CYCLIC BEHAVIOR OF THE CESÀRO OPERATOR ON $L_2(0, \infty)$

M. GONZÁLEZ AND F. LEÓN-SAAVEDRA

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ABSTRACT. In this paper we study the cyclic properties of the infinite continuous Cesàro operator defined on $L^2(0, \infty)$ by $(C_\infty f)(x) = \frac{1}{x} \int_0^x f(s) ds$. Despite this operator being cyclic, we show that it is not supercyclic; even more, it is not weakly supercyclic. These results complement some recent ones on the cyclic behavior of Cesàro operators.

1. INTRODUCTION AND MAIN RESULTS

The continuous Cesàro operators C_∞ and C_1 are defined on $L^2(0, \infty)$ and $L^2(0, 1)$, respectively, by the expression

$$(Cf)(x) := \frac{1}{x} \int_0^x f(s) ds.$$

Also, the discrete Cesàro operator C_0 is defined on the sequence space ℓ^2 by

$$C_0(x_1, x_2, \dots) = (x_1, (x_1 + x_2)/2, (x_1 + x_2 + x_3)/3, \dots).$$

It turns out that all three Cesàro operators are bounded linear transformations on their respective Hilbert spaces. Boundedness for C_0 and C_∞ was provided by Hardy, Littlewood and Pólya (see [6] Chapter IX). We refer the reader to the paper [4] by Brown, Halmos and Shields, where the authors, among other results, provide a less computational proof of the boundedness of the operators and determine their norms and various parts of the spectrum. Since the original paper [4], a large number of new results on Cesàro operators have appeared in the literature. In [5], the fine structure of the spectrum of C_0 as an operator in ℓ_p ($1 < p < \infty$) has been determined. Moreover, C_0 has been widely generalized and studied on different spaces of analytic functions.

A cyclic vector x for an operator T on a Banach space X is one whose orbit under the operator has a dense linear span. Supercyclicity was introduced in the seventies by Hilden and Wallen ([8]), defining x supercyclic for T if the projective orbit, that is, the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$, is dense in X . More recently, the notion of weak supercyclicity was introduced. A vector x is weakly supercyclic for

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T if the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is weakly dense in X . The existence of weakly supercyclic operators which are not norm supercyclic was established in [14].

Our main result is connected with the study of orbits of bilateral shifts. The hypercyclic and supercyclic behavior of unilateral and bilateral weighted shifts were studied by H. Salas in [12] and [13]. In [12] it is shown that the identity plus a unilateral weighted backward shift is always hypercyclic. This result was a fundamental piece in proving that any separable Banach space supports a hypercyclic operator (see [1] and [3]). As far as we know, the analogous result for bilateral weighted shifts is completely unknown: it is very difficult to compute the corresponding orbits.

Let W denote a simple bilateral shift, and let q be a polynomial with real coefficients. Then we show that $q(W)$ is not weakly supercyclic on $\ell^p(\mathbb{Z})$, for $1 \leq p \leq 2$. In particular, this is true for $I + W$. This result is sharp, because recently S. Shkarin (see [16]) proved that the unweighted bilateral shift W is weakly supercyclic on $\ell^p(\mathbb{Z})$, for $p > 2$. This paper arises from [11], where the authors studied the cyclic behavior of Cesàro type operators. They showed that C_0 and C_1 have opposite cyclic behavior: C_0 is not weakly supercyclic while C_1 is hypercyclic; that is, the orbit of some vector is dense. However, the techniques they used cannot be extended for C_∞ .

Here is an outline of the contents of the paper. In Section 2 we include a brief survey of prerequisites and tools that we will need later, and in Section 3 we prove that C_∞ is not weakly supercyclic on $L^2(0, \infty)$. We do not know if the analogous result for $p \neq 2$ is true. Finally, we show that the rationally strictly cyclic weighted shifts are not weakly supercyclic.

2. PREREQUISITES AND TOOLS

2.1. $I - C_\infty$ is a bilateral shift. An operator W on a Hilbert space H is a simple bilateral shift if H has an orthonormal basis $\{\cdots, e_{-2}, e_{-1}, e_0, e_1, e_2, \cdots\}$ such that $We_n = e_{n+1}$ for all n . It was proved by Brown, Halmos and Shields ([4], Theorem 5) that the operator $I - C_\infty^*$ is a simple bilateral shift. If W is a simple bilateral shift with respect to the orthonormal basis $\{e_k\}_{k=-\infty}^\infty$, then its adjoint satisfies $W^*e_n = e_{n-1}$, for all n . Then it is a simple bilateral shift with respect to the orthonormal basis $\{\cdots, e_2, e_1, e_0, e_{-1}, e_{-2}, \cdots\}$. Hence, $I - C_\infty$ is a simple bilateral shift.

To keep our exposition complete, we will describe the orthonormal basis $\{e_k\}_{k=-\infty}^\infty$ for which $I - C_\infty^*$ is a simple bilateral shift. Note that the operator $W = I - C_\infty^*$ has the following explicit representation:

$$Wf(x) = f(x) - \int_x^\infty \frac{1}{y} f(y) dy.$$

Let e_0 denote the characteristic function of $(0, 1)$. Then $e_n = W^n e_0$, for $n \geq 0$. The negative terms of the basis can be easily computed using the symmetry R , defined by

$$(Rf)(x) = -\frac{1}{x} f\left(\frac{1}{x}\right).$$

Thus $e_{-n-1} = Re_n$, for $n \geq 0$, and the symmetry R satisfies the following properties:

- (1) $Re_0 = W^{-1}e_0$,
- (2) $RW = W^{-1}R$, and

$$(3) \ R(\bigvee\{e_n : n \geq 0\}) = \{e_n : n \geq 0\}^\perp.$$

2.2. The commutant of a weighted bilateral shift. A description of the commutant of the bilateral weighted shift and other related results can be found in the excellent survey by A. Shields ([15]). Given an orthonormal basis $\{e_n\}_{n=-\infty}^\infty$ and a sequence of weights $\{w_n\}_{n=-\infty}^\infty$, we define a weighted bilateral shift by $We_n = w_n e_{n+1}$, where the weights satisfy $w_n > 0$ for all n and form a bounded sequence. The operator W can be identified with a multiplication operator M_z (multiplication by z) on the Hilbert space $L^2(\beta)$ of formal Laurent series $f(z) = \sum_{n=-\infty}^\infty \hat{f}(n)z^n$ (whether or not the series converges for any values of z) such that

$$\sum |\hat{f}(n)|^2 \beta(n)^2 < \infty,$$

where the sequence $\beta(n)$ is defined by recurrence by $\beta(n+1) = w_n \beta(n)$. When the weight sequence is $\beta(n) = 1$ for all n , M_z is the unweighed bilateral shift, and it is defined on the space $L^2(1)$.

Let us consider the multiplication of formal Laurent series $fg = h$:

$$\left(\sum \hat{f}(n)z^n\right) \left(\sum \hat{g}(n)z^n\right) = \sum \hat{h}(n)z^n,$$

where the numbers $\hat{h}(n)$ can be obtained by discrete convolution:

$$\hat{h}(n) = \sum_k \hat{f}(k)\hat{g}(n-k).$$

Then, the commutant of W , denoted by $L^\infty(\beta)$, consists of those formal Laurent series $\varphi(z) = \sum \hat{\varphi}(n)z^n$ for which $\varphi L^2(\beta) \subset L^2(\beta)$. Thus, an element of the commutant of W can be identified with a multiplication operator M_φ .

2.3. The positive supercyclicity theorem. The proof of our main result is based on two ideas. Let us suppose that x is a weakly supercyclic vector for $T \in B(X)$. Then the set

$$\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$$

must be weakly dense in X . The first idea is to find an element M in the commutant of T such that Mx fails to be weakly supercyclic.

To see that Mx is not weakly supercyclic, we will resort to the Positive Supercyclicity Theorem, discovered in [9] and developed later on other settings in [10]. This theorem asserts that, under certain conditions, supercyclicity and weak supercyclicity are equivalent to positive supercyclicity and positive weak supercyclicity, respectively. Namely, the statement below summarizes the second tool that we will need:

Theorem 2.1. *Let T be a bounded linear operator defined on a separable Banach space X . Let us suppose that $\sigma_p(T^*) = \emptyset$. Then T is supercyclic (respectively, weakly supercyclic) if and only if for any supercyclic (respectively, weakly supercyclic) vector x the orbit*

$$\{rT^n x : r > 0, n \in \mathbb{N}\}$$

is dense (respectively, weakly dense) in X .

We observe that Theorem 2.1 actually simplifies the notion of supercyclicity. It was originally proved for supercyclicity in [9], and later was proved for weak supercyclicity in [10].

3. THE MAIN RESULT

Theorem 3.1. *Let $1 \leq p \leq 2$ and let W be a simple bilateral shift on $\ell^p(\mathbb{Z})$. Suppose that Q is a polynomial with real coefficients. Then $Q(W)$ is not weakly supercyclic on $\ell^p(\mathbb{Z})$.*

Proof. Note that it is enough to prove the case $p = 2$. Indeed, $p \leq 2$ implies $\ell^p(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$ and any weakly dense subset of $\ell^p(\mathbb{Z})$ is also weakly dense in $\ell^2(\mathbb{Z})$.

Since $p = 2$, we can consider the representation of W as the multiplication operator M_z acting on $L^2(1)$. Suppose that $g \in L^2(1)$ is a weakly supercyclic vector for $Q(M_z)$. Let us denote $\varphi(z) := \overline{g(\bar{z})} \in L^2(1)$.

In general, the multiplication operator M_φ does not belong to the commutant of M_z , because in general, M_φ could be unbounded on $L^2(1)$. However, let us see that M_φ maps boundedly $L^2(1)$ into $L^2(1/n)$.

Indeed, if $f \in L^2(1)$, then

$$\varphi f(z) = \sum_{n=-\infty}^{\infty} \widehat{h}(n) z^n$$

and, applying the Cauchy-Schwarz inequality in the third line below, we get

$$\begin{aligned} |\widehat{h}(n)| &= \left| \sum_k \widehat{\varphi}(k) \widehat{f}(n-k) \right| \\ &\leq \sum_k |\widehat{\varphi}(k)| |\widehat{f}(n-k)| \\ &\leq \left(\sum_k |\widehat{\varphi}(k)|^2 \right)^{1/2} \left(\sum_k |\widehat{f}(k)|^2 \right)^{1/2} \\ &= \|\varphi\|_{L^2(1)} \|f\|_{L^2(1)}. \end{aligned}$$

Therefore

$$\sum_n |\widehat{h}(n)|^2 \frac{1}{n^2} \leq \sum_n \|\varphi\|_{L^2(1)}^2 \|f\|_{L^2(1)}^2 \frac{1}{n^2} \leq C \|\varphi\|_{L^2(1)}^2 \|f\|_{L^2(1)}^2,$$

for some constant $C > 0$, and the boundedness of $M_\varphi : L^2(1) \rightarrow L^2(1/n)$ follows directly from the above equation.

On the other hand, from the properties of the discrete convolution, we derive

$$\widehat{h}(n) = \sum_k \overline{\widehat{f}(k)} \widehat{f}(n-k) \in \mathbb{R}, \quad \text{for all } n \in \mathbb{Z}.$$

Let us denote by R the weak closure of the range of M_φ , and let us suppose that the set

$$\{rQ(M_z)^n g : r \in \mathbb{R}_+ \ n \in \mathbb{N}\}$$

is weakly dense in $L^2(\mathbb{Z})$. Then its image under M_φ must be weakly dense on R . However, since the vector $M_\varphi g$ has real coefficients, the set

$$M_\varphi \{rQ(M_z)^n g : r \in \mathbb{R}_+ \ n \in \mathbb{N}\} = \{rQ(M_z)^n M_\varphi g : r \in \mathbb{R}_+ \ n \in \mathbb{N}\}$$

is non-trivial and it consists of vectors with real coefficients. Therefore this set cannot be weakly dense in R , a contradiction. $\square$

The previous result can be adapted for bilateral weighted shifts for which the weight sequence $\{w_n\}_{n \in \mathbb{Z}}$ satisfies a growth condition which guarantees the application of our argument.

Corollary 3.2. *Let W be a weighted bilateral shift with associated weight sequence $\{w_n\}_{n \in \mathbb{Z}}$. Suppose that*

$$\sup_n \sup_k \frac{1}{\beta(k)\beta(n-k)} < \infty.$$

Then, given a polynomial with real coefficients Q , the operator $Q(W)$ is not weakly supercyclic on $\ell^p(\mathbb{Z})$ for $1 \leq p \leq 2$.

Proof. Let $f \in L^p(\beta)$. In accordance with Theorem 3.1, it is enough to show that the multiplication operator

$$M_{\bar{f}} : L^2(\beta) \rightarrow L^2(1/n)$$

is bounded. Since $M_{\bar{f}}g = \sum \hat{h}(n)z^n$ with

$$\hat{h}(n) = \sum_k \bar{\hat{f}}(k)\hat{g}(n-k),$$

applying the Cauchy-Schwarz inequality we get

$$\begin{aligned} |\hat{h}(n)| &= \sum_k |\bar{\hat{f}}(k)| |\hat{f}(n-k)| \beta(k)\beta(n-k) \frac{1}{\beta(k)\beta(n-k)} \\ &\leq \text{Const.} \|\bar{\hat{f}}\|_{L^2(\beta)} \|g\|_{L^2(\beta)}. \end{aligned}$$

Therefore

$$\sum_n |\hat{h}(n)|^2 \frac{1}{n^2} \leq \sum_n \|\bar{\hat{f}}\|_{L^2(\beta)}^2 \|g\|_{L^2(\beta)}^2 \frac{1}{n^2} \leq \text{Const.} \|\bar{\hat{f}}\|_{L^2(\beta)}^2 \|g\|_{L^2(\beta)}^2,$$

which proves the desired result. $\square$

Corollary 3.3. *The infinite Cesàro operator C_∞ is not weakly supercyclic on $L^2(0, \infty)$.*

Remarks 3.4. (a) Theorem 3.1 is sharp. As we mentioned in the introduction, S. Shkarin (see [16]) proved recently that the unweighted bilateral shift W is weakly supercyclic on $\ell^p(\mathbb{Z})$ for $p > 2$.

(b) The weight condition in Corollary 3.2 does not include the case when $\{w_n\}$ converges to zero, i.e., when W is compact. It would be interesting to know in such a case when $Q(W)$ is weakly supercyclic.

(c) It is worth noting that one could also demonstrate that $p(W)$ is not weakly supercyclic for any polynomial p (even with complex coefficients) by applying a result of F. Bayart and E. Matheron [2]. Namely, Bayart and Matheron proved that

- (a) A normal operator which is not a scalar multiple of a unitary operator is not weakly supercyclic.
- (b) If a unitary operator is weakly supercyclic, then its spectral measure is continuous and purely singular.

Then one could observe that W is unitarily equivalent to the operator $Mf(z) = zf(z)$ of multiplication by the argument acting on $L_2(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with the Lebesgue measure. Hence for each polynomial p , $p(W)$ is unitarily equivalent to multiplication by $p(z)$ acting on $L_2(\mathbb{T})$. If $|p|$ is not a constant on $\mathbb{T}$, then $p(W)$ is normal and not a scalar multiple of a unitary operator. Therefore $p(W)$ is not weakly supercyclic, according to (1). If p is a constant, then $p(W) = cI$ is not even cyclic. Finally, let p be non-constant and $|p|$ be constant on $\mathbb{T}$. Without loss of generality, $|p| = 1$ on $\mathbb{T}$. Then $p(W)$ is a unitary operator with purely absolutely continuous spectral measure. Thus $p(W)$ is not weakly supercyclic according to (2).

4. AN EXAMPLE: RATIONALLY STRICTLY CYCLIC BILATERAL SHIFTS

As we can see in the previous section, our technique relies on the size of the commutant of the operator. In this section we will study the case when the commutant of the bilateral weighted shift W is the biggest possible, that is, when W is rationally strictly cyclic.

Let W be a bilateral weighted shift with weight sequence $\{w_n\}$ defined on a Hilbert space H . Denote by $\mathcal{A}_W$ the weak operator closure of the rational functions in W . The operator W is said to be rationally strictly cyclic if there exists $x \in H$ such that $\{Ax : A \in \mathcal{A}_W\} = H$. This provides a natural class of examples of cyclic operators which are not weakly supercyclic.

Theorem 4.1. *Let W be a rationally strictly cyclic bilateral shift on $\ell^p(\mathbb{Z})$. If Q is a polynomial with real coefficients, then the operator $Q(W)$ is not weakly supercyclic.*

Proof. Let us consider W as multiplication by z on the space $L^p(\beta)$, with β the sequence defined by $\{w_n\}$. Let $g \in L^p(\beta)$ be a weakly supercyclic vector for M_z . It is well known that M_z is rationally strictly cyclic if and only if $L^p(\beta) = L^\infty(\beta)$ (see [15], §9). Then, the complex conjugate φ of g , defined by $\varphi(z) = \overline{g(\bar{z})}$, belongs to $L^p(\beta) = L^\infty(\beta)$; therefore, the multiplication operator M_φ maps boundedly $L^p(\beta)$ into $L^p(\beta)$ and it commutes with M_z .

We can suppose without loss of generality that $\sigma_p(M_z^*) = \emptyset$. Otherwise, by circular symmetry (see [15], §5), the point spectrum of M_z^* contains more than one point, and this implies by a result of D. Herrero ([7]) that M_z is not weakly supercyclic.

Therefore, applying Theorem 2.1, we conclude that if the set

$$\{rQ(M_z)^n g : r \in \mathbb{R}_+, n \in \mathbb{N}\}$$

is weakly dense in $L^p(\beta)$, then its image under M_φ must be weakly dense on the rank of M_φ :

$$M_\varphi \{rQ(M_z)^n g : r \in \mathbb{R}_+, n \in \mathbb{N}\} = \{rQ(M_z)^n M_\varphi g : r \in \mathbb{R}_+, n \in \mathbb{N}\}.$$

However, this set is non-trivial and it consists only of vectors with real coefficients, a contradiction. $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CANTABRIA, FACULTAD DE CIENCIAS, AVDA. DE LOS CASTROS S/N, E-39071-SANTANDER, SPAIN

E-mail address: gonzalelem@unican.es

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CÁDIZ, AVDA. DE LA UNIVERSIDAD S/N, E-11405-JEREZ DE LA FRONTERA, SPAIN

E-mail address: fernando.leon@uca.es