REGULARITY AND SEGRE-VERONESE EMBEDDINGS

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ABSTRACT. This paper studies the regularity of certain coherent sheaves that arise naturally from Segre-Veronese embeddings of a product of projective spaces. We give an explicit formula for the regularity of these sheaves and show that their regularity is subadditive. We then apply our results to study the Tate resolutions of these sheaves.

1. REGULARITY

We will use the following generalized concept of regularity.

Definition 1.1. Fix a projective variety X and line bundles L, B on X such that B is generated by global sections. Then a coherent sheaf $\mathcal{F}$ on X is *L -regular with respect to B* provided that

$$H^i(X, \mathcal{F} \otimes L \otimes B^{\otimes(-i)}) = 0 \quad \text{for all } i > 0.$$

Here $B^{\otimes(-i)}$ denotes the dual of $B^{\otimes i}$.

This notion of regularity is a special case the multigraded regularity for sheaves introduced in [6], which in turn is a modification of the regularity defined in [8] for multigraded modules over a polynomial ring.

We can relate Definition 1.1 to the Castelnuovo-Mumford regularity of a sheaf on projective space as follows. Suppose that B is very ample and $\mathcal{L} = i_*\mathcal{F}$, where $i : X \rightarrow \mathbb{P}^N$ is the projective embedding given by B . Then, given an integer p , the isomorphism

$$H^i(\mathbb{P}^N, \mathcal{L}(p-i)) \simeq H^i(X, \mathcal{F} \otimes B^{\otimes p} \otimes B^{\otimes(-i)})$$

shows that $\mathcal{F}$ is $B^{\otimes p}$ -regular with respect to B if and only if $\mathcal{L}$ is p -regular as a coherent sheaf on $\mathbb{P}^N$.

Now fix r -tuples $\underline{l} := (l_1, \dots, l_r)$ and $\underline{d} := (d_1, \dots, d_r)$ of positive integers. These give the product variety

$$X := \mathbb{P}^{l_1} \times \dots \times \mathbb{P}^{l_r}$$

of dimension $n := \sum_{i=1}^r l_i$ and the $\underline{d}$ -uple Segre-Veronese embedding

$$\nu_{\underline{d}} : X \longrightarrow \mathbb{P}^N, \quad N = \prod_{k=1}^r \binom{l_k + d_k}{d_k} - 1.$$

The r -tuples $\underline{l}$ and $\underline{d}$ will be fixed for the remainder of the paper.

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Given $\underline{m} := (m_1, \dots, m_r)$, consider the line bundle

$$\mathcal{O}_X(\underline{m}) = \mathcal{O}_X(m_1, \dots, m_r)$$

defined by

$$\mathcal{O}_X(\underline{m}) := p_1^* \mathcal{O}_{\mathbb{P}^{l_1}}(m_1) \otimes \cdots \otimes p_r^* \mathcal{O}_{\mathbb{P}^{l_r}}(m_r),$$

where $p_i : \mathbb{P}^{l_1} \times \cdots \times \mathbb{P}^{l_r} \rightarrow \mathbb{P}^{l_i}$ is the projection. In this paper, we will study the regularity (in the sense of Definition 1.1) of $\mathcal{O}_X(\underline{m})$ with respect to the line bundles

$$\begin{aligned} L &:= \mathcal{O}_X(\underline{p}) = \mathcal{O}_X(p_1, \dots, p_r), \quad p_i \in \mathbb{Z}, \\ B &:= \mathcal{O}_X(\underline{d}) = \mathcal{O}_X(d_1, \dots, d_r). \end{aligned}$$

For any nonempty subset $J \subseteq \{1, \dots, r\}$ let l_J denote the sum $\sum_{j \in J} l_j$. Here is our first main result.

Theorem 1.2. *Let $L = \mathcal{O}_X(\underline{p})$ and $B = \mathcal{O}_X(\underline{d})$ be as above. Then $\mathcal{O}_X(\underline{m})$ is L -regular with respect to B if and only if*

$$(1.1) \quad \max_{k \in J} \{p_k + m_k + l_k - l_J d_k\} \geq 0$$

for all nonempty subsets $J \subseteq \{1, \dots, r\}$.

Proof. Observe that

$$(1.2) \quad H^i(X, \mathcal{O}_X(\underline{m}) \otimes L \otimes B^{\otimes(-i)}) = H^i(X, \mathcal{O}_X(\underline{m} + \underline{p} - i\underline{d}))$$

and also that

$$H^i(X, \mathcal{O}_X(\underline{m} + \underline{p} - i\underline{d})) = \bigoplus_{i_1 + \cdots + i_r = i} \bigotimes_{k=1}^r H^{i_k}(\mathbb{P}^{l_k}, \mathcal{O}_{\mathbb{P}^{l_k}}(m_k + p_k - i d_k))$$

by the Künneth formula. Since $H^{i_k}(\mathbb{P}^{l_k}, \mathcal{O}_{\mathbb{P}^{l_k}}(j)) = 0$ when $i_k \neq 0, l_k$, we may assume that $i = l_J$ for some $\emptyset \neq J \subseteq \{1, \dots, r\}$ and that

$$i_k = \begin{cases} l_k, & k \in J, \\ 0, & k \notin J. \end{cases}$$

First suppose that (1.1) is satisfied for all $J \neq \emptyset$. Given such a J , pick $k \in J$ such that

$$p_k + m_k + l_k - l_J d_k \geq 0.$$

Then $m_k + p_k - l_J d_k \geq -l_k$, so that $H^{l_k}(\mathbb{P}^{l_k}, \mathcal{O}_{\mathbb{P}^{l_k}}(m_k + p_k - l_J d_k)) = 0$ by standard vanishing theorems for line bundles on projective space. By the above analysis, it follows easily that (1.2) vanishes for $i > 0$.

Next suppose that

$$(1.3) \quad \max_{k \in J} \{p_k + m_k + l_k - l_J d_k\} < 0$$

for some $J \neq \emptyset$. Among all such subsets J , pick one of maximum cardinality. For this J , we will prove that (1.2) is nonzero when $i = l_J$. By the Künneth formula, it suffices to show that

$$(1.4) \quad \begin{aligned} H^{l_k}(\mathbb{P}^{l_k}, \mathcal{O}_{\mathbb{P}^{l_k}}(m_k + p_k - l_J d_k)) &\neq 0, & k \in J, \\ H^0(\mathbb{P}^{l_k}, \mathcal{O}_{\mathbb{P}^{l_k}}(m_k + p_k - l_J d_k)) &\neq 0, & k \notin J. \end{aligned}$$

The first line of (1.4) is easy, since by Serre duality,

$$H^{l_k}(\mathbb{P}^{l_k}, \mathcal{O}_{\mathbb{P}^{l_k}}(m_k + p_k - l_J d_k)) \simeq H^0(\mathbb{P}^{l_k}, \mathcal{O}_{\mathbb{P}^{l_k}}(-m_k - p_k + l_J d_k - l_k - 1))^*.$$

This is nonzero provided $-m_k - p_k + l_J d_k - l_k - 1 \geq 0$, which for $k \in J$ follows immediately from (1.3). For the second line of (1.4), suppose that $k \notin J$. By the maximality of J , we must have

$$(1.5) \quad \max_{s \in J \cup \{k\}} \{p_s + m_s + l_s - l_{J \cup \{k\}} d_s\} \geq 0.$$

Note that $l_{J \cup \{k\}} = l_J + l_k$. If the maximum in (1.5) occurs at an element of J , say $s \in J$, then

$$p_s + m_s + l_s - (l_J + l_k) d_s \geq 0,$$

which is impossible since $s \in J$ and (1.3) imply that $p_s + m_s + l_s - l_J d_s < 0$. Hence the maximum occurs at k , so that

$$p_k + m_k + l_k - (l_J + l_k) d_k \geq 0.$$

This implies

$$p_k + m_k - l_J d_k \geq l_k d_k - l_k \geq 0.$$

It follows that the second line of (1.4) is nonzero, as desired. $\square$

If we fix the sheaf $\mathcal{O}_X(\underline{m})$ and let $L = \mathcal{O}_X(\underline{p})$ vary over all $\underline{p} \in \mathbb{Z}^r$, we get the *regularity set*

$$\text{Reg}(\mathcal{O}_X(\underline{m})) = \{\underline{p} \in \mathbb{Z}^r \mid \mathcal{O}_X(\underline{m}) \text{ is } \mathcal{O}_X(\underline{p})\text{-regular for } B = \mathcal{O}_X(\underline{d})\}.$$

This set is easy to describe. Given a permutation σ in the symmetric group S_r , let $J(\sigma, k) \subseteq \{1, \dots, r\}$ denote the subset

$$J(\sigma, k) = \{\sigma(i) \mid i \geq \sigma^{-1}(k)\}, \quad 1 \leq k \leq r.$$

Thus

$$J(\sigma, \sigma(1)) = \{\sigma(1), \sigma(2), \dots, \sigma(r)\}$$

$$J(\sigma, \sigma(2)) = \{\sigma(2), \dots, \sigma(r)\}$$

$$\vdots$$

$$J(\sigma, \sigma(r)) = \{\sigma(r)\}.$$

Then define

$$\underline{p}_\sigma = (-m_1 - l_1 + l_{J(\sigma, 1)} d_1, \dots, -m_r - l_r + l_{J(\sigma, r)} d_r).$$

Proposition 1.3. *The regularity set of $\mathcal{O}_X(\underline{m})$ is the union of $r!$ translates of $\mathbb{N}^r$ given by*

$$\text{Reg}(\mathcal{O}_X(\underline{m})) = \bigcup_{\sigma \in S_r} (\underline{p}_\sigma + \mathbb{N}^r).$$

Proof. First suppose that $\mathcal{O}_X(\underline{m})$ is $\mathcal{O}_X(\underline{p})$ -regular for B . We build a permutation $\sigma \in S_r$ as follows. Theorem 1.2 tells us that (1.1) holds for all $J \neq \emptyset$. Hence, for $J = \{1, \dots, r\}$, there is k_1 such that

$$p_{k_1} + m_{k_1} + l_{k_1} - l_J d_{k_1} \geq 0.$$

Setting $\sigma(1) = k_1$, this becomes

$$p_{k_1} \geq -m_{k_1} - l_{k_1} + l_{J(\sigma, k_1)} d_{k_1}.$$

Applying (1.1) to $J = \{1, \dots, r\} \setminus \{\sigma(1)\}$ gives $k_2 \neq k_1$ with

$$p_{k_2} + m_{k_2} + l_{k_2} - l_J d_{k_2} \geq 0.$$

Setting $\sigma(1) = k_2$, this becomes

$$p_{k_2} \geq -m_{k_2} - l_{k_2} + l_{J(\sigma, k_2)} d_{k_2}.$$

Continuing in this way gives $\sigma \in S_r$ with $\underline{p} \in \underline{p}_\sigma + \mathbb{N}^r$.

Conversely, let $\underline{p} \in \underline{p}_\sigma + \mathbb{N}^r$ and take any nonempty $J \subseteq \{1, \dots, r\}$. Write $J = \{\sigma(i_1), \dots, \sigma(i_t)\}$ for $i_1 < \dots < i_t$. Then set $k = \sigma(i_1) \in J$ and observe that

$$J \subseteq \{\sigma(i_1), \sigma(i_1 + 1), \dots, \sigma(r)\} = J(\sigma, \sigma(i_1)) = J(\sigma, k).$$

This implies that $l_J \leq l_{J(\sigma, k)}$. Since $\underline{p} \in \underline{p}_\sigma + \mathbb{N}^r$, we obtain

$$p_k \geq -m_k - l_k + l_{J(\sigma, k)} d_k \geq -m_k - l_k + l_J d_k,$$

so that $p_k + m_k + l_k - l_J d_k \geq 0$. This proves that (1.1) holds for J , and since $J \neq \emptyset$ was arbitrary, we have $\underline{p} \in \text{Reg}(\mathcal{O}_X(\underline{m}))$ by Theorem 1.2. $\square$

We next apply our results to study the Castelnuovo-Mumford regularity of $\mathcal{O}_X(\underline{m})$ under the projective embedding given by the Segre-Veronese map

$$\nu_{\underline{d}} : \mathbb{P}^{l_1} \times \dots \times \mathbb{P}^{l_r} \longrightarrow \mathbb{P}^N$$

coming from $\mathcal{O}_X(\underline{d})$. This gives the sheaf $\mathcal{F}(\underline{m}) := \nu_{\underline{d}*} \mathcal{O}_X(\underline{m})$ on $\mathbb{P}^N$.

Theorem 1.4. *The sheaf $\mathcal{F}(\underline{m}) = \nu_{\underline{d}*} \mathcal{O}_X(\underline{m})$ is p -regular on $\mathbb{P}^N$ if and only if*

$$p \geq \max_{J \neq \emptyset} \left\{ \min_{k \in J} \left\{ l_J - \left\lfloor \frac{m_k + l_k}{d_k} \right\rfloor \right\} \right\}.$$

Hence the regularity of $\mathcal{F}(\underline{m})$ is given by

$$\text{reg}(\mathcal{F}(\underline{m})) = \max_{J \neq \emptyset} \left\{ \min_{k \in J} \left\{ l_J - \left\lfloor \frac{m_k + l_k}{d_k} \right\rfloor \right\} \right\}.$$

Proof. Let $B = \mathcal{O}_X(\underline{d})$. As noted earlier, $\mathcal{F}(\underline{m})$ is p -regular if and only if $\mathcal{O}_X(\underline{m})$ is $B^{\otimes p}$ -regular with respect to B . Since $B^{\otimes p} = \mathcal{O}_X(p\underline{d})$, Theorem 1.2 implies that $\mathcal{F}(\underline{m})$ is p -regular if and only if

$$\max_{k \in J} \{pd_k + m_k + l_k - l_J d_k\} \geq 0$$

for all subsets $J \neq \emptyset$ of $\{1, \dots, r\}$. This is equivalent to

$$p \geq \min_{k \in J} \left\{ l_J - \frac{m_k + l_k}{d_k} \right\}$$

for all $J \neq \emptyset$. From here, the first assertion of the theorem follows easily, and the second assertion is immediate. $\square$

Example 1.5. When $r = 2$, the usual Segre embedding $\nu : \mathbb{P}^a \times \mathbb{P}^b \rightarrow \mathbb{P}^{ab+a+b}$ gives $\mathcal{F}(k, l) = \nu_* \mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b}(k, l)$. Since $\underline{d} = (1, 1)$, Theorem 1.4 implies that

$$\begin{aligned} \text{reg}(\mathcal{F}(k, l)) &= \max\{a - \lfloor k + a \rfloor, b - \lfloor l + b \rfloor, \min\{a + b - \lfloor k + a \rfloor, a + b - \lfloor l + b \rfloor\}\} \\ &= \max\{-k, -l, \min\{b - k, a - l\}\} \\ &= \max\{-\min\{k, l\}, \min\{b - k, a - l\}\}. \end{aligned}$$

This is the regularity formula from [4, Lem. 3.1].

Example 1.6. Let $\mathcal{I}$ be the ideal sheaf of $Y = \nu_{\underline{d}}(X) \subset \mathbb{P}^N$. Following [1], set $q_k = \lfloor \frac{l_k+1}{d_k} \rfloor$ and $q_0 = \min_{1 \leq k \leq r} \{q_k\}$, and define

$$\lambda = \begin{cases} n+2-q_0, & q_0 = q_k \text{ for some } k \text{ with } d_k | l_k + 1, \\ n+1-q_0, & \text{otherwise.} \end{cases}$$

Recall that $n = \sum_{k=1}^r l_k = \dim(Y)$. Then [1, Lem. 3.4] asserts that $\mathcal{I}$ is λ -regular.

We can derive this from Theorem 1.4 as follows. We know that $\mathcal{I}$ is λ -regular if and only if $\mathcal{O}_Y$ is $(\lambda-1)$ -regular. Note also that λ can be defined more simply as

$$\lambda = n+1 - \min_{1 \leq k \leq r} \left\lfloor \frac{l_k}{d_k} \right\rfloor.$$

If $J \neq \emptyset$, it follows easily that

$$\lambda-1 \geq l_J - \min_{k \in J} \left\lfloor \frac{l_k}{d_k} \right\rfloor \geq l_J - \max_{k \in J} \left\lfloor \frac{l_k}{d_k} \right\rfloor = \min_{k \in J} \left\{ l_J - \left\lfloor \frac{l_k}{d_k} \right\rfloor \right\}.$$

By Theorem 1.4, we obtain $\lambda-1 \geq \text{reg}(\mathcal{F}(\underline{0})) = \text{reg}(\mathcal{O}_Y)$, so that $\lambda \geq \text{reg}(\mathcal{I})$, as claimed.

It is also easy to see that $\lambda-1 > \text{reg}(\mathcal{O}_Y)$ can occur. For example, if $r=2$ and $\underline{d} = (1, 1)$, then one can show without difficulty that

$$\lambda-1 = \max\{l_1, l_2\} \geq \min\{l_1, l_2\} = \text{reg}(\mathcal{O}_Y).$$

2. SUBADDITIVITY

In this section we study how the regularity of $\mathcal{O}_X(\underline{m})$ and $\mathcal{O}_X(\underline{m}')$ compares to the regularity of the tensor product

$$\mathcal{O}_X(\underline{m}) \otimes \mathcal{O}_X(\underline{m}') = \mathcal{O}_X(\underline{m} + \underline{m}').$$

We first consider regularity as defined in Definition 1.1.

Theorem 2.1. *Let $B = \mathcal{O}_X(\underline{d})$. If $\mathcal{O}_X(\underline{m})$ is $\mathcal{O}_X(\underline{p})$ -regular for B and $\mathcal{O}_X(\underline{m}')$ is $\mathcal{O}_X(\underline{p}')$ -regular for B , then $\mathcal{O}_X(\underline{m} + \underline{m}')$ is $\mathcal{O}_X(\underline{p} + \underline{p}')$ -regular for B .*

Proof. We will use Theorem 1.2. Given a nonempty subset $J \subseteq \{1, \dots, r\}$, it suffices to find $k \in J$ such that

$$(2.1) \quad p_k + p'_k + m_k + m'_k + l_k - l_J d_k \geq 0.$$

Using (1.1) for this J and the sheaves $\mathcal{O}_X(\underline{m})$ and $\mathcal{O}_X(\underline{p})$, we know that

$$\max_{s \in J} \{p_s + m_s + l_s - l_J d_s\} \geq 0.$$

Hence we can find $k \in J$ such that

$$(2.2) \quad p_k + m_k + l_k - l_J d_k \geq 0.$$

Then using (1.1) for $\{k\}$ and the sheaves $\mathcal{O}_X(\underline{m}')$ and $\mathcal{O}_X(\underline{p}')$, we also have

$$p'_k + m'_k + l_k - l_k d_k \geq 0.$$

Since $l_k \leq l_k d_k$, this implies that

$$(2.3) \quad p'_k + m'_k \geq 0.$$

Adding (2.2) and (2.3), we obtain the desired inequality (2.1). $\square$

Theorem 2.1 gives the following result concerning the Castelnuovo-Mumford regularity of the sheaves $\mathcal{F}(\underline{m}) = \nu_{\underline{d}*} \mathcal{O}_X(\underline{m})$ on projective space.

Theorem 2.2. *Given any $\underline{m}, \underline{m}' \in \mathbb{Z}^r$, we have*

$$\operatorname{reg}(\mathcal{F}(\underline{m})) + \operatorname{reg}(\mathcal{F}(\underline{m}')) \geq \operatorname{reg}(\mathcal{F}(\underline{m}) \otimes \mathcal{F}(\underline{m}')).$$

Proof. It suffices to show that if $\mathcal{F}(\underline{m})$ is p -regular and $\mathcal{F}(\underline{m}')$ is p' -regular, then $\mathcal{F}(\underline{m} + \underline{m}')$ is $(p + p')$ -regular. This follows immediately from Theorem 2.1 and the already-noted equivalence

$$\mathcal{F}(\underline{n}) \text{ is } q\text{-regular} \iff \mathcal{O}_X(\underline{n}) \text{ is } \mathcal{O}_X(q\underline{d})\text{-regular for } B = \mathcal{O}_X(\underline{d}). \quad \square$$

In general, regularity is not subadditive; i.e., given coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{P}^N$, the inequality

$$(2.4) \quad \operatorname{reg}(\mathcal{F}) + \operatorname{reg}(\mathcal{G}) \geq \operatorname{reg}(\mathcal{F} \otimes \mathcal{G})$$

may fail. Here is an example due to Chardin.

Example 2.3. Let $R = k[x, y, z, t]$ and consider the ideals $I_n = \langle z^n, t^n \rangle$ and $J_m = \langle x^{m-1}t - y^{m-1}z \rangle$. When $n, m \geq 3$, Example 1.13.6 of [2] implies that

$$(2.5) \quad \operatorname{reg}(I_n) + \operatorname{reg}(J_m) = m + 2n - 1 < \operatorname{reg}(I_n + J_m) = mn - 1.$$

As noted by Chardin [3], this remains true when we work in the larger ring $S = k[x, y, z, t, u, v]$. The key point is that the ideals I_n , J_m , and $I_n + J_m$ are saturated in S .

Now let $\mathcal{F}$ and $\mathcal{G}$ be the coherent sheaves associated to S/I_n and S/J_m respectively. Then $\mathcal{F} \otimes \mathcal{G}$ is the sheaf associated to $S/(I_n + J_m)$. Since I_n and J_m are saturated in S , (2.5) easily implies that

$$\operatorname{reg}(\mathcal{F}) + \operatorname{reg}(\mathcal{G}) = m + 2n - 3 < \operatorname{reg}(\mathcal{F} \otimes \mathcal{G}) = mn - 2$$

when $n, m \geq 3$. This shows that subadditivity fails in general.

However, there are certain situations where (2.4) does hold, such as when $\mathcal{F}$ or $\mathcal{G}$ is locally free (see [7, Prop. 1.8.9]). Theorem 2.1 shows that the sheaves $\mathcal{F}(\underline{m})$ give another class of coherent sheaves for which regularity is subadditive.

3. TATE RESOLUTIONS

By [5], a coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}(W)$ has a Tate resolution

$$\cdots \longrightarrow T^p(\mathcal{F}) \longrightarrow T^{p+1}(\mathcal{F}) \longrightarrow \cdots$$

of free graded E -modules, $E = \bigwedge W^*$, with terms

$$(3.1) \quad T^p(\mathcal{F}) = \bigoplus_i \widehat{E}(i - p) \otimes H^i(\mathcal{F}(p - i)).$$

Here, $\widehat{E} = \operatorname{Hom}_K(E, K)$ is the dual over the base field K .

Standard vanishing theorems imply that

$$T^p(\mathcal{F}) = \widehat{E}(-p) \otimes H^0(\mathcal{F}(p)) \quad \text{for } p \gg 0.$$

Furthermore, if $\mathcal{F} = i_*\mathcal{E}$ for a locally free sheaf $\mathcal{E}$ on an irreducible Cohen-Macaulay subvariety $Y \hookrightarrow \mathbb{P}(W)$, then Serre duality and the same vanishing theorems imply that

$$T^p(\mathcal{F}) = \widehat{E}(n - p) \otimes H^n(\mathcal{F}(p - n)) \quad \text{for } p \ll 0.$$

In this situation, we define

$$\begin{aligned} p^+ &= \min\{p \mid T^p(\mathcal{F}) = \widehat{E}(-p) \otimes H^0(\mathcal{F}(p))\} \\ p^- &= \max\{p \mid T^p(\mathcal{F}) = \widehat{E}(n - p) \otimes H^n(\mathcal{F}(p - n))\}. \end{aligned}$$

In general, the differentials in the Tate resolution are hard to describe. The exceptions are the “horizontal” maps

$$\widehat{E}(i-p) \otimes H^i(\mathcal{F}(p-i)) \longrightarrow \widehat{E}(i-p-1) \otimes H^i(\mathcal{F}(p+1-i)),$$

which can be written down explicitly (see [5]). It follows that the Tate resolution is known for $p \geq p^+$ and $p \leq p^-$. In other words, the interesting part of the Tate resolution lies in the range $p^- \leq p \leq p^+$.

For $X = \mathbb{P}^{l_1} \times \cdots \times \mathbb{P}^{l_r}$ and the Segre-Veronese embedding $\nu_{\underline{d}} : X \rightarrow \mathbb{P}^N = \mathbb{P}(W)$ coming from $\mathcal{O}_X(\underline{d})$, we have the sheaf $\mathcal{F}(\underline{m})$, $\underline{m} = (m_1, \dots, m_r)$. When $r = 2$ and $\underline{d} = (1, 1)$, the Tate resolution, including differentials, is described in [4]. In the general case considered here, we restrict ourselves to giving explicit formulas for p^+ and p^- . This determines the length of the interesting part of the Tate resolution.

To state our formulas, recall that $\underline{l} = (l_1, \dots, l_r)$ and $\underline{d} = (d_1, \dots, d_r)$. We also set $\underline{1} = (1, \dots, 1)$ and

$$\widetilde{\underline{m}} = -\underline{m} + n\underline{d} - \underline{l} - \underline{1} = (-m_1 + nd_1 - l_1 - 1, \dots, -m_r + nd_r - l_r - 1).$$

Theorem 3.1. *For the sheaf $\mathcal{F}(\underline{m})$, we have*

$$\begin{aligned} p^+ &= \operatorname{reg}(\mathcal{F}(\underline{m})) = \max_{J \neq \emptyset} \left\{ \min_{k \in J} \left\{ l_J - \left\lfloor \frac{m_k + l_k}{d_k} \right\rfloor \right\} \right\} \\ p^- &= -\operatorname{reg}(\mathcal{F}(\widetilde{\underline{m}})) = -\max_{J \neq \emptyset} \left\{ \min_{k \in J} \left\{ \left\lceil \frac{m_k + 1}{d_k} \right\rceil - l_{J^c} \right\} \right\}, \end{aligned}$$

where J^c denotes the complement of $J \subseteq \{1, \dots, r\}$.

Proof. The equality $p^+ = \operatorname{reg}(\mathcal{F}(\underline{m}))$ is immediate from (3.1) and the definition of regularity. Then Theorem 1.4 gives the desired formula for p^+ . Since $\widetilde{\underline{m}} = -\underline{m} + n\underline{d} - \underline{l} - \underline{1}$, Theorem 1.4 also implies that

$$\begin{aligned} \operatorname{reg}(\mathcal{F}(\widetilde{\underline{m}})) &= \max_{J \neq \emptyset} \left\{ \min_{k \in J} \left\{ l_J - \left\lfloor \frac{(-m_k + nd_k - l_k - 1) + l_k}{d_k} \right\rfloor \right\} \right\} \\ &= \max_{J \neq \emptyset} \left\{ \min_{k \in J} \left\{ \left\lceil \frac{m_k + 1}{d_k} \right\rceil - l_{J^c} \right\} \right\}, \end{aligned}$$

where the last equality follows easily using $l_J - n = -l_{J^c}$.

It remains to prove $p^- = -\operatorname{reg}(\mathcal{F}(\widetilde{\underline{m}}))$. Observe that $p \leq p^-$ if and only if

$$(3.2) \quad H^{n-i}(\mathcal{F}(\underline{m})(p - (n-i))) = 0 \quad \text{for all } i > 0.$$

By Serre duality on $X = \mathbb{P}^{l_1} \times \cdots \times \mathbb{P}^{l_r}$,

$$\begin{aligned} H^{n-i}(\mathcal{F}(\underline{m})(p - (n-i))) &= H^{n-i}(X, \mathcal{O}_X(\underline{m} + (p - (n-i))\underline{d})) \\ &= H^i(X, \mathcal{O}_X(-\underline{m} - (p - n + i)\underline{d} - \underline{l} - \underline{1}))^* \\ &= H^i(\mathcal{F}(-\underline{m} + (n - p - i)\underline{d} - \underline{l} - \underline{1}))^* \\ &= H^i(\mathcal{F}(\widetilde{\underline{m}})(-p - i))^*, \end{aligned}$$

so that (3.2) is equivalent to

$$H^i(\mathcal{F}(\widetilde{\underline{m}})(-p - i)) = 0 \quad \text{for all } i > 0.$$

This vanishing is equivalent to $-p \geq \operatorname{reg}(\mathcal{F}(\widetilde{\underline{m}}))$, so that

$$p \leq p^- \iff -p \geq \operatorname{reg}(\mathcal{F}(\widetilde{\underline{m}})).$$

Hence $p^- = -\operatorname{reg}(\mathcal{F}(\widetilde{\underline{m}}))$, as desired. $\square$

Example 3.2. When $\underline{d} = (1, \dots, 1)$ and $\underline{l} = (l, \dots, l)$, we may assume that $\underline{m} = (m_1, \dots, m_r)$ with $m_1 \leq \dots \leq m_r$. Then one can show without difficulty that

$$p^+ = \max_{1 \leq i \leq r} \{(i-1)l - m_i\}$$

$$p^- = \min_{1 \leq i \leq r} \{(i-1)l - m_i\} - 1.$$

This makes it easy to compute $p^+ - p^-$, which is the length of the interesting part of the Tate resolution. For instance, if $m_1 = \dots = m_{r-1} = 0$ and $m_r = m$, where $m \geq (r-1)l$, then

$$p^+ - p^- = m - l + 1,$$

which can be arbitrarily large. On the other hand, if $\underline{m}$ is more balanced, say $m_1 = \dots = m_r = m$, then

$$p^+ - p^- = (r-1)l + 1,$$

independent of m .

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