BOUNDARY POINTS AS LIMIT FUNCTIONS OF ITERATED HOLOMORPHIC FUNCTION SYSTEMS

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(Communicated by Jane M. Hawkins)

This paper is dedicated to my mother.

ABSTRACT. We show that if c is a boundary point of a non-relatively compact subdomain of the unit disk, there always exists an iterated holomorphic function system with the constant c as a limit function.

1. INTRODUCTION

Let Ω be a hyperbolic domain in the plane. Suppose $\{f_n\}$ is a sequence of holomorphic maps from Ω into a subdomain $X \subset \Omega$.

Consider the compositions

$$(1.1) \quad F_n = f_1 \circ f_2 \circ \dots \circ f_{n-1} \circ f_n.$$

The sequence $\{F_n\}$ is called an *iterated function system*. $\{F_n\}$ is a family of holomorphic maps from Ω into the subdomain $X \subset \Omega$. Since X is a hyperbolic domain, by Montel's theorem, the iterated function system $\{F_n\}$ is a normal family. Therefore, there is a subsequence of the sequence $\{F_n\}$ which converges locally uniformly to a holomorphic map F or to infinity. The limit functions are either open maps from Ω into X or constants in $\overline{X}$.

Definition 1.1. A subdomain X of a domain Ω is called relatively compact in Ω if there exists a compact set K such that $X \subset K \subset \Omega$. Otherwise, X is called non-relatively compact in Ω .

Suppose Ω is the unit disk Δ . Lorentzen [7] and Gill [3] showed that if X is relatively compact in Δ , then every iterated function system converges to a unique constant which is inside X not on its boundary. Therefore, non-relative compactness of X in Δ is a necessary condition in order to have a boundary point of X as a limit function of an iterated function system. In this paper, we want to show that for any point c on the boundary of X , the non-relative compactness of X in Δ is a sufficient condition for the existence of an iterated function system that has the constant c as a limit function.

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This paper is organized as follows. In section 2, the key facts are stated and proved. Section 3 is devoted to the statement and the proof of the main theorem and an interesting corollary.

2. KEY FACTS

First we state a lemma of Keen and Lakic which is used later in this section to prove the key theorem. We will bring the proof of the lemma, and in the remark following the proof we explain how the lemma can be used in proving the key theorem.

Lemma 2.1 (Keen-Lakic). *Suppose X is a non-relatively compact subdomain of the unit disk Δ . Let $a_1, \dots, a_n$ be n distinct points in $\Delta \setminus \{0\}$. Then there exist a holomorphic function $f : \Delta \rightarrow \Delta$ and points $x_1, \dots, x_n \in X$ such that $f(x_i) = a_i/x_i$, for all $i = 1, \dots, n$.*

Proof. The proof will make use of the automorphisms

$$A(a, z) = \frac{z - a}{1 - \bar{a}z}.$$

Step 1. Since X is not relatively compact we may choose an $x_1 \in X$ such that $|x_1| > |a_1|$. Let $g_1(z)$ be a self-map of the unit disk to be determined. Define

$$f(z) = \frac{A(x_1, z)g_1(A(x_1, z)) + \frac{a_1}{x_1}}{1 + \frac{\bar{a}_1}{\bar{x}_1}A(x_1, z)g_1(A(x_1, z))}.$$

It follows that $f(x_1) = a_1/x_1$ as required. Because we want to work inductively we rewrite this definition implicitly as follows:

$$(2.1) \quad A(x_1, z)g_1(A(x_1, z)) = A\left(\frac{a_1}{x_1}, f(z)\right).$$

If $n = 1$ we set $g_1(z) \equiv 0$ and we are done. From now on we assume that $n > 1$ and that we have chosen x_1 .

Step 2. Before we proceed, we set up some further notation:

For $1 \leq j \leq k \leq n$, set $a_{jk} = A(x_j, x_k)$. Next, for $k = 2, \dots, n$, set

$$(2.2) \quad b_{1k} = A\left(\frac{a_1}{x_1}, \frac{a_k}{x_k}\right).$$

For $j = 2, \dots, n-1$ and $k = j, j+1, \dots, n$ set

$$(2.3) \quad b_{jk} = A\left(\frac{b_{(j-1)j}}{a_{(j-1)j}}, \frac{b_{(j-1)k}}{a_{(j-1)k}}\right).$$

For our construction to work we need to choose the x_i so that the following inequalities hold:

$$(2.4) \quad \left| \frac{a_i}{x_i} \right| < 1, \quad i = 1, \dots, n.$$

In step 1 we chose x_1 so this holds for $i = 1$.

For all j, k such that $j < k$ we also need to have

$$(2.5) \quad \left| \frac{b_{jk}}{a_{jk}} \right| < 1.$$

To see that we can choose the x_i 's so that these inequalities are all satisfied, note first that for fixed j , and all $k > j$, $|x_k| \rightarrow 1$ implies $|a_{jk}| \rightarrow 1$.

Next, as $|x_j| \rightarrow 1$,

$$\limsup |b_{1j}| \leq \left| A\left(\frac{a_1}{x_1}, a_j e^{\theta_j}\right) \right| = B_{1j} < 1,$$

where θ_j is chosen so that $\arg a_j e^{\theta_j} = \arg \frac{a_1}{x_1} + \pi$ and B_{1j} is maximal.

Choose x_1 so that, if the remaining $|x_i|$'s are close enough to 1, inequalities (2.4) and (2.5) hold with $j = 1$.

We now find bounds

$$\limsup_{|x_j| \rightarrow 1} |b_{2j}| \leq \left| A\left(\frac{b_{12}}{a_{12}}, B_{1j} e^{\theta_j}\right) \right| = B_{2j} < 1,$$

where again θ_j is chosen to maximize B_{2j} .

We repeat this process, choosing $x_3, \dots, x_{n-1}, x_n$, in turn so that all the inequalities above hold.

Step 3. Define the functions $g_k(z) : \Delta \rightarrow \Delta$, $k = 2, \dots, n$ recursively by

$$(2.6) \quad A(x_k, z)g_k(A(x_k, z)) = A\left(\frac{b_{(k-1)k}}{a_{(k-1)k}}, g_{(k-1)}(A(x_{(k-1)}, z))\right).$$

Now take $g_n(z)$ to be the function $g_n(z) \equiv 0$. Then work back through equations (2.6) to obtain the functions g_1 and f .

Check that $f(x_i) = \frac{a_i}{x_i}$ for $i = 1, \dots, n$, so that the points x_i and the function f are as required by the lemma. $\square$

Remark 2.2. In the lemma above, n points $a_1, \dots, a_n$ were given and we chose points $x_1, \dots, x_n$ which satisfy inequalities 2.4 and 2.5. Now suppose we add one point a_{n+1} to the list of the given points $a_1, \dots, a_n$ ($a_{n+1} \neq a_i$, for $1 \leq i \leq n$). The first n chosen points $x_1, \dots, x_n$ don't need to change because we only need to choose x_{n+1} to satisfy the following inequalities:

$$(2.7) \quad \left| \frac{a_{n+1}}{x_{n+1}} \right| < 1,$$

$$(2.8) \quad \left| \frac{b_{j(n+1)}}{a_{j(n+1)}} \right| < 1,$$

for $j \leq n$. By an argument similar to the one that we made in the proof of the lemma above and by using the non-relative compactness of X in Δ , we can choose x_{n+1} close enough to the boundary of the unit disk that all the inequalities above are satisfied. In the case that n points $a_1, \dots, a_n$ are given and we choose $x_1, \dots, x_n$, the function f is constructed in terms of $a_1, \dots, a_n$ and $x_1, \dots, x_n$. When a_{n+1} is added to the list of given points and we choose one more point x_{n+1} , the new constructed function, say g , is given in terms of $a_1, \dots, a_{n+1}$ and $x_1, \dots, x_{n+1}$. The function g is different from f . However, at each x_i , $1 \leq i \leq n$, f and g have the same value; that is, $f(x_i) = g(x_i) = a_i/x_i$, for $1 \leq i \leq n$. In addition, $g(x_{n+1}) = a_{n+1}/x_{n+1}$.

Theorem 2.1 (The Key Theorem). *Let X be a non-relatively compact subdomain of Δ . Let $a_1, a_2, a_3, \dots$ be a sequence of infinitely many distinct points in $\Delta \setminus \{0\}$. Then there exists a holomorphic function $f : \Delta \rightarrow \Delta$ and a sequence of points $x_1, x_2, x_3, \dots \in X$ such that for all $i = 1, 2, 3, \dots$, $f(x_i) = a_i$ and $f(0) = 0$.*

Proof. First, we construct a sequence of functions $\{f_n\}_{n=1}^\infty$ in the following way: Consider the first point of this sequence, a_1 . By Lemma 2.1, there exist a point $x_1 \in X$ and a function $f_1 : \Delta \rightarrow \Delta$ such that $f_1(x_1) = a_1/x_1$. Next, consider the first two points of the sequence, a_1 and a_2 . By the remark above, x_1 does not need to change and there exist a point x_2 and a function $f_2 : \Delta \rightarrow \Delta$ such that $f_2(x_1) = a_1/x_1$ and $f_2(x_2) = a_2/x_2$. In general, in order to construct f_n , we consider the first n points of the sequence, $a_1, \dots, a_n$. By the remark above, the points $x_1, \dots, x_{n-1}$ that were selected for constructing f_{n-1} , do not need to change and there exist a point x_n and a function $f_n : \Delta \rightarrow \Delta$ such that $f_n(x_i) = a_i/x_i$, for $1 \leq i \leq n$. Therefore, we have a sequence of functions $f_1, f_2, f_3, \dots$ and a sequence of points $x_1, x_2, x_3, \dots \in X$ such that for every fixed n ,

$$f_n(x_i) = a_i/x_i,$$

for $i \leq n$. That is, $f_1(x_1) = a_1/x_1$,

$$f_2(x_1) = a_1/x_1, f_2(x_2) = a_2/x_2,$$

$$f_3(x_1) = a_1/x_1, f_3(x_2) = a_2/x_2, f_3(x_3) = a_3/x_3,$$

$$\vdots$$

$$f_n(x_1) = a_1/x_1, f_n(x_2) = a_2/x_2, f_n(x_3) = a_3/x_3, \dots, f_n(x_n) = a_n/x_n.$$

Now, for every $n \geq 1$, define

$$h_n(z) = zf_n(z).$$

It is clear that $h_n(0) = 0$ and for every fixed n ,

$$h_n(x_i) = a_i$$

for $i \leq n$.

The sequence $h_1, h_2, h_3, \dots$ is a normal family, so by Montel's Theorem there is a subsequence which converges locally uniformly to a holomorphic map f from the unit disk to the unit disk with $f(0) = 0$ and $f(x_i) = a_i$ for $i = 1, 2, 3, \dots$ $\square$

3. THE MAIN THEOREM

Theorem 3.1. *Suppose X is a non-relatively compact subdomain of Δ . Then, for any boundary point $c \in \partial X$, there is an iterated function system with c as a limit function.*

Proof. Let $c_0, c_1, c_2, c_3, \dots$ be a sequence of distinct points in X which converges to c . We claim that there exists a function $f_1 : \Delta \rightarrow X$ and a sequence of points $c_{11}, c_{12}, c_{13}, \dots \in X$ such that for all $i = 1, 2, 3, \dots$, $f_1(c_{1i}) = c_i$ and $f_1(0) = c_0$.

Now, we prove this claim. Since X is a hyperbolic domain, it has a holomorphic covering $g : \Delta \rightarrow X$ such that $g(0) = c_0$. Let $a_1, a_2, a_3, \dots$ be a sequence in Δ that is mapped onto the sequence $c_1, c_2, c_3, \dots$ by g . By Theorem 2.1, there is a function $h : \Delta \rightarrow \Delta$ and a sequence of points $c_{11}, c_{12}, c_{13}, \dots \in X$ such that for all $i = 1, 2, 3, \dots$, $h(c_{1i}) = a_i$ and $h(0) = 0$. Define $f_1(z) = g \circ h(z)$. It is clear that f_1 is a function from the unit disk into X with $f_1(c_{1i}) = c_i$ and $f_1(0) = c_0$. This proves our claim. By the same argument there is a function $f_2 : \Delta \rightarrow X$ and a sequence of points $c_{212}, c_{213}, c_{214}, \dots \in X$ such that for all $i = 2, 3, 4, \dots$, $f_2(c_{21i}) = c_{1i}$ and $f_2(0) = c_{11}$.

By using the same argument repeatedly, we can obtain a sequence of functions $f_1, f_2, f_3, \dots$ from Δ to X such that for every fixed n ,

$$f_n(c_{n(n-1)\dots 21i}) = c_{(n-1)\dots 21i},$$

for $i = n, n+1, n+2, \dots$ and $f_n(0) = c_{(n-1)(n-2)\dots 1(n-1)}$.

Now, let's look at the backward compositions and their values at 0:

$$F_1(0) = f_1(0) = c_0,$$

$$F_2(0) = f_1 \circ f_2(0) = c_1,$$

$$F_3(0) = f_1 \circ f_2 \circ f_3(0) = c_2,$$

$$\text{and in general, } F_n(0) = f_1 \circ f_2 \circ \dots \circ f_n(0) = c_{n-1}.$$

Since the sequence c_n converges to c , any convergent subsequence of F_n will also converge to c . $\square$

Corollary 3.1. *Suppose X is a non-relatively compact subdomain of Δ . Then there is an iterated function system $\{F_n\}$ with the following property: If p is a point in $\overline{X}$, it can be realized by one of the accumulation points of $\{F_n\}$. In other words, there is an iterated function system such that any point in $\overline{X}$ can be realized by one of its limit functions.*

Proof. Let $\{c_n\}$ be a sequence consisting of all rational points (i.e. points with rational coordinates) in X . Using the same argument that we made in the proof of the previous theorem, there exists an iterated function system $\{F_n\}$ with the property that $F_n(0) = c_{n-1}$. Let p be a point in $\overline{X}$. Since $\{c_n\}$ is dense in $\overline{X}$, there is a subsequence $\{c_{n_j}\}$ which converges to p . Now, consider the corresponding subsequence $\{F_{n_j+1}\}$. We have $F_{n_j+1}(0) = c_{n_j}$. Consequently, $\{F_{n_j+1}(0)\}$ converges to p . Therefore, the limit of the subsequence $\{F_{n_j+1}\}$, which is a limit function of $\{F_n\}$, realizes p . $\square$

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