THE CRYSTAL $\mathcal{B}(\infty)$ AND EXTENDED NAKAJIMA MONOMIALS

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ABSTRACT. We construct the set of *extended Nakajima monomials* containing monomials realizing the irreducible highest weight crystals, introduced by Nakajima. We present explicit descriptions of a crystal $\mathcal{B}_\infty$ and the crystal $\mathcal{B}(\infty)$ in this language for the case $A_n^{(1)}$.

1. INTRODUCTION

The quantum group $U_q(\mathfrak{g})$ is a q -deformation of the universal enveloping algebra over a Lie algebra $\mathfrak{g}$, and crystal bases reveal the structure of $U_q(\mathfrak{g})$ -modules in a very simplified form. As these $U_q(\mathfrak{g})$ -modules are known to be q -deformations of modules over the original Lie algebras, knowledge of these structures also affects the study of Lie algebras.

The crystal base of the negative part $U_q^-(\mathfrak{g})$ of a quantum group, denoted by $\mathcal{B}(\infty)$, has received attention since the very birth of crystal base theory [3]. This is not only because it is an essential part of the grand loop argument proving the existence of crystal bases, but because it gives insight into the structure of the quantum group itself. The present paper addresses the problem of describing the crystal base $\mathcal{B}(\infty)$.

Nakajima introduced a certain set of monomials [10]. Kashiwara and Nakajima independently defined a crystal structure on it and also gave a monomial realization for the irreducible highest weight crystal $\mathcal{B}(\lambda)$ [4, 10]. Starting from their results, we can argue that it is not possible to find $\mathcal{B}(\infty)$ within the set of Nakajima monomials with their given crystal structure. Hence, in this work, we first construct the set of *extended Nakajima monomials* and develop a crystal structure on it. Actually, the set of Nakajima monomials can be embedded as a subcrystal in this set of extended Nakajima monomials. Thus, the monomial theory developed for the crystal $\mathcal{B}(\lambda)$ can easily be transferred to the extended monomial set. Next, we give an explicit monomial description of the limit $\mathcal{B}_\infty$ of the coherent family of perfect crystals $\mathcal{B}_l$ appearing in [2] and the crystal $\mathcal{B}(\infty)$ for the $A_n^{(1)}$ -type.

Let us briefly remark on a work that followed naturally from the present paper. Using extended Nakajima monomials introduced in the present work,¹ we were able

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¹A previous version of this paper [6] was available when the works [8, 9] were being prepared.

to give explicit descriptions of the crystal $\mathcal{B}(\infty)$ for all classical finite types and the G_2 -type [8, 9]. While, in this paper, we had to rely on the path realization [2] of $\mathcal{B}(\infty)$ for the $A_n^{(1)}$ -type to obtain the monomial description for $\mathcal{B}(\infty)$, the results for all classical finite types and the G_2 -type relied on the Young tableau realization [1, 7].

Now, we start our current work by introducing the notation. Let I be a finite index set, and let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix. The free abelian group $P^\vee = (\bigoplus_{i \in I} \mathbf{Z} h_i) \oplus (\bigoplus_{j=1}^{\text{corank } A} \mathbf{Z} d_j)$ is the dual weight lattice. Let $\Pi = \{\alpha_i | i \in I\}$ denote the set of simple roots and $\Pi^\vee = \{h_i | i \in I\}$ the set of simple coroots. The weight lattice $P = \{\lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subset \mathbf{Z}\}$ with $\mathfrak{h} = \mathbf{Q} \otimes_{\mathbf{Z}} P^\vee$ contains the set $P^+ = \{\lambda \in P | \lambda(h_i) \in \mathbf{Z}_{\geq 0} \text{ for all } i \in I\}$ of dominant integral weights. The quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ defined as above is said to form a *Cartan datum* associated with the generalized Cartan matrix A .

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. We denote by $U_q(\mathfrak{g})$ the quantum group associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ and let e_i, f_i ($i \in I$), q^h ($h \in P^\vee$) be the generators of the quantum group. We also denote by $U_q^-(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by f_i ($i \in I$) and by $\mathcal{B}(\infty)$ the crystal base of $U_q^-(\mathfrak{g})$. Throughout this paper, a $U_q(\mathfrak{g})$ -crystal will refer to a (abstract) crystal associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. The crystal base $\mathcal{B}(\infty)$ of $U_q^-(\mathfrak{g})$ is a $U_q(\mathfrak{g})$ -crystal.

2. EXTENDED NAKAJIMA MONOMIALS

Let $\mathcal{M}^\mathcal{E}$ be a certain set of formal monomials in the variables $Y_i(m)^{(1,0)}$ and $Y_i(m)^{(0,1)}$ ($i \in I, m \in \mathbf{Z}$) given by

$$(2.1) \quad \mathcal{M}^\mathcal{E} = \left\{ \prod_{(i,m) \in I \times \mathbf{Z}} Y_i(m)^{y_i(m)} \left| \begin{array}{l} y_i(m) = (y_i^0(m), y_i^1(m)) \in \mathbf{Z} \times \mathbf{Z} \\ \text{vanish except at finitely many } (i, m) \end{array} \right. \right\}.$$

The product of monomials $Y_i(m)^{(u,v)}$ and $Y_i(m)^{(u',v')}$ is set to $Y_i(m)^{(u+u', v+v')}$, for $(u, v), (u', v') \in \mathbf{Z} \times \mathbf{Z}$. We give the lexicographic order to the set $\mathbf{Z} \times \mathbf{Z}$ of variable exponents.

Fix any set of integers $c = (c_{ij})_{i \neq j \in I}$ such that $c_{ij} + c_{ji} = 1$, and set

$$(2.2) \quad A_i(m)^{\pm 1} = Y_i(m)^{(0, \pm 1)} Y_i(m+1)^{(0, \pm 1)} \prod_{j \neq i} Y_j(m + c_{ji})^{(0, \pm \langle h_j, \alpha_i \rangle)}.$$

We define the following maps to give a crystal structure on $\mathcal{M}^\mathcal{E}$. For each monomial $M = \prod_{(i,m) \in I \times \mathbf{Z}} Y_i(m)^{y_i(m)} \in \mathcal{M}^\mathcal{E}$, we set

$$\begin{aligned} \tilde{\text{wt}}(M) &= \sum_{i \in I} \left(\sum_{m \in \mathbf{Z}} y_i(m) \right) \Lambda_i, \\ \tilde{\varphi}_i(M) &= \max \left\{ \sum_{k \leq m} y_i(k) \mid m \in \mathbf{Z} \right\}, \quad \tilde{\varepsilon}_i(M) = \max \left\{ - \sum_{k > m} y_i(k) \mid m \in \mathbf{Z} \right\}. \end{aligned}$$

Notice that the coefficients of $\tilde{\text{wt}}(M)$ are pairs of integers. In this setting, we have $\tilde{\varphi}_i(M) \geq (0, 0)$, $\tilde{\varepsilon}_i(M) \geq (0, 0)$, and $\tilde{\text{wt}}(M) = \sum_i (\tilde{\varphi}_i(M) - \tilde{\varepsilon}_i(M))\Lambda_i$. We set

$$(2.3) \quad \text{wt}(M) = \sum_{i \in I} \left(\sum_{m \in \mathbf{Z}} y_i^1(m) \right) \Lambda_i,$$

$$(2.4) \quad \varphi_i(M) = \sum_{k \leq m} y_i^1(k) \quad \text{where } \tilde{\varphi}_i(M) = \sum_{k \leq m} (y_i^0(k), y_i^1(k)),$$

$$(2.5) \quad \varepsilon_i(M) = - \sum_{k > m} y_i^1(k) \quad \text{where } \tilde{\varepsilon}_i(M) = - \sum_{k > m} (y_i^0(k), y_i^1(k)).$$

Then trivially we have $\text{wt}(M) = \sum_i (\varphi_i(M) - \varepsilon_i(M))\Lambda_i$. From the above definition, $Y_i(m)^{(0,1)}$ has weight Λ_i , and so $A_i(m)$ has weight α_i . We define the action of Kashiwara operators as

$$(2.6) \quad \tilde{f}_i(M) = \begin{cases} 0 & \text{if } \tilde{\varphi}_i(M) = (0, 0), \\ A_i(m_f)^{-1}M & \text{if } \tilde{\varphi}_i(M) > (0, 0), \end{cases}$$

$$(2.7) \quad \tilde{e}_i(M) = \begin{cases} 0 & \text{if } \tilde{\varepsilon}_i(M) = (0, 0), \\ A_i(m_e)M & \text{if } \tilde{\varepsilon}_i(M) > (0, 0), \end{cases}$$

$$(2.8) \quad m_f = \min \left\{ m \mid \tilde{\varphi}_i(M) = \sum_{k \leq m} y_i(k) \right\}, \quad m_e = \max \left\{ m \mid \tilde{\varepsilon}_i(M) = - \sum_{k > m} y_i(k) \right\}.$$

Note that $y_i(m_f) > (0, 0)$, $y_i(m_f + 1) \leq (0, 0)$, $y_i(m_e + 1) < (0, 0)$, and $y_i(m_e) \geq (0, 0)$.

Most parts of proving $\mathcal{M}^{\mathcal{E}}$ to be a crystal are straightforward. It is straightforward to show

$$\varepsilon_i(\tilde{f}_i M) = \varepsilon_i(M) + 1, \quad \varphi_i(\tilde{e}_i M) = \varphi_i(M) + 1 \quad \text{if } \tilde{f}_i M, \tilde{e}_i M \in \mathcal{M}^{\mathcal{E}},$$

which is one of the conditions that should be satisfied by a crystal, and this can then be used to show

$$\tilde{f}_i(M) = M' \text{ if and only if } M = \tilde{e}_i(M') \text{ for } M, M' \in \mathcal{M}^{\mathcal{E}}, i \in I.$$

Now, we have the following result.

Theorem 2.1. *For any fixed set of integers $c = (c_{ij})_{i \neq j \in I}$ such that $c_{ij} + c_{ji} = 1$, the Kashiwara operators defined in (2.6) and (2.7), together with the maps wt , φ_i , ε_i ($i \in I$) of (2.3) to (2.5), define a $U_q(\mathfrak{g})$ -crystal structure on the set $\mathcal{M}^{\mathcal{E}}$.*

We refer to an element of the set $\mathcal{M}^{\mathcal{E}}$ as an *extended Nakajima monomial* and denote by $\mathcal{M}_c^{\mathcal{E}}$ the set $\mathcal{M}^{\mathcal{E}}$ subject to the crystal structure depending on the set c , as given above.

We can obtain the following facts about the relations among the set of extended Nakajima monomials with different crystal structures.

Proposition 2.2.

- (1) *For any fixed families of integers $(m_i)_{i \in I}$ and $c = (c_{ij})_{i \neq j \in I}$ such that $c_{ij} + c_{ji} = 1$, let us set $c' = (c'_{ij})_{i \neq j \in I}$ by $c'_{ij} = c_{ij} + m_i - m_j$. Then the map $\tau : \mathcal{M}_c^{\mathcal{E}} \rightarrow \mathcal{M}_{c'}^{\mathcal{E}}$ given by $Y_i(m)^{(0,1)} \mapsto Y_i(m + m_i)^{(0,1)}$ and $Y_i(m)^{(1,0)} \mapsto Y_i(m + m_i)^{(1,0)}$ gives a crystal isomorphism.*

- (2) For Kac-Moody algebras corresponding to the Dynkin diagrams containing no loops, all the different crystal structures on the set of extended Nakajima monomials, corresponding to different choices of the set c , are isomorphic.

Proof. (1) Since the set $c' = (c'_{ij})_{i \neq j \in I}$ satisfies the condition $c'_{ij} + c'_{ji} = 1$, the crystal $\mathcal{M}_{c'}^{\mathcal{E}}$ is well defined. The outputs of the functions wt , φ_i , and ε_i , defined in (2.3), (2.4), and (2.5), do not depend on any fixed integer m . Thus, to show that the map τ is a crystal isomorphism, it is enough to show that the map τ commutes with the Kashiwara operators:

$$\begin{array}{ccc} \prod_{(i,m)} Y_i(m)^{y_i(m)} & \xrightarrow{\tau} & \prod_{(i,m)} Y_i(m+m_i)^{y_i(m)} \\ \tilde{f}_k \downarrow & & \downarrow \tilde{f}_k \\ A_k(m_f)^{-1} \left(\prod_{(i,m)} Y_i(m)^{y_i(m)} \right) & \xrightarrow{\tau} & A_k(m_f+m_k)^{-1} \left(\prod_{(i,m)} Y_i(m+m_i)^{y_i(m)} \right). \end{array}$$

Since

$$\begin{aligned} A_k(m_f) &= Y_k(m_f)^{(0,1)} Y_k(m_f+1)^{(0,1)} \prod_{j \neq k} Y_j(m_f+c_{jk})^{(0, \langle h_j, \alpha_k \rangle)}, \\ A_k(m_f+m_k) &= Y_k(m_f+m_k)^{(0,1)} Y_k(m_f+m_k+1)^{(0,1)} \\ &\quad \cdot \prod_{j \neq k} Y_j(m_f+m_k+c'_{jk})^{(0, \langle h_j, \alpha_k \rangle)}, \end{aligned}$$

and $c'_{jk} = c_{jk} + m_j - m_k$, the above diagram commutes.

(2) Let us now consider the crystal structure given to the set of extended Nakajima monomials for Kac-Moody algebras corresponding to the Dynkin diagrams containing no loops.

For any two fixed sets $c = (c_{ij})_{i \neq j \in I}$ and $c' = (c'_{ij})_{i \neq j \in I}$, there always exists the set of integers $(m_i)_{i \in I}$ satisfying the conditions $c'_{uv} = c_{uv} + m_u - m_v$, where the indices run over all $u \neq v \in I$, for which the Cartan matrix entry a_{uv} is non-zero. With the set $(m_i)_{i \in I}$, we define the map $\mathcal{M}_c^{\mathcal{E}} \rightarrow \mathcal{M}_{c'}^{\mathcal{E}}$ by setting $Y_i(m)^{(0,1)} \mapsto Y_i(m+m_i)^{(0,1)}$ and $Y_i(m)^{(1,0)} \mapsto Y_i(m+m_i)^{(1,0)}$. Then the crystals $\mathcal{M}_c^{\mathcal{E}}$ and $\mathcal{M}_{c'}^{\mathcal{E}}$ are isomorphic under this map. $\square$

Remark 2.3. The set of monomials $\prod_{(i,m)} Y_i(m)^{(y_i^0(m), y_i^1(m))} \in \mathcal{M}_c^{\mathcal{E}}$ with $y_i^0(m) = 0$ for all $(i, m) \in I \times \mathbf{Z}$ is a subcrystal of $\mathcal{M}_c^{\mathcal{E}}$, which is isomorphic to the Nakajima monomial set $\mathcal{M}_c$ introduced in [4] under the natural identification. Viewing $\mathcal{M}_c$ as a subcrystal of $\mathcal{M}_c^{\mathcal{E}}$, the monomial theory developed for irreducible highest weight crystals can easily be transferred to that on the extended monomial set.

3. DESCRIPTION OF THE CRYSTAL $\mathcal{B}_{\infty}$

We give an explicit monomial description of the crystal $\mathcal{B}_{\infty}$ for the $A_n^{(1)}$ case appearing in [2], by relating it to the description of $\mathcal{B}_{\infty}$ given therein, i.e.,

$$(3.1) \quad \mathcal{B}_{\infty} = \left\{ (u_0, \dots, u_n) \in \mathbf{Z}^{n+1} \mid \sum_{i=0}^n u_i = 0 \right\} \text{ with } b_{\infty} = (0, \dots, 0).$$

For $b = (u_0, \dots, u_n) \in \mathcal{B}_{\infty}$, the action of the Kashiwara operators $\tilde{f}_i$ is given by

$$\tilde{f}_i b = \begin{cases} (u_0, \dots, u_{i-1}, u_i - 1, u_{i+1} + 1, u_{i+2}, \dots, u_n) & \text{for } i = 0, \dots, n-1, \\ (u_0 + 1, u_1, \dots, u_{n-1}, u_n - 1) & \text{for } i = n. \end{cases}$$

For the remaining maps describing the crystal structure on $\mathcal{B}_{\infty}$, see [2].

We take the set $c = (c_{ij})_{i \neq j \in I = \{0,1,\dots,n\}}$ to be $c_{0n} = 0$, $c_{n0} = 1$, $c_{ij} = 0$ when $i > j$ and $(i, j) \neq (n, 0)$, $c_{ij} = 1$ when $i < j$ and $(i, j) \neq (0, n)$. Then for $i \in I = \{0, 1, \dots, n\}$ and $m \in \mathbf{Z}$,

$$A_i(m)^{\pm 1} = Y_i(m)^{(0,\pm 1)} Y_i(m+1)^{(0,\pm 1)} Y_{i-1}(m+1)^{(0,\mp 1)} Y_{i+1}(m)^{(0,\mp 1)},$$

where the subscript j in $Y_j(k)^{(p,q)}$ is read modulo $(n+1)$.

Remark 3.1. Since the Dynkin diagram corresponding to Kac-Moody algebras of type $A_n^{(1)}$ ($n \geq 2$) has a loop, all the different crystal structures on the set of extended Nakajima monomials, corresponding to different choices of the set c , are not always isomorphic. The theory to be developed on the fixed $\mathcal{M}_c^\mathcal{E}$ can similarly be developed on all such crystals isomorphic to $\mathcal{M}_c^\mathcal{E}$ under the isomorphism given in Proposition 2.2 (2).

From now on, we shall omit c and use the notation $\mathcal{M}^\mathcal{E}$ instead of $\mathcal{M}_c^\mathcal{E}$, since we have already fixed the set c . For convenience, we use the following notation.

Definition 3.2. For $(v_0, \dots, v_n), (u_0, \dots, u_n) \in \mathbf{Z}^{n+1}$, and $m \in \mathbf{Z}$, set

$$X_{(u_0, \dots, u_n)}^{(v_0, \dots, v_n)}(m) = Y_0(m)^{(v_0, u_0)} Y_1(m)^{(v_1, u_1)} \dots Y_n(m)^{(v_n, u_n)} \\ \cdot \left(Y_n(m+1)^{(-v_0, -u_0)} Y_0(m+1)^{(-v_1, -u_1)} \dots Y_{n-1}(m+1)^{(-v_n, -u_n)} \right).$$

Fix any integer m . The set consisting of all monomials of the form $X_{(u_0, \dots, u_n)}^{(1, \dots, 1)}(m)$ with $\sum_{i=0}^n u_i = 0$ will be denoted by $\mathcal{M}(m)_\infty$.

Theorem 3.3. For each $m \in \mathbf{Z}$, $\mathcal{M}(m)_\infty$ forms a $U_q(A_n^{(1)})$ -subcrystal of $\mathcal{M}^\mathcal{E}$ which is isomorphic to $\mathcal{B}_\infty$.

Proof. For any monomial $M = X_{(u_0, \dots, u_n)}^{(1, \dots, 1)}(m) \in \mathcal{M}(m)_\infty$ and $i \in I$, since $\tilde{\varphi}_i(M) = (1, u_i) > (0, 0)$, we have

$$(3.2) \quad \begin{aligned} \tilde{f}_i(M) &= A_i(m)^{-1} X_{(u_0, \dots, u_n)}^{(1, \dots, 1)}(m) \\ &= X_{(0, \dots, 0, -1, 1, 0, \dots, 0)}^{(0, \dots, 0)}(m) X_{(u_0, \dots, u_n)}^{(1, \dots, 1)}(m) \\ &= X_{(u_0, \dots, u_{i-1}, u_i-1, u_{i+1}+1, u_{i+2}, \dots, u_n)}^{(1, \dots, 1)}(m) \in \mathcal{M}(m)_\infty. \end{aligned}$$

Here, the subscript $(0, \dots, 0, -1, 1, 0, \dots, 0)$ of a monomial appearing in the equation (3.2) has -1 as the i th entry and 1 as the $(i+1)$ th. Since $\tilde{\varepsilon}_i(M) = (1, u_{i+1}) > (0, 0)$ for $0 \leq i \leq n-1$ and $\tilde{\varepsilon}_n(M) = (1, u_0) > (0, 0)$,

$$(3.3) \quad \begin{aligned} \tilde{e}_i(M) &= A_i(m) X_{(u_0, \dots, u_n)}^{(1, \dots, 1)}(m) \\ &= X_{(0, \dots, 0, 1, -1, 0, \dots, 0)}^{(0, \dots, 0)}(m) X_{(u_0, \dots, u_n)}^{(1, \dots, 1)}(m) \\ &= X_{(u_0, \dots, u_{i-1}, u_i+1, u_{i+1}-1, u_{i+2}, \dots, u_n)}^{(1, \dots, 1)}(m) \in \mathcal{M}(m)_\infty. \end{aligned}$$

Thus, $\mathcal{M}(m)_\infty$ is a subcrystal of $\mathcal{M}^\mathcal{E}$. In particular, the result of $\tilde{f}_i, \tilde{e}_i$ ($i \in I$) cannot be 0 for every vector of $\mathcal{M}(m)_\infty$.

Recall that every element $b \in \mathcal{B}_\infty$ is of the form $b = (u_0, \dots, u_n) \in \mathbf{Z}^{n+1}$ with $\sum_{i=0}^n u_i = 0$. We define a canonical map $\phi_m : \mathcal{M}(m)_\infty \rightarrow \mathcal{B}_\infty$ given by

$$(3.4) \quad X_{(u_0, \dots, u_n)}^{(1, \dots, 1)}(m) \mapsto (u_0, \dots, u_n).$$

FIGURE 1. Part of crystal $\mathcal{M}(0)_\infty$ for $A_2^{(1)}$ type

It is obvious that this map is well-defined and bijective. From the above result concerning the Kashiwara action on the monomial M , we can show in a straightforward manner that ϕ_m commutes with the Kashiwara operators. Other parts of the proof are similar or easy. $\square$

Remark 3.4. As we can see in equations (3.2) and (3.3), $A_i(m)^{-1} = X_{(u_0, \dots, u_n)}^{(0, \dots, 0)}(m)$, where $u_i = -1$, $u_{i+1} = 1$, and all other entries of the subscript are 0. This monomial expression for $A_i(m)$ is very useful when computing the Kashiwara action on monomials of the forms $X_{(u_0, \dots, u_n)}^{(1, \dots, 1)}(m)$ and $X_{(u_0, \dots, u_n)}^{(0, \dots, 0)}(m)$.

Figure 1 is a drawing of a portion of the crystal $\mathcal{M}(0)_\infty$ for type $A_2^{(1)}$.

4. DESCRIPTION OF THE CRYSTAL $\mathcal{B}(\infty)$

We give an explicit description of the crystal $\mathcal{B}(\infty)$ for $A_n^{(1)}$ type, in terms of extended monomials, by giving a crystal isomorphism with the path description of $\mathcal{B}(\infty)$. The following is the path realization of the crystal $\mathcal{B}(\infty)$.

Theorem 4.1 ([2]). *The crystal $\mathcal{B}(\infty)$ is isomorphic to the crystal*

$$\mathcal{P}(\infty) = \left\{ \bigotimes_{k=0}^{\infty} P(k) \in \cdots \otimes \mathcal{B}_\infty \otimes \mathcal{B}_\infty \otimes \mathcal{B}_\infty \mid P(k) = b_\infty \text{ for all } k \gg 0 \right\}$$

of all paths in $\mathcal{B}_\infty$ as a $U_q(\mathfrak{g})$ -crystal, where $\mathfrak{g}$ is a classical affine Lie algebra. The isomorphism maps u_∞ to the ground-state path $P_\infty = (\cdots \otimes b_\infty \otimes b_\infty \otimes b_\infty)$.

The set we define below was originally obtained by applying Kashiwara actions $\tilde{f}_i$ ($i \in I$) continuously on the single element $\prod_{i \in I} Y_i(0)^{(1,0)} \in \mathcal{M}^\mathcal{E}$.

Definition 4.2. Consider elements of $\mathcal{M}^\mathcal{E}$ having the form

$$(4.1) \quad M = \prod_{i \in I = \{0, 1, \dots, n\}} \left(Y_i(0)^{(1, u_i^0)} \prod_{1 \leq m \leq r} Y_i(m)^{(0, u_i^m)} \right),$$

where r is any nonnegative integer, and which satisfies the following conditions:

- (1) $\sum_{i=0}^n u_i^m = 0$ for each $0 \leq m \leq r$.
- (2) For each $s \in I$, $\sum u_i^m = 0$, where the sum is over all $i \in I$ and $0 \leq m \leq r$ satisfying $i + m \equiv s \pmod{n+1}$.

The empty product appearing when $r=0$ is understood to be 1, in which case we have

$$(4.2) \quad M = \prod_{i \in I} Y_i(0)^{(1,0)}.$$

FIGURE 2. Part of the monomial set $\mathcal{M}(\infty)$ for $A_2^{(1)}$ type

We denote by $\mathcal{M}(\infty)$ the set of all monomials of this form and by M_∞ the monomial of (4.2).

Figure 2 is the top part of the monomial set $\mathcal{M}(\infty)$ for the $A_2^{(1)}$ case. We now introduce new expressions for elements of $\mathcal{M}(\infty)$, using elements of the perfect crystal $\mathcal{M}(m)_\infty$. These expressions allow us to relate monomials to paths in $\mathcal{P}(\infty)$ naturally.

Lemma 4.3. *Each element of $\mathcal{M}(\infty)$ may also be written uniquely in the form*

$$(4.3) \quad M = \prod_{m=0}^{\infty} X_{(v_0^m, \dots, v_n^m)}^{(1, \dots, 1)}(m)$$

with each $X_{(v_0^m, \dots, v_n^m)}^{(1, \dots, 1)}(m) \in \mathcal{M}(m)_\infty$ and with $(v_0^m, \dots, v_n^m) = (0, \dots, 0)$ for all $m \gg 0$. Conversely, any element of this form is an element of $\mathcal{M}(\infty)$.

Proof. Given any monomial $M = \prod_{i \in I} (Y_i(0)^{(1, u_i^0)} \prod_{1 \leq m \leq r} Y_i(m)^{(0, u_i^m)}) \in \mathcal{M}(\infty)$, through routine computation, we can obtain the expression

$$(4.4) \quad M = Y_0(r)^{(1, v_0^r)} \dots Y_n(r)^{(1, v_n^r)} \prod_{m=0}^{r-1} X_{(v_0^m, \dots, v_n^m)}^{(1, \dots, 1)}(m),$$

where $v_i^m = \sum_{(s,t) \in I \times \{0, \dots, m\}, s+t \equiv i+m \pmod{n+1}} u_s^t$ for each $(i, m) \in I \times \{0, \dots, r\}$. Since $M \in \mathcal{M}(\infty)$, using the two conditions appearing in Definition 4.2, we obtain $\sum_{i=0}^n v_i^m = 0$ for each $0 \leq m \leq r-1$ and $(v_0^r, \dots, v_n^r) = (0, \dots, 0)$. This expression for M may also easily be transformed to the product form (4.3) with $(v_0^m, \dots, v_n^m) = (0, \dots, 0)$ for all $m \geq r$. Specifically, M_∞ corresponds to $\prod_{m=0}^{\infty} X_{(0, \dots, 0)}^{(1, \dots, 1)}(m)$.

Conversely, given any expression of the form (4.3), if $(v_0^m, \dots, v_n^m) = (0, \dots, 0)$ for all $m \geq r$, then it can easily be transformed to the following form through

infinite cancellation:

(4.5)

$$\begin{aligned}
M &= Y_0(r)^{(1,0)} \cdots Y_n(r)^{(1,0)} \prod_{m=0}^{r-1} X_{(v_0^m, \dots, v_n^m)}^{(1, \dots, 1)}(m) \\
&= Y_0(0)^{(1, v_0^0)} Y_0(1)^{(0, v_0^1 - v_1^0)} \cdots Y_0(r-1)^{(0, v_0^{r-1} - v_1^{r-2})} Y_0(r)^{(0, -v_1^{r-1})} \\
&\quad \cdots \\
&\quad \cdot Y_{n-1}(0)^{(1, v_{n-1}^0)} Y_{n-1}(1)^{(0, v_{n-1}^1 - v_n^0)} \cdots Y_{n-1}(r-1)^{(0, v_{n-1}^{r-1} - v_n^{r-2})} Y_{n-1}(r)^{(0, -v_n^{r-1})} \\
&\quad \cdot Y_n(0)^{(1, v_n^0)} Y_n(1)^{(0, v_n^1 - v_0^0)} \cdots Y_n(r-1)^{(0, v_n^{r-1} - v_0^{r-2})} Y_n(r)^{(0, -v_0^{r-1})}.
\end{aligned}$$

It is now straightforward to check that $M \in \mathcal{M}(\infty)$.

We have thus shown that $\mathcal{M}(\infty)$ consists of elements of the form (4.3). The uniqueness part may be proved through simple computation. $\square$

Now, we translate the Kashiwara operator actions (2.6), (2.7) into a form suitable for the new monomial expression of $\mathcal{M}(\infty)$.

Lemma 4.4. *The set $\mathcal{M}(\infty)$ is closed under the action given below. Fix an element $M = \prod_{m=0}^{\infty} X_{(v_0^m, \dots, v_n^m)}^{(1, \dots, 1)}(m) \in \mathcal{M}(\infty)$:*

- (1) *Under each m th component $X(m) = X_{(v_0^m, \dots, v_n^m)}^{(1, \dots, 1)}(m)$ of M , write $|\varepsilon_i X(m)|$ -many 1's or $\bar{1}$'s depending on whether $\varepsilon_i X(m)$ is positive or negative, followed by $|\varphi_i X(m)|$ -many 0's or $\bar{0}$'s depending on whether $\varphi_i X(m)$ is positive or negative.*
- (2) *If $\bar{1}$'s (or $\bar{0}$'s) were written under $X(m)$, then change and write it as 0's (resp. 1's) under $X(m+1)$ (resp. $X(m-1)$). In the case of $\bar{0}$'s written under $X(0)$, we simply remove them.*
- (3) *From this sequence of 1's and 0's, successively cancel out each $(0, 1)$ -pair to obtain a sequence of 1's followed by 0's (reading from left to right). This sequence of remaining 1's and 0's is called the i -signature of M .*
- (4) *We define $\tilde{f}_i M$ (or $\tilde{e}_i M$) to be the monomial obtained from M by replacing the component $X(m)$ corresponding to the left-most 0 (resp. the right-most 1) remaining with $\tilde{f}_i X(m)$ (resp. $\tilde{e}_i X(m)$).*
- (5) *We define $\tilde{e}_i M$ to be zero if no 1 remains. Let $\tilde{f}_i$ act on $X(0)$ if no 0 remains for $M \in \mathcal{M}(\infty)$.*

Proof. Recall that each component $X(m)$ of $M \in \mathcal{M}(\infty)$ is an element of $\mathcal{M}(m)_{\infty}$ and so $\tilde{f}_i X(m)$ and $\tilde{e}_i X(m)$ are also in $\mathcal{M}(m)_{\infty}$. This means that the above actions satisfy the properties $\tilde{f}_i \mathcal{M}(\infty) \subset \mathcal{M}(\infty)$, $\tilde{e}_i \mathcal{M}(\infty) \subset \mathcal{M}(\infty) \cup \{0\}$ for all $i \in I$. $\square$

Lemma 4.5. *The operation given in Lemma 4.4 is just another expression for the Kashiwara operators given on $\mathcal{M}^{\mathcal{E}}$, restricted to $\mathcal{M}(\infty)$.*

Proof. From (3.2), we can see that for each $M \in \mathcal{M}(\infty)$, $\tilde{f}_i M$ defined in Lemma 4.4 can also be expressed in the form $MA_i(m)^{-1}$ if the i -signature of M contains at least one 0 and $X(m)$ is the component corresponding to the left-most 0 in the i -signature of M . It can be written in the form $MA_i(0)^{-1}$ if the i -signature of M contains no 0.

To show that this operation is just another interpretation of the Kashiwara operator $\tilde{f}_i$ given on $\mathcal{M}^{\mathcal{E}}$, restricted to $\mathcal{M}(\infty)$, it is enough to show that m_f defined

in (2.8) for each M is equal to m , with $X(m)$ corresponding to the left-most 0 in the i -signature of M , and that if no 0 remains in the i -signature of M , then $m_f = 0$.

Given a monomial $M \in \mathcal{M}(\infty)$, we can express it in the following two forms:

(4.6)

$$\begin{aligned} M &= \prod_{m=0}^{\infty} X_{(v_0^m, \dots, v_n^m)}^{(1, \dots, 1)}(m) \\ &= Y_0(0)^{(1, v_0^0)} Y_0(1)^{(0, v_0^1 - v_1^0)} \dots Y_0(r-1)^{(0, v_0^{r-1} - v_1^{r-2})} Y_0(r)^{(0, -v_1^{r-1})} \\ &\quad \dots \\ &\quad \cdot Y_{n-1}(0)^{(1, v_{n-1}^0)} Y_{n-1}(1)^{(0, v_{n-1}^1 - v_n^0)} \dots Y_{n-1}(r-1)^{(0, v_{n-1}^{r-1} - v_n^{r-2})} Y_{n-1}(r)^{(0, -v_n^{r-1})} \\ &\quad \cdot Y_n(0)^{(1, v_n^0)} Y_n(1)^{(0, v_n^1 - v_0^0)} \dots Y_n(r-1)^{(0, v_n^{r-1} - v_0^{r-2})} Y_n(r)^{(0, -v_0^{r-1})}. \end{aligned}$$

If the i -signature of M contains at least one 0 and $X(m)$ is the component corresponding to the left-most 0 in the i -signature of M , then we can obtain

$$m_f = \min \left\{ j \mid \max \left\{ \sum_{k \leq j} y_i(k) \mid j \in \mathbf{Z} \right\} \right\} = m,$$

where $y_i(k)$ is the exponent of $Y_i(k)$ appearing in M given by expression (4.6). If the i -signature of M contains no 0, then

$$m_f = \min \left\{ j \mid \max \left\{ \sum_{k \leq j} y_i(k) \mid j \in \mathbf{Z} \right\} \right\} = 0.$$

The proof for the statements concerning $\tilde{e}_i$ may be done in a similar manner. $\square$

From the above lemmas, we obtain the following result.

Proposition 4.6. *The set $\mathcal{M}(\infty)$ forms a $U_q(A_n^{(1)})$ -subcrystal of $\mathcal{M}^{\mathcal{E}}$.*

Now, we will show that $\mathcal{M}(\infty)$ is an explicit description of $\mathcal{B}(\infty)$ by giving a crystal isomorphism.

Theorem 4.7. *We have a $U_q(A_n^{(1)})$ -crystal isomorphism*

$$\mathcal{M}(\infty) \xrightarrow{\sim} \mathcal{B}(\infty)$$

which maps M_{∞} onto u_{∞} .

Proof. Recall that the set $\mathcal{P}(\infty)$ of all paths in $\mathcal{B}_{\infty}$ with ground-state path P_{∞} gives a description of the crystal $\mathcal{B}(\infty)$ (see (3.1) and Theorem 4.1). Now, it suffices to show that $\mathcal{M}(\infty) \cong \mathcal{P}(\infty)$. We define a map $\psi : \mathcal{M}(\infty) \rightarrow \mathcal{P}(\infty)$ by setting

$$(4.7) \quad \prod_{m=0}^{\infty} X_{(v_0^m, \dots, v_n^m)}^{(1, \dots, 1)}(m) \mapsto \bigotimes_{m=0}^{\infty} \phi_m(X_{(v_0^m, \dots, v_n^m)}^{(1, \dots, 1)}(m)).$$

Here, we are using the crystal isomorphism $\phi_m : \mathcal{M}(m)_{\infty} \mapsto \mathcal{B}_{\infty}$ given in (3.4). Note that the highest weight vector M_{∞} of $\mathcal{M}(\infty)$ is mapped onto the ground-state path P_{∞} . It is obvious that ψ is well-defined and that it is actually bijective.

The fact that ψ is determined by ϕ_m and the fact that the action of Kashiwara operators in Lemma 4.4 on $\mathcal{M}(\infty)$ follows the process for defining it on $\mathcal{P}(\infty)$ (see Lemma 2.2 of [7]) imply naturally that the map ψ commutes with the Kashiwara operators. Other parts of the proof are similar or easy. $\square$

We remark that we can give infinitely many more monomial descriptions of the crystals $\mathcal{B}_\infty$ and $\mathcal{B}(\infty)$ in a manner similar to that given in this work.

We close this paper with an example. We will illustrate the correspondence between $\mathcal{M}(\infty)$ and $\mathcal{P}(\infty)$ for type $A_2^{(1)}$ with simple examples.

Example 4.8. The monomial of $\mathcal{M}(\infty)$,

$$M = Y_0(0)^{(1,1)} Y_1(0)^{(1,-1)} Y_2(0)^{(1,0)} \\ \cdot Y_0(1)^{(0,1)} Y_1(1)^{(0,-1)} \cdot Y_0(2)^{(0,1)} Y_1(2)^{(0,-1)},$$

can also be expressed as

$$M = \left(Y_0(2)^{(1,0)} Y_1(2)^{(1,0)} Y_2(2)^{(1,0)} \right) X_{(0,-1,1)}^{(1,1,1)}(1) X_{(1,-1,0)}^{(1,1,1)}(0) \\ = \left(\prod_{m=2}^{\infty} X_{(0,0,0)}^{(1,1,1)}(m) \right) X_{(0,-1,1)}^{(1,1,1)}(1) X_{(1,-1,0)}^{(1,1,1)}(0).$$

We have the following path as the image of M under ψ defined through (4.7):

$$\psi(M) = \cdots \otimes (0, 0, 0) \otimes (0, -1, 1) \otimes (1, -1, 0) \in \mathcal{P}(\infty).$$

For the path $P = \cdots \otimes (0, 0, 0) \otimes \cdots \otimes (0, 0, 0) \otimes (-1, -1, 2) \in \mathcal{P}(\infty)$, we have the following monomial as its image under ψ^{-1} :

$$\psi^{-1}(P) = \left(\prod_{m=1}^{\infty} X_{(0,0,0)}^{(1,1,1)}(m) \right) X_{(-1,-1,2)}^{(1,1,1)}(0) \\ = \left(Y_0(1)^{(1,0)} Y_1(1)^{(1,0)} Y_2(1)^{(1,0)} \right) X_{(-1,-1,2)}^{(1,1,1)}(0) \\ = Y_0(1)^{(0,1)} Y_1(1)^{(0,-2)} Y_2(1)^{(0,1)} Y_0(0)^{(1,-1)} Y_1(0)^{(1,-1)} Y_2(0)^{(1,2)}.$$

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