \dots, k_s\}.$$

Each group $H^{k_i}(M, \mathbb{Z}/2)$, $1 \leq i \leq s$, is generated by spherical cohomology classes. Hence, with notation as in Lemma 2.1, there exist a positive integer d_i and a smooth map $f_i : M \rightarrow B^{k_i}(d_i)$ satisfying

$$(b_1) \quad f_i(M) \subseteq B_0^{k_i}(d_i) \text{ and } H^{k_i}(M, \mathbb{Z}/2) = f_i^*(H^{k_i}(B^{k_i}(d_i), \mathbb{Z}/2)).$$

The equality in (b_1) implies that the induced homomorphism

$$(b_2) \quad (f_i)_* : H_{k_i}(M, \mathbb{Z}/2) \rightarrow H_{k_i}(B^{k_i}(d_i), \mathbb{Z}/2) \text{ is injective.}$$

Moreover, we can find such a map f_i with the additional property that it is constant on some open subset U of M which has a nonempty intersection with each connected component of M . Let C be a 1-dimensional smooth submanifold of M such that $C \subseteq U$, each connected component of M contains a connected component of C , and the normal bundle of C_i in M is trivial. In particular, $f_i|_C : C \rightarrow B^{k_i}(d_i)$ is a constant map.

Set $\bar{M} := (M \times \{0\}) \cup (M \times \{1\})$ and let $e : \bar{M} \rightarrow \mathbb{R}^n$ be a smooth embedding, $n \geq 2m + 1$, such that

$$Y := e((C \times \{0\}) \cup (C \times \{1\}))$$

is a nonsingular irreducible algebraic subset of $\mathbb{R}^n$. Let $\bar{f}_i : e(\bar{M}) \rightarrow B^{k_i}(d_i)$ be defined by $\bar{f}_i(e(x, 0)) = \bar{f}_i(e(x, 1)) = f_i(x)$ for all x in M . By construction, the map

$$\bar{f} = (\bar{f}_1, \dots, \bar{f}_s) : e(\bar{M}) \rightarrow B := B^{k_1}(d_1) \times \dots \times B^{k_s}(d_s)$$

represents 0 in the unoriented bordism group $\mathcal{N}_*(B)$. Moreover, the normal bundle of Y in $e(\bar{M})$ is trivial, and $\bar{f}|_Y : Y \rightarrow B$ is a constant map. By [3, Theorem 2.8.4], there exist a nonnegative integer p , a nonsingular algebraic subset X of $\mathbb{R}^{n+p}$, a smooth diffeomorphism $\psi : e(\bar{M}) \rightarrow X$, and a regular map

$$g = (g_1, \dots, g_s) : X \rightarrow B$$

such that $Y \subseteq X$, $\psi(y) = y$ for all y in Y , and g is homotopic to $\bar{f} \circ \psi^{-1}$ (we regard $\mathbb{R}^n$ as a subset of $\mathbb{R}^{n+p}$, identifying $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^n \times \mathbb{R}^p = \mathbb{R}^{n+p}$). Setting $X_0 := \psi(e(M \times \{0\}))$, we obtain a nice Nash algebraic model (X, X_0) of M . Moreover, X is irreducible since Y is irreducible. In particular, $H_k^{\text{alg}}(X_0, \mathbb{Z}/2) = 0$ for $k \geq m$.

It remains to prove that $H_{k_i}^{\text{alg}}(X_0, \mathbb{Z}/2) = 0$ for $1 \leq i \leq s$. Define $\varphi : M \rightarrow X_0$ by $\varphi(x) = \psi(e(x, 0))$ for all x in M . Then φ is a smooth diffeomorphism, and $g_i|_{X_0}$ is homotopic to $(f \circ \psi^{-1})|_{X_0} = f_i \circ \varphi^{-1}$. In particular, $g_i(X_0) \subseteq B_0^{k_i}(d_i)$ and, according to (b_2) , the induced homomorphism

$$(g_i|_{X_0})_* : H_{k_i}(X_0, \mathbb{Z}/2) \rightarrow H_{k_i}(B^{k_i}(d_i), \mathbb{Z}/2)$$

is injective. Hence, making use of Lemma 2.1, we get $H_{k_i}^{\text{alg}}(X_0, \mathbb{Z}/2) = 0$, as required. $\square$

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