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EXPLICIT ORTHOGONAL POLYNOMIALS FOR RECIPROCAL POLYNOMIAL WEIGHTS ON $(-\infty, \infty)$

D. S. LUBINSKY

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ABSTRACT. Let S be a polynomial of degree $2n+2$, that is, positive on the real axis, and let $w = 1/S$ on $(-\infty, \infty)$. We present an explicit formula for the nth orthogonal polynomial and related quantities for the weight w. This is an analogue for the real line of the classical Bernstein-Szegő formula for $(-1, 1)$.

1. THE RESULT

The Bernstein-Szegő formula provides an explicit formula for orthogonal polynomials for a weight of the form $\sqrt{1-x^2}/S(x)$, $x \in (-1, 1)$, where S is a polynomial positive in $(-1, 1)$, possibly with at most simple zeros at ± 1. It plays a key role in asymptotic analysis of orthogonal polynomials.

In this paper, we present an explicit formula for the nth degree orthogonal polynomial for weights w on the whole real line of the form

$$w = 1/S, \tag{1.1}$$

where S is a polynomial of degree $2n+2$, positive on $\mathbb{R}$. In addition, we give representations for the $(n+1)$st reproducing kernel and Christoffel function. We present elementary proofs, although they follow partly from the theory of de Branges spaces [1]. The formulae do not seem to be recorded in de Branges' book nor in the orthogonal polynomial literature [2], [3], [7], [8], [9]. We believe they will be useful in analyzing orthogonal polynomials for weights on $\mathbb{R}$.

Recall that we may define orthonormal polynomials $\{p_m\}_{m=0}^n$, where

$$p_m(x) = \gamma_m x^m + \cdots, \qquad \gamma_m > 0, \tag{1.2}$$

satisfying

$$\int_{-\infty}^{\infty} p_j p_k w = \delta_{jk}.$$

Because the denominator S in w has degree $2n+2$, orthogonal polynomials of degree higher than n are not defined. The $(n+1)$st reproducing kernel for w is

$$K_{n+1}(x, y) = \sum_{j=0}^{n} p_j(x) p_j(y). \tag{1.3}$$

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Inasmuch as S is a positive polynomial, we can write

$$S(z) = E(z)\overline{E(\bar{z})}, \tag{1.4}$$

where E is a polynomial of degree $n+1$, with all zeros in the lower half-plane $\{z : \operatorname{Im} z < 0\}$. We ensure E is unique by normalizing E so that

$$E(i) \text{ is real and positive.} \tag{1.5}$$

Write

$$E(z) = \sum_{j=0}^{n+1} e_j z^j, \qquad S(z) = \sum_{j=0}^{2n+2} s_j z^j \tag{1.6}$$

and

$$E^*(z) = \overline{E(\bar{z})}. \tag{1.7}$$

Denote the first difference of a function f by

$$[f, t, x] = \frac{f(t) - f(x)}{t - x}. \tag{1.8}$$

We shall need various Cauchy principal value integrals: for real x and suitable functions h,

$$PV_x \int_{-\infty}^{\infty} \frac{h(t)}{t-x} dt = \lim_{\varepsilon \to 0+} \int_{|t-x| \geq \varepsilon} \frac{h(t)}{t-x} dt;$$

$$PV_\infty \int_{-\infty}^{\infty} h(t)\ dt = \lim_{R \to \infty} \int_{-R}^{R} h(t)\ dt;$$

$$PV_{x,\infty} \int_{-\infty}^{\infty} \frac{h(t)}{t-x} dt = \lim_{\varepsilon \to 0+, R \to \infty} \int_{|t| \leq R, |t-x| \geq \varepsilon} \frac{h(t)}{t-x} dt.$$

With the above assumptions on w, we prove:

Theorem 1. (a) *For* $\operatorname{Im} z > 0$,

$$E(z) = \exp\left(-\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1+tz}{t-z} \frac{\log w(t)}{1+t^2} dt \right) \tag{1.9}$$

and

$$e_{n+1} = s_{2n+2}^{1/2} (-i)^{n+1} \exp\left(\frac{1}{2\pi i} PV_\infty \int_{-\infty}^{\infty} \frac{\log w(t)}{1+t^2} t\ dt \right). \tag{1.10}$$

(b) *For* $z \neq v$,

$$K_{n+1}(z, v) = \frac{i}{2\pi} \frac{E(z) E^*(v) - E^*(z) E(v)}{z - v}, \tag{1.11}$$

$$K_{n+1}(z, z) = \frac{i}{2\pi} \left(E'(z) E^*(z) - E(z) E^{*\prime}(z) \right). \tag{1.12}$$

(c)

$$\gamma_n = \left\{ \frac{1}{\pi} \operatorname{Im} \left(\overline{e_{n+1}} e_n \right) \right\}^{1/2}, \tag{1.13}$$

and

$$p_n(z) = -\frac{1}{\gamma_n} \frac{i}{2\pi} \left(\overline{e_{n+1}} E(z) - e_{n+1} E^*(z) \right). \tag{1.14}$$

Theorem 2. *For $x \in \mathbb{R}$,*
(a)

$$p_n(x)\, w(x)^{1/2} = \frac{s_{2n+2}^{1/2}}{\pi\gamma_n} \cos\left(\frac{n\pi}{2} + \frac{1}{2\pi} PV_{x,\infty} \int_{-\infty}^{\infty} \frac{\log w(t)}{t-x} dt\right). \tag{1.15}$$

(b)

$$\begin{aligned}\pi K_{n+1}(x,x)\, w(x) = &- \frac{1}{2\pi}\int_{-\infty}^{\infty} [\log w, t, x] \frac{t}{1+t^2} dt \\ &- \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}[\log w, t, x] \frac{1+tx}{1+t^2} dt.\end{aligned} \tag{1.16}$$

(c) *If $s_{2n+1} = 0$,*

$$\gamma_n = \frac{1}{\pi}\left\{\frac{s_{2n+2}}{2}\int_{-\infty}^{\infty} \log\left[\frac{S(t)}{s_{2n+2}t^{2n+2}}\right] dt\right\}^{1/2}. \tag{1.17}$$

Remarks. (a) The function E is a Szegő/outer function associated with w for the upper half-plane. It has been used in the relative asymptotics of G. Lopez Lagomasino [6] and in the orthogonal rational functions of Bultheel et al. [2].

(b) It is easily seen that for $\operatorname{Im} z > 0$,

$$E^*(z) = CE(z) \prod_{a:E(a)=0} \frac{z-\bar{a}}{z-a}, \tag{1.18}$$

where

$$C = \frac{\overline{e_{n+1}}}{e_{n+1}} = (-1)^{n+1} \exp\left(-\frac{1}{\pi i} PV_{\infty} \int_{-\infty}^{\infty} \frac{\log w(t)}{1+t^2} t\, dt\right).$$

(c) Of course if S is even, then s_{2n+1} is 0. The latter condition ensures that the integral in (1.17) converges.

(d) Explicit formulae for the Christoffel function $K_n(x,x)^{-1}$ for Bernstein-Szegő weights appear in [3], [5], [7], [8], [9], [10]. We will present one application of (1.11)–(1.12) in Section 3.

2. Proofs

As we noted above, our original proofs arose from de Branges spaces, but we present elementary proofs. Let us choose E satisfying (1.4) and (1.5).

Proof of (1.9) *of Theorem* 1(a). Let H denote the right-hand side of (1.9) so that

$$H(z) = \exp\left(-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1+tz}{t-z} \frac{\log w(t)}{1+t^2} dt\right).$$

Then for $z = x + iy$,

$$\begin{aligned}\log|H(z)| &= -\operatorname{Re}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1+tz}{t-z}\frac{\log w(t)}{1+t^2} dt\right] \\ &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log|E(t)|}{(t-x)^2+y^2} dt \\ &= \log|E(z)|,\end{aligned} \tag{2.1}$$

by a Theorem in [4, p. 47]. This may be applied as $E(z)$ is analytic and non-zero in the closed upper half-plane, and $\log|E(z)|$ is $O(\log|z|)$ as $|z| \to \infty$. Since H/E

is analytic there, we deduce that for some C with $|C| = 1$, $E = CH$. Now by hypothesis, $E(i)$ is real and positive, while

$$H(i) = \exp\left(-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\log w(t)}{1+t^2}dt\right) > 0,$$

so $C = 1$. □

Proof of (1.10) *of Theorem* 1(a). We first show that

$$1 - iz = \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\left(1+t^2\right)}{1+t^2}\frac{1+tz}{t-z}dt\right), \quad \operatorname{Im} z > 0. \tag{2.2}$$

Indeed, $1 - iz$ serves as the Szegő function for the weight $1/\left(1+t^2\right)$, so (1.9) of Theorem 1 applied to the weight $1/\left(1+t^2\right)$ gives this identity. Then for $\operatorname{Im} z > 0$,

$$E(z)/(1-iz)^{n+1} = \exp\left(I_1 + I_2\right), \tag{2.3}$$

where

$$I_1 = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\left[w(t)\,s_{2n+2}\left(1+t^2\right)^{n+1}\right]}{1+t^2}\frac{1+tz}{t-z}dt,$$
$$I_2 = \frac{\log s_{2n+2}}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{1+t^2}\frac{1+tz}{t-z}dt.$$

The integrand in I_2 has simple poles in the upper half-plane at i and z and is $O\left(t^{-2}\right)$ as $|t| \to \infty$, so the residue calculus gives

$$I_2 = \frac{\log s_{2n+2}}{2}. \tag{2.4}$$

Next, $\log\left[w(t)\,s_{2n+2}\left(1+t^2\right)^{n+1}\right] = O\left(\frac{1}{t}\right)$ as $|t| \to \infty$. Thus the integrand in I_1 is bounded in absolute value for $z = iy$, $y \geq 1$ and all t by

$$C\frac{1}{\left(1+t^2\right)\left(1+|t|\right)}\frac{1+|t|\,y}{|t|+y} \leq \frac{C}{1+t^2}.$$

Here C is independent of t and z. We may then apply Lebesgue's Dominated Convergence Theorem to I_1, with $z = iy$, $y \to \infty$, to deduce that

$$\begin{aligned} I_1 &\to \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\left[w(t)\,s_{2n+2}\left(1+t^2\right)^{n+1}\right]}{1+t^2}t\;dt \\ &= \frac{1}{2\pi i}PV_\infty\int_{-\infty}^{\infty}\frac{\log w(t)}{1+t^2}t\;dt, \end{aligned} \tag{2.5}$$

as

$$PV_\infty\int_{-\infty}^{\infty}\frac{t}{1+t^2}\;dt = 0 = PV_\infty\int_{-\infty}^{\infty}\frac{\log\left(1+t^2\right)}{1+t^2}t\;dt,$$

the integrands being odd. Substituting (2.5) and (2.4) into (2.3) and also letting $z = iy$, $y \to \infty$, in the left-hand side there, gives (1.10). □

Proof of Theorem 1(b). We need prove only (1.11), for (1.12) then follows by l'Hospital's rule. Set

$$G(u,v) = \frac{i}{2\pi}\frac{E(u)\,E^*(v) - E^*(u)\,E(v)}{u-v}.$$

Observe that for fixed v, $G(u,v)$ is a polynomial of degree at most n in u. Assume that P is a polynomial of degree $\leq n$ and that $\operatorname{Im} u > 0$. Now for real t, $w(t) = 1/(E(t)E^*(t))$, so

$$\int_{-\infty}^{\infty} P(t)\,G(u,t)\,w(t)\,dt$$
$$= \frac{i}{2\pi}\left(E^*(u)\int_{-\infty}^{\infty}\frac{P(t)}{E^*(t)(t-u)}dt - E(u)\int_{-\infty}^{\infty}\frac{P(t)}{E(t)(t-u)}dt\right). \tag{2.6}$$

Recall that E has all its zeros in the lower half-plane, so E^* has all its zeros in the upper half-plane. Then the integrand $\frac{P(t)}{E^*(t)(t-u)}$ in the first integral is analytic in the closed lower half-plane and is $O\left(|t|^{-2}\right)$ as $|t| \to \infty$. By Cauchy's integral theorem, the first integral is 0. Next, the integrand $\frac{P(t)}{E(t)(t-u)}$ in the second integral is analytic in the closed upper half-plane, except for a simple pole at u (unless $P(u) = 0$) and is $O\left(|t|^{-2}\right)$ as $|t| \to \infty$. The residue theorem shows that

$$\int_{-\infty}^{\infty}\frac{P(t)}{E(t)(t-u)}dt = 2\pi i\frac{P(u)}{E(u)}.$$

Substituting this into (2.6) gives

$$\int_{-\infty}^{\infty} P(t)\,G(u,t)\,w(t)\,dt = P(u)$$

for $\operatorname{Im} u > 0$. As both sides are polynomials in u, analytic continuation gives it for all u. Finally, (1.11) follows from the uniqueness of reproducing kernels:

$$K_{n+1}(u,v) = \int_{-\infty}^{\infty} K_{n+1}(t,v)\,G(u,t)\,w(t)\,dt = G(u,v).$$

□

Proof of Theorem 1(c). We note that since p_{n+1} is not defined, we cannot use the Christoffel-Darboux formula for K_{n+1}. However, we can use it for K_n:

$$K_{n+1}(u,v) = \frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(u)\,p_{n-1}(v) - p_n(v)\,p_{n-1}(u)}{u-v} + p_n(u)\,p_n(v).$$

Multiplying by $u - v$ leads to

$$\frac{\gamma_{n-1}}{\gamma_n}\left(p_n(u)\,p_{n-1}(v) - p_n(v)\,p_{n-1}(u)\right) + (u-v)\,p_n(u)\,p_n(v)$$
$$= (u-v)\,K_{n+1}(u,v) = \frac{i}{2\pi}\left(E(u)\,E^*(v) - E^*(u)\,E(v)\right),$$

by (1.11). Now we compare the coefficients of u^{n+1} on both sides above:

$$\gamma_n p_n(v) = \frac{i}{2\pi}\left(e_{n+1}E^*(v) - \overline{e_{n+1}}E(v)\right), \tag{2.7}$$

giving (1.14). For (1.13), we compare the coefficients of v^n on both sides above:

$$\gamma_n^2 = \frac{i}{2\pi}\left(e_{n+1}\overline{e_n} - \overline{e_{n+1}}e_n\right).$$

(Note that the coefficient of v^{n+1} on the right-hand side in (2.7) is zero.) □

Proof of Theorem 2(a). From (1.14), for real x,

$$\pi\gamma_n p_n(x) = \operatorname{Im}\left(\overline{e_{n+1}}E(x)\right).$$

We take non-tangential boundary values $z \to x$ from the upper half-plane in (1.9). The Sokhotsky-Plemelj formulae give

$$E(x) = \exp\left(-\frac{1}{2\pi i}PV_x\int_{-\infty}^{\infty}\frac{\log w(t)}{1+t^2}\frac{1+tx}{t-x}dt - \frac{1}{2}\log w(x)\right), \tag{2.8}$$

and this and (1.10) give

$$\begin{aligned}\pi\gamma_n p_n(x) w(x)^{1/2} &= s_{2n+2}^{1/2}\operatorname{Im}\Bigg[i^{n+1}\exp\Bigg(-\frac{1}{2\pi i}PV_\infty\int_{-\infty}^{\infty}\frac{\log w(t)}{1+t^2}t\ dt \\ &\qquad - \frac{1}{2\pi i}PV_x\int_{-\infty}^{\infty}\frac{\log w(t)}{1+t^2}\frac{1+tx}{t-x}dt\Bigg)\Bigg] \\ &= s_{2n+2}^{1/2}\operatorname{Im}\left[i^{n+1}\exp\left(-\frac{1}{2\pi i}PV_{x,\infty}\int_{-\infty}^{\infty}\frac{\log w(t)}{t-x}dt\right)\right].\end{aligned}$$

□

Proof of Theorem 2(b). For real x, and E as above, we define a phase function φ (cf. [1, p. 54]) by

$$E(x) = |E(x)|\,e^{-i\varphi(x)}. \tag{2.9}$$

Here, as in [1, p. 54], φ is an increasing differentiable function. We have, as there,

$$K_{n+1}(x,x) = \frac{1}{\pi}|E(x)|^2\varphi'(x) = \frac{1}{\pi}w(x)^{-1}\varphi'(x). \tag{2.10}$$

Indeed, for real x,

$$E^*(x) = |E(x)|\,e^{i\varphi(x)},$$

so for real $t \neq x$, (1.11) gives

$$K_{n+1}(x,t) = \frac{|E(x)|\,|E(t)|}{\pi}\frac{\sin(\varphi(x)-\varphi(t))}{x-t}.$$

L'Hospital's rule gives the first equality in (2.10). Next, from (2.8) and the definition of φ, we have for some constant C independent of x,

$$\varphi(x) = -\frac{1}{2\pi}PV_x\int_{-\infty}^{\infty}\frac{\log w(t)}{1+t^2}\frac{1+tx}{t-x}dt + C. \tag{2.11}$$

The residue theorem shows that for $\operatorname{Im} z > 0$,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{1+t^2}\frac{1+tz}{t-z}dt = \frac{1}{2}, \tag{2.12}$$

so also for real x, the Sokhotsky-Plemelj formulae give

$$\frac{1}{2\pi i}PV_x\int_{-\infty}^{\infty}\frac{1}{1+t^2}\frac{1+tx}{t-x}dt + \frac{1}{2} = \frac{1}{2};$$

thus

$$\frac{1}{2\pi i}PV_x\int_{-\infty}^{\infty}\frac{1}{1+t^2}\frac{1+tx}{t-x}dt = 0. \tag{2.13}$$

Hence we may write

$$\begin{aligned}\varphi(x) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\log w(t)-\log w(x)}{t-x}\frac{1+tx}{1+t^2}dt+C\\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}[\log w,t,x]\,\frac{1+tx}{1+t^2}dt+C,\end{aligned}$$

where the integral is now a Lebesgue integral. Then

$$\varphi'(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}[\log w,t,x]\,\frac{t}{1+t^2}dt-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}[\log w,t,x]\,\frac{1+tx}{1+t^2}dt.$$

The interchange of derivative and integral is justified by uniform in x (and absolute) convergence of the differentiated integrals. Finally, apply (2.10). □

Proof of Theorem 2(c). We compute γ_n by comparing both sides of (2.10) as $x\to\infty$. First observe that if $a>0$ and

$$w_a(x) = \left(x^2+a^2\right)^{-(n+1)},$$

then the Szegő/outer function E_a for the weight w_a is given by

$$E_a(z) = (a-iz)^{n+1} \text{ and } E_a^*(z) = (a+iz)^{n+1}.$$

If $K_{n+1}(w_a,\cdot,\cdot)$ denotes the kernel for w_a, (1.11) leads to

$$K_{n+1}(w_a,x+iy,x-iy) = \frac{\left(x^2+(a+y)^2\right)^{n+1}-\left(x^2+(a-y)^2\right)^{n+1}}{4\pi y}.$$

Letting $y\to 0+$ and using l'Hospital's rule give

$$K_{n+1}(w_a,x,x) = \frac{n+1}{\pi}a\left(x^2+a^2\right)^n$$

and

$$K_{n+1}(w_a,x,x)\,w_a(x) = \frac{(n+1)\,a}{\pi\left(x^2+a^2\right)}. \tag{2.14}$$

Next, if we write

$$E_a(x) = |E_a(x)|\,e^{-i\varphi_a(x)},$$

then, as in (2.11),

$$\varphi_a(x) = -\frac{1}{2\pi}PV_x\int_{-\infty}^{\infty}\frac{\log w_a(t)}{1+t^2}\frac{1+tx}{t-x}dt+C_a. \tag{2.15}$$

Let

$$g_a(t) = \log\left[w(t)\,s_{2n+2}/w_a(t)\right] = \log\left[\frac{s_{2n+2}\left(t^2+a^2\right)^{n+1}}{S(t)}\right].$$

In view of (2.11), (2.13) and (2.15), we may then write

$$\varphi(x)-\varphi_a(x) = -\frac{1}{2\pi}PV_x\int_{-\infty}^{\infty}\frac{g_a(t)}{1+t^2}\frac{1+tx}{t-x}dt+C-C_a, \tag{2.16}$$

and then (2.14), followed by (2.10) and (2.16), gives

$$\begin{aligned}\pi K_{n+1}(x,x)\,w(x) - \frac{(n+1)\,a}{x^2+a^2} &= \pi K_{n+1}(x,x)\,w(x) - \pi K_{n+1}(w_a,x,x)\,w_a(x)\\ &= \varphi'(x) - \varphi_a'(x)\\ &= \frac{d}{dx}\left[-\frac{1}{2\pi}PV_x\int_{-\infty}^{\infty}\frac{g_a(t)}{1+t^2}\frac{1+tx}{t-x}dt\right].\end{aligned} \tag{2.17}$$

Since $s_{2n+1}=0$, it is easily seen that for each $j \geq 0$,

$$g_a^{(j)}(t) = O(|t|^{-j-2}) \text{ as } |t| \to \infty. \tag{2.18}$$

As

$$\frac{1}{1+t^2}\frac{1+tx}{t-x} = \frac{1}{t-x} - \frac{t}{1+t^2},$$

the decay of g_a at ∞ enables us to deduce that

$$\pi K_{n+1}(x,x)\,w(x) - \frac{(n+1)\,a}{x^2+a^2} = \frac{d}{dx}\left[-\frac{1}{2\pi}PV_x\int_{-\infty}^{\infty}\frac{g_a(t)}{t-x}dt\right]. \tag{2.19}$$

It is well known that the derivative of a Cauchy principal value is a Hadamard finite part integral, but we sketch what we need here. Fix x, let $R > |x|$, and split

$$PV_x\int_{-\infty}^{\infty}\frac{g_a(t)}{t-x}dt = PV_x\left(\int_{-R}^{R} + \int_{\mathbb{R}\setminus[-R,R]}\right)\frac{g_a(t)}{t-x}dt =: F_R(x) + G_R(x).$$

Here, because the differentiated integrand has uniformly convergent integral,

$$G_R'(x) = \int_{\mathbb{R}\setminus[-R,R]}\frac{g_a(t)}{(t-x)^2}dt.$$

Note too that $G_R'(x) \to 0$ as $R \to \infty$. Next, adding and subtracting a principal value integral give

$$F_R(x) = \int_{-R}^{R}\frac{g_a(t)-g_a(x)}{t-x}dt + g_a(x)\ln\left|\frac{R-x}{R+x}\right|,$$

so, again, as the differentiated integrand has uniformly convergent integral,

$$\begin{aligned}F_R'(x) &= \int_{-R}^{R}\frac{g_a(t)-g_a(x)-g_a'(x)(t-x)}{(t-x)^2}dt\\ &\quad + g_a'(x)\ln\left|\frac{R-x}{R+x}\right| + g_a(x)\left(\frac{1}{x-R}-\frac{1}{x+R}\right)\\ &= PV_x\int_{-R}^{R}\frac{g_a(t)-g_a(x)}{(t-x)^2}dt + g_a(x)\left(\frac{1}{x-R}-\frac{1}{x+R}\right).\end{aligned}$$

As $R \to \infty$, the decay of g_a at ∞ ensures that

$$F_R'(x) \to PV_x\int_{-\infty}^{\infty}\frac{g_a(t)-g_a(x)}{(t-x)^2}dt.$$

We deduce that

$$\frac{d}{dx}\left[PV_x\int_{-\infty}^{\infty}\frac{g_a(t)}{t-x}dt\right] = PV_x\int_{-\infty}^{\infty}\frac{g_a(t)-g_a(x)}{(t-x)^2}dt.$$

Thus, from (2.19),

$$\pi x^2 K_{n+1}(x,x)\,w(x) - \frac{(n+1)\,ax^2}{x^2+a^2} = -\frac{x^2}{2\pi} PV_x \int_{-\infty}^{\infty} \frac{g_a(t)-g_a(x)}{(t-x)^2}dt$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} h_a(t,x)\,dt, \tag{2.20}$$

where

$$h_a(t,x) = \begin{cases} \frac{x^2[g_a(t)-g_a(x)]}{(t-x)^2}, & t \notin \left[\frac{x}{2}, \frac{3x}{2}\right], \\ \frac{x^2\left[g_a(t)-g_a(x)-g_a'(x)(t-x)\right]}{(t-x)^2}, & t \in \left[\frac{x}{2}, \frac{3x}{2}\right]. \end{cases}$$

Observe that for each fixed t,

$$\lim_{x\to\infty} h_a(t,x) = g_a(t).$$

(We use (2.18) for this.) We next obtain an integrable bound on $h_a(t,x)$ that is independent of large x. If $t \in \left(-\infty, \frac{x}{2}\right)$,

$$|h_a(t,x)| \leq C\,|g_a(t)| + \frac{C}{1+t^2}.$$

Similarly if $t \in \left(\frac{3x}{2}, \infty\right)$, this bound holds. If $t \in \left[\frac{x}{2}, \frac{3x}{2}\right]$, then for some ξ in this interval, (2.18) shows that

$$|h_a(t,x)| = \frac{x^2}{2}\,|g_a''(\xi)| \leq \frac{C}{1+t^2}.$$

In all occurrences, C is independent of x and t. It follows that we may apply Lebesgue's Dominated Convergence Theorem to the integral on the right-hand side of (2.20) and let $x \to \infty$ on both sides to deduce that

$$\frac{\pi\gamma_n^2}{s_{2n+2}} - (n+1)\,a = -\frac{1}{2\pi}\int_{-\infty}^{\infty} g_a(t)\,dt.$$

Now we let $a \to 0+$ and use the definition of g_a (and an easier Dominated Convergence) to deduce that

$$\frac{\pi\gamma_n^2}{s_{2n+2}} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \log\left[\frac{s_{2n+2}t^{2n+2}}{S(t)}\right]dt.$$

$\square$

3. An application to reciprocal entire weights

Suppose $z_j = x_j + iy_j$, $j \geq 1$, with all $y_j < 0$ and

$$\sum_{j=1}^{\infty}\left(\frac{x_j}{|z_j|}\right)^2 < \infty. \tag{3.1}$$

Let

$$E(z) = \prod_{j=1}^{\infty}\left(1 - \frac{z}{z_j}\right) \text{ and } E_n(z) = \prod_{j=1}^{n+1}\left(1 - \frac{z}{z_j}\right), \quad n \geq 1.$$

Assume that E is entire, and let

$$W = \frac{1}{|E|^2} \text{ and } w_n = \frac{1}{|E_n|^2}, \quad n \geq 1.$$

For real x, it is easily seen that

$$\frac{w_n}{W}(x) \geq \prod_{j=n+2}^{\infty} \left(1 - \left(\frac{x_j}{|z_j|}\right)^2\right) =: \rho_n.$$

Let $K_{n+1}(W, \cdot, \cdot)$ and $K_{n+1}(w_n, \cdot, \cdot)$ denote the nth reproducing kernels for W and w_n respectively. This last inequality and extremal properties of K_{n+1} yield

$$K_{n+1}(W, z, \bar{z}) \geq \rho_n^{-1} K_{n+1}(w_n, z, \bar{z}) \text{ for all } z \in \mathbb{C}.$$

In view of (3.1), $\rho_n \to 1$ as $n \to \infty$. Then the explicit formula (1.11) for $K_{n+1}(w_n, z, \bar{z})$ and the fact that $E_n \to E$ as $n \to \infty$ give, for non-real z,

$$\liminf_{n\to\infty} K_{n+1}(W, z, \bar{z}) \geq \frac{i}{2\pi} \frac{E(z) E^*(\bar{z}) - E^*(z) E(\bar{z})}{z - \bar{z}}. \tag{3.2}$$

For real z, we instead use (1.12). Now let $\mathcal{H}(E)$ be the de Branges space corresponding to E. This consists [1, p. 50 ff.] of all entire functions g for which both g/E and g^*/E belong to the Hardy 2 space of the upper half-plane $H^2(\mathbb{C}^+)$, with

$$\int_{-\infty}^{\infty} \left|\frac{g}{E}\right|^2 < \infty.$$

The reproducing kernel for this space is [1, p. 51]

$$K(z, v) = \frac{i}{2\pi} \frac{E(z) E^*(v) - E^*(z) E(v)}{z - v}, \quad z \neq v,$$

with a confluent form when $z = v$. Moreover, for such g, we have [1, p. 53]

$$|g(z)|^2 \leq K(z, \bar{z}) \int_{-\infty}^{\infty} \left|\frac{g}{E}\right|^2, \; z \in \mathbb{C}.$$

Since $\mathcal{H}(E)$ contains all polynomials, we may apply this last inequality to $g(t) = K_{n+1}(W, t, \bar{z})$ for fixed z and deduce that

$$K_{n+1}(W, z, \bar{z})^2 \leq K(z, \bar{z}) \int_{-\infty}^{\infty} |K_{n+1}(W, t, \bar{z})|^2 W(t)\, dt = K(z, \bar{z}) K_{n+1}(W, z, \bar{z}),$$

so

$$K_{n+1}(W, z, \bar{z}) \leq K(z, \bar{z}).$$

Together with (3.2), this yields, for non-real z,

$$\lim_{n\to\infty} K_n(W, z, \bar{z}) = K(z, \bar{z}) = \frac{i}{2\pi} \frac{E(z) E^*(\bar{z}) - E^*(z) E(\bar{z})}{z - \bar{z}}.$$

Similarly, for x real,

$$\lim_{n\to\infty} K_n(W, x, x) = K(x, x) = \frac{i}{2\pi} \left(E'(x) E^*(x) - E(x) E^{*\prime}(x)\right).$$

In particular, as this is finite, the moment problem corresponding to W is indeterminate (cf. [3]).

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School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160

E-mail address: `lubinsky@math.gatech.edu`