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SUBANALYTIC BLOW-$\mathcal{C}^m$ FUNCTIONS

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Abstract. We describe rings of subanalytic functions which become continuously differentiable after finitely many local blowings-up with analytic centers.

1. Introduction

An arc-analytic function is a function that is analytic along every analytic arc. Arc-analytic functions were introduced in [13] and have been successfully studied in various papers under the additional hypothesis that they are semialgebraic or subanalytic; see for example [2, 5, 14, 15]. For instance every blow-analytic function in the sense of Kuo [11] is subanalytic and arc-analytic. Also a weak version of the inverse is true [2, 17]: every subanalytic and arc-analytic function becomes analytic after composition with some finite sequences of local blowings-up with smooth analytic centers. In general an arc-analytic function is not analytic. However, if we assume that a function is C^∞ along each C^∞ arc, then this function is actually of the class C^∞; this is due to J. Boman [4].

Let $m > 0$ be an integer. In the present paper we investigate subanalytic functions which become m times continuously differentiable after composition with a finite sequence of local blowings-up. In analogy to Kuo's notation, we call such a function a *blow-$\mathcal{C}^m$ function*. For an introduction of the notion and major properties of blowings-up in the subanalytic setting, see [1]. A general introduction to semialgebraic and subanalytic geometry is provided by [18].

Throughout the paper, every manifold is assumed to be of pure dimension, Hausdorff and equipped with a countable basis for its topology. We consider continuously differentiable versions of the concept of arc-analyticity. There are three versions we will discuss: Let M be a real analytic manifold. A function $f : M \to \mathbb{R}$ is called

(a) a $\mathcal{C}^m_\omega$ function if f is $\mathcal{C}^m$-smooth along all analytic arcs,
(b) a $\mathcal{C}^m_{m,sub}$ function if f is $\mathcal{C}^m$-smooth along all subanalytic $\mathcal{C}^m$ arcs,
(c) a $\mathcal{C}^m_m$ function if f is $\mathcal{C}^m$-smooth along all $\mathcal{C}^m$ arcs.

Every $\mathcal{C}^m_m$ function is $\mathcal{C}^m_{m,sub}$-smooth, and every $\mathcal{C}^m_{m,sub}$ function is $\mathcal{C}^m_\omega$-smooth. We will show that these inclusions are proper even in the subanalytic category. In general, a $\mathcal{C}^m_\omega$ function is not necessarily continuous; see the example of [3]. But

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subanalytic $\mathcal{C}^m_\omega$ functions are continuous. This enables us to study them with the help of Parusinski's Rectilinearization Theorem. As an application of this theorem, we prove the following theorem.

Theorem 1.1. *Let M be a real analytic manifold, and let $f : M \to \mathbb{R}$ be a subanalytic $\mathcal{C}^m_\omega$ function. Then f is blow-$\mathcal{C}^m$.*

A $\mathcal{C}^m$*-singular point* of a function f is a point at which f is not $\mathcal{C}^m$-smooth. The centers of a blowing-up are always analytic manifolds whose dimension is bounded by $\dim(M) - 2$. By [19] (see also [2, 12]), the set of $\mathcal{C}^m$-singular points of a subanalytic function is again subanalytic. Hence, we obtain the following statement.

Theorem 1.2. *Let M be a real analytic manifold, and let $f : M \to \mathbb{R}$ be a subanalytic arc-$\mathcal{C}^m$ function. Then the set S of $\mathcal{C}^m$-singular points of f is subanalytic and satisfies*

$$\dim(S) \leq \dim(M) - 2.$$

In Section 2, we briefly recall Parusiński's Rectilinearization Theorem and some facts about subanalytic Peano differentiable functions which we need to investigate the examples presented in Section 3. In Section 4 we prove Theorem 1.1.

2. Basics

We will use Parusiński's Rectilinearization Theorem; cf. [17, Theorem 2.7].

Theorem 2.1 (Parusiński)**.** *Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a continuous subanalytic function. Then there exist a locally finite collection Ψ of real analytic morphisms $\phi_\alpha : W_\alpha \to \mathbb{R}^n$ such that*

(a) *each W_α contains a compact subset K_α such that $\bigcup_\alpha \phi_\alpha(K_\alpha)$ is a neighbourhood of* $\mathrm{cl}(U)$*;*

(b) *for each α there exist $r_i \in \mathbb{N}$, $i = 1, \ldots, n$, such that $\phi_\alpha = \sigma_\alpha \circ \psi_\alpha$, where σ_α is the composition of a finite sequence of local blowings-up with analytic center and*

$$\psi_\alpha(x) = (\varepsilon_1 x_1^{r_1}, \ldots, \varepsilon_n x_n^{r_n})$$

for some $\varepsilon_i = \pm 1$;

(c) *for any choice of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{1, -1\}^n$ and ψ_α as in Theorem* 2.1 (b)*, the composition $f \circ \sigma_\alpha \circ \psi_\alpha$ is analytic.*

We will give examples to distinguish the notions of differentiability along curves. This requires the concept of *Peano differentiable functions*.

Definition 2.2. Let $U \subset \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}$ is called m times Peano differentiable, in short $f \in \mathcal{P}^m(U, \mathbb{R})$, if for every $u \in U$ there is a polynomial p such that

$$f(x) - f(u) = p(x - u) + o(\|x - u\|^m) \text{ as } x \to u.$$

By Taylor's Theorem, every $\mathcal{C}^m$ function is m times Peano differentiable. The sets of $\mathcal{C}^m$-singular points of $\mathcal{P}^m$ functions have been studied in [9] (see also [7]) for the o-minimal context. Every continuous subanalytic function is locally definable in the o-minimal structure $\mathbb{R}_{an}$ consisting of all *globally subanalytic* sets; cf. [6, page 506]. A subanalytic set $A \subset \mathbb{R}^n$ is called *globally subanalytic* if $\tau_n(A)$ is subanalytic where

$$\tau_n(x) = \left(\frac{x_1}{\sqrt{1 + x_1^2}}, \ldots, \frac{x_n}{\sqrt{1 + x_n^2}} \right);$$

see for example [6, page 506]. The theorems in [9] are stated for o-minimal expansions of real closed fields. However, for the subanalytic category the result of our interest (cf. [9, Theorem 1.1]) reads as follows:

Theorem 2.3. *Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}$ be a subanalytic $\mathcal{P}^m$ function. Then the set S of $\mathcal{C}^m$-singular points is subanalytic and*

$$\dim(S) \leq n - 2.$$

In particular unary subanalytic $\mathcal{P}^m$ functions are $\mathcal{C}^m$-smooth. Note that every subanalytic $\mathcal{P}^m$ function is actually $\mathcal{C}^m_\omega$. Hence from Theorem 1.1 follows:

Corollary 2.4. *Every subanalytic $\mathcal{P}^m$ function is blow-$\mathcal{C}^m$.*

3. Examples

Next we discuss the announced examples.

Example 3.1. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a semialgebraic $\mathcal{C}^m$ function that vanishes outside of $(0, 2)$ and for which $\varphi(1) = 1$. Let

$$A := \left\{(x, y) \in \mathbb{R}^2 : \ x > 0, \ x^{m+1/2} < y < 3x^{m+1/2}\right\}.$$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function

$$f(x, y) = \begin{cases} x^{m/2+1/8}\varphi\left(\dfrac{y}{x^{m+1/2}} - 1\right), & \text{if } (x, y) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then f is a semialgebraic $\mathcal{C}^m_\omega$ function that is not $\mathcal{C}^m_{m,sub}$-smooth.

Proof. First we prove that f is $\mathcal{C}^m_\omega$-smooth.

Outside of $(0, 0)$ the function f is $\mathcal{C}^m$-smooth. It remains to study the origin. Let

$$\phi = (\phi_1, \phi_2) : (-1, 1) \to \mathbb{R}^2$$

be an analytic curve with $\phi(0) = (0, 0)$.

Assume that $\phi_1'(0) = 0$, and that $\phi_1(t) > 0$ for $t > 0$ small enough. Then

$$\phi_1(t) \text{ is } O\left(t^2\right) \text{ as } t \to 0$$

so that

$$f \circ \phi(t) \text{ is } O\left(t^{m+1/4}\right) \text{ as } t \to 0.$$

Hence $f \circ \phi$ is m times Peano differentiable at $t = 0$. Note that $f \circ \phi(t)$ restricted to $(-1/2, 1/2)$ is $\mathbb{R}_{an}$-definable. Thus $f \circ \phi$ is $\mathcal{C}^m$-smooth in some pointed neighbourhood of 0. By Theorem 2.3, the function $f \circ \phi$ is $\mathcal{C}^m$-smooth.

If $\phi_1'(0) \neq 0$, then we claim that $f \circ \phi$ is locally zero at 0. Again we may assume that $\phi_1(t) > 0$ for $t > 0$ sufficiently small. If $\phi_2'(0) > 0$, then the germ of ϕ at 0^+ lies above A, and if $\phi_2'(0) \leq 0$, then the germ lies below A. In both cases, the function $f \circ \phi(t) = 0$ for t sufficiently close to 0.

Hence f is a $\mathcal{C}^m_\omega$ function.

To see that f is not a $\mathcal{C}^m_{m,sub}$ function we show that f is not $\mathcal{C}^m$-smooth along the semialgebraic $\mathcal{C}^m$ curve $\phi : (-1, 1) \to \mathbb{R}^2$ given by

$$\phi(t) := \begin{cases} \left(t, 2t^{m+1/2}\right), & \text{if } t > 0, \\ (t, 0), & \text{if } t \leq 0. \end{cases}$$

The composition $f \circ \phi(t) = 0$ for $t \le 0$. But for $t > 0$,

$$f \circ \phi(t) = t^{m/2+1/8},$$

which cannot be extended to 0 as a $\mathcal{C}^m$ function. $\square$

Remark 3.1. By the previous example we see that the class of subanalytic $\mathcal{C}^m_\omega$ functions is not closed under compositions. The classes of subanalytic $\mathcal{C}^m_m$ and $\mathcal{C}^m_{m,sub}$ functions are closed under compositions.

Example 3.3. Let the semialgebraic function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} y^{m+1}\varphi\left(\dfrac{x}{y^{2m^2}} - 2\right), & y > 0, \\ 0, & y \le 0, \end{cases}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is defined by

$$\varphi(t) := \begin{cases} t\left(1 - t^2\right)^{m+1}, & \text{if } t \in (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

Then f is m times Peano differentiable. Thus f is $\mathcal{C}^m_{m,sub}$-smooth. But f is not a $\mathcal{C}^m_m$ function.

Proof. The function f is $\mathcal{C}^m$-smooth outside of the origin. The function φ is bounded, so that

$$f(x,y) \text{ is } o\left(\|(x,y)\|^m\right) \text{ as } (x,y) \to (0,0).$$

Hence f is m times Peano differentiable, so f is $\mathcal{C}^m_{m,sub}$-smooth.

Next we present a $\mathcal{C}^m$ curve along which f is not even $\mathcal{C}^1$-smooth. Let $\phi : (-1,1) \to \mathbb{R}^2$ be the curve given by

$$\phi(t) := \left(2t^{2m^2} + t^{2m^2+1}\sin\left(t^{-m-1/2}\right), t\right).$$

It is straightforward to verify that ϕ is a $\mathcal{C}^m$ curve. We note the first derivative of the first component of ϕ for $t > 0$:

$$\phi_1'(t) = 2m^2 t^{2m^2-1}\left(1 + \sin\left(t^{-m-\frac{1}{2}}\right)\right) - \left(m + \frac{1}{2}\right) t^{2m^2-m-\frac{1}{2}} \cos\left(t^{-m-\frac{1}{2}}\right). \tag{3.1}$$

The partial derivative of f with respect to y is continuous. Hence it suffices to study $\partial f/\partial x$. For $y > 0$,

$$\frac{\partial f}{\partial x}(x,y) = y^{-2m^2+m}\varphi'\left(xy^{-2m^2} - 2\right). \tag{3.2}$$

Note that $\varphi'(0) = 1$. Hence, combining the equations (3.1) and (3.2) we can write $(f \circ \phi)'(t)$ for positive t as follows:

$$\begin{aligned}(f \circ \phi)'(t) =& \frac{\partial f}{\partial y}(\phi(t))\phi_2'(t) + \frac{\partial f}{\partial x}(\phi(t))\phi_1'(t) \\ =& \frac{\partial f}{\partial y}(\phi(t)) + \varphi'\left(t\sin\left(t^{-m-\frac{1}{2}}\right)\right) 2m^2 t^{m-1}\left(1 + \sin\left(t^{-m-\frac{1}{2}}\right)\right) \\ &+ \varphi'\left(t\sin\left(t^{-m-\frac{1}{2}}\right)\right)\left(-\left(m + \frac{1}{2}\right) t^{-\frac{1}{2}} \cos\left(t^{-m-\frac{1}{2}}\right)\right).\end{aligned}$$

The first two summands are bounded, while the third summand is not locally bounded at $t = 0$. Thus $f \circ \phi(t)$ is not continuously differentiable at $t = 0$. $\square$

4. Proof of the main theorem

We prepare the proof of Theorem 1.1 by the following observation. Let $B_1(0)$ denote the open unit-ball in $\mathbb{R}^n$.

Lemma 4.1. *Let $f, g : B_1(0) \to \mathbb{R}$ be $\mathcal{C}^m$ functions. Assume that the function $f : B_1(0) \to \mathbb{R}$,*

$$F(x) := \begin{cases} f(x), & \text{if } x_1 \leq 0, \\ g(x), & \text{if } x_1 > 0, \end{cases}$$

is $\mathcal{C}^m$-smooth along every line segment contained in $B_1(0)$. Then F is $\mathcal{C}^m$-smooth.

Proof. We may assume that g vanishes identically. Then, it remains to prove that for every $\xi \in B_1(0) \cap \{x_1 = 0\}$ and every $\alpha \in \mathbb{N}^n$ with $\alpha_1 + \cdots + \alpha_n \leq m$,

$$D_\alpha f(\xi) = 0,$$

because in this case, the Hestenes Lemma (cf. [10], [20]) implies that F is $\mathcal{C}^m$-smooth.

We express $D_\alpha f(\xi)$ as a linear combination of higher-order directional derivatives. This is possible by [8, Proof of Theorem 1.4]. All directional derivatives of F at ξ vanish, as $F = 0$ for $x_1 > 0$. But $f = F$ for $x_1 \leq 0$, so that every directional derivative of f at ξ vanishes. Thus F is $\mathcal{C}^m$-smooth. □

Lemma 4.2. *Any subanalytic $\mathcal{C}^m_\omega$ function $f : M \to \mathbb{R}$ is continuous.*

Proof. Assume that f is not continuous at a. Then, by the curve selection (cf. [16], [2], [6, 1.17]), there exists an analytic map $\gamma : (-1, 1) \to M$ with $\gamma(0) = a$ and $\gamma(t) \neq a$ for $t \neq 0$ such that $\lim_{t \searrow 0} f \circ \gamma(t) \neq f(a)$. But $f \circ \gamma(0) = f(a)$ and $f \circ \gamma$ is at least continuous. Thus f must be continuous. □

Proof of Theorem 1.1. The problem is local, so that we may assume that $M = U \subset \mathbb{R}^n$ is a neighbourhood of the origin. Since every $\mathcal{C}^m_\omega$ function is continuous, we can apply Parusiński's theorem to f and U and obtain a family $\{\phi_\alpha = \sigma_\alpha \circ \psi_\alpha\}$ which satisfies the conclusion of Theorem 2.1. Let $\sigma = \sigma_\alpha$, and fix the corresponding $r_i \in \mathbb{N}$. Then, for each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{1, -1\}^n$ and ψ as defined in Theorem 2.1 (b), the function $f \circ \sigma \circ \psi$ is analytic. Hence

$$f \circ \sigma \circ \psi(x) = \sum_\beta a_\beta \prod_{i=1}^n x_i^{\beta_i}.$$

On the quadrant $Q_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon_i x_i \geq 0 \text{ for } i = 1, \ldots, n\}$ we have

$$f \circ \sigma(x) = \sum_\beta a_\beta \prod_{i=1}^n (\varepsilon_i x_i)^{\beta_i / r_i}.$$

But σ is analytic; hence $f \circ \sigma$ is a $\mathcal{C}^m_\omega$ function. Assume that there is a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n)$ with

$$\frac{\gamma_1}{r_1} + \cdots + \frac{\gamma_n}{r_n} < m$$

such that at least one of the r_i does not divide γ_i, and $a_\gamma \neq 0$. Then, for generic $c = (c_1, \ldots, \hat{c}_i, \ldots, c_n)$ with $\varepsilon_j c_j \geq 0$ for $j \neq i$, the function

$$f \circ \sigma(c_1, \ldots, c_{i-1}, \varepsilon_i t, c_{i+1}, \ldots, c_n)$$

has the Puiseux expansion with non-zero coefficient at t^{γ_i/r_i}. This contradicts the fact that $f \circ \sigma$ is $\mathcal{C}^m_\omega$-smooth.

Therefore, the function $f \circ \sigma$ restricted to Q_ε is $\mathcal{C}^m$-smooth. By [21], it extends to a $\mathcal{C}^m$ function F_ε defined on some open neighbourhood of Q_ε. Thus, $f \circ \sigma$ is the gluing of the F_ε restricted to Q_ε. Recall that $f \circ \sigma$ is $\mathcal{C}^m_\omega$-smooth. Lemma 4.1 implies that $f \circ \sigma$ is $\mathcal{C}^m$-smooth in the set

$$U \setminus \bigcup_{\ell \neq k} \{x \in \mathbb{R}^n : x_\ell = x_k = 0\}.$$

The derivatives of $f \circ \sigma$ extend continuously to U, so again by [21], the function $f \circ \sigma$ is $\mathcal{C}^m$-smooth. □

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