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THE NUMBER OF MINIMAL RIGHT IDEALS OF βG

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ABSTRACT. Let G be an infinite Abelian group of cardinality κ and let βG denote the Stone-Čech compactification of G as a discrete semigroup. We show that βG contains 2^{2^κ} many minimal right ideals.

Given a discrete semigroup S, the operation can be naturally extended to the Stone-Čech compactification βS of S making βS a compact right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$, the right translation

$$\beta S \ni x \mapsto xp \in \beta S$$

is continuous, and for each $a \in S$, the left translation

$$\beta S \ni x \mapsto ax \in \beta S$$

is continuous.

We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S, and $S^* = \beta S \setminus S$. The topology of βS is generated by taking as a base the subsets of the form

$$\overline{A} = \{p \in \beta S : A \in p\},$$

where $A \subseteq S$. For $p, q \in \beta S$, the ultrafilter pq has a base consisting of subsets of the form

$$\bigcup\{xB_x : x \in A\}$$

where $A \in p$ and $B_x \in q$.

The semigroup βS is interesting both for its own sake and for its applications to combinatorial number theory and to topological dynamics. An elementary introduction to βS can be found in [5].

As any compact Hausdorff right topological semigroup does, βS has a smallest two-sided ideal $K(\beta S)$ which is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. The intersection of a minimal right ideal and a minimal left ideal is a group, and all these groups are isomorphic. The idempotents of a minimal right (left) ideal form a right (left) zero semigroup, that is, one satisfying the identity $xy = y$ (respectively, $xy = x$). See [5, Sections 1.7 and 2.2] for detailed information about this.

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In [2] and [1], respectively, it was shown that the semigroup $\beta\mathbb{N}$ contains 2^{2^ω} minimal left ideals and 2^{2^ω} minimal right ideals. The first result has been fully extended to an arbitrary infinite discrete cancellative semigroup S by proving that βS contains $2^{2^{|S|}}$ minimal left ideals [5, Theorem 6.42]. However, the problem of counting the minimal right ideals of βS turned out to be more difficult. It was only established that for every infinite discrete cancellative semigroup S, βS contains at least 2^{2^ω} minimal right ideals [5, Corollary 6.41].

The aim of this note is to prove the following result.

Theorem 1. *For every infinite discrete Abelian group G of cardinality κ, βG contains 2^{2^κ} many minimal right ideals.*

The proof of Theorem 1 involves some additional concepts.

Recall that the *Bohr compactification* of a topological group G is a compact topological group bG together with a continuous homomorphism $e : G \to bG$ such that $e(G)$ is dense in bG and the following universal property holds: For every continuous homomorphism $h : G \to K$ from G into a compact topological group K there is a continuous homomorphism $h^b : bG \to K$ such that $h = h^b \circ e$. In the case where G is a discrete Abelian group, the Bohr compactification can be naturally defined in terms of the Pontrjagin duality as follows. Let $\hat{G}$ be the dual group of G and let $\hat{G}_d$ be the group $\hat{G}$ reendowed with the discrete topology. Then bG is the dual group of $\hat{G}_d$. The mapping $e : G \to bG$ is given by $e(x)(\chi) = \chi(x)$, where $x \in G$ and $\chi \in \hat{G}_d$. It is injective. (See [4, 26.11 and 26.12].)

We say that filters $\mathcal{F}$ and $\mathcal{G}$ on a set X are *incompatible* if there are $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that $A \cap B = \emptyset$. A filter $\mathcal{F}$ on a topological space X is *open* if $\mathcal{F}$ has a base of open subsets of X.

In order to prove Theorem 1, we show the following.

Theorem 2. *For every infinite discrete Abelian group G of cardinality κ, there are 2^{2^κ} many pairwise incompatible open filters on bG converging to zero.*

Before proving Theorem 2, let us show how it implies Theorem 1.

Proof of Theorem 1. Let $\mathcal{T}$ denote the Bohr topology on G, that is, the one induced by the mapping $e : G \to bG$, and let $\mathcal{F}$ be the neighborhood filter of zero of $(G, \mathcal{T})$. By Theorem 2, there are 2^{2^κ} pairwise incompatible open filters on bG converging to zero. Considering the restriction of the filters to $e(G)$, we conclude that there are pairwise incompatible open filters $\mathcal{F}_\alpha$ ($\alpha < 2^{2^\kappa}$) on $(G, \mathcal{T})$ converging to zero, that is to say, containing $\mathcal{F}$. Define a closed subset S of G^* and for each $\alpha < 2^{2^\kappa}$, a closed subset J_α of S by

$$S = \bigcap_{U \in \mathcal{F}} \overline{U \setminus \{0\}} \quad \text{and} \quad J_\alpha = \bigcap_{U \in \mathcal{F}_\alpha} \overline{U \setminus \{0\}}.$$

(Here, $\overline{U \setminus \{0\}} = \{p \in \beta G : U \setminus \{0\} \in p\}$.) Equivalently, S and J_α consist of all nonprincipal ultrafilters on G containing $\mathcal{F}$ and $\mathcal{F}_\alpha$, respectively. We claim that S is a subsemigroup of G^* and for each $\alpha < 2^{2^\kappa}$, J_α is a right ideal of S.

To see that S is a subsemigroup, let $p, q \in S$. We have to show that for every $U \in \mathcal{F}$, one has $U \setminus \{0\} \in p + q$. Without loss of generality one may assume that U is open. For every $x \in U \setminus \{0\}$, choose $V_x \in \mathcal{F}$ such that $x + V_x \subseteq U \setminus \{0\}$. Then $U \setminus \{0\} = \bigcup_{x \in U \setminus \{0\}} (x + V_x)$. Since $U \setminus \{0\} \in p$ and $V_x \in q$, we obtain that

$U \setminus \{0\} \in p+q$. The same argument shows that for every $p \in R_\alpha$, $q \in S$ and $U \in \mathcal{F}_\alpha$, one has $U \setminus \{0\} \in p+q$, which witnesses that J_α is a right ideal.

Since the filters $\mathcal{F}_\alpha$ are pairwise incompatible, the right ideals J_α are pairwise disjoint. Taking a minimal right ideal in each J_α, we obtain that there are 2^{2^κ} minimal right ideals of S. Furthermore, since $(G, \mathcal{T})$ is a subgroup of a compact topological group, S contains all the idempotents of G^* [6, Lemma 3], in particular, the idempotents of $K(\beta G)$. (Note that $K(\beta G) \subseteq G^*$, because G^* is an ideal of βG.) Consequently, $S \cap K(\beta G) \neq \emptyset$. But then, by [5, Theorem 1.65],

$$K(S) = K(\beta G) \cap S.$$

It follows from this that every minimal right ideal R of S is contained in a minimal right ideal R' of βG, and the correspondence $R \mapsto R'$ is injective.

Indeed, $K(\beta G)$ is a union of minimal right ideals, so there is a minimal right ideal R' of βG such that $R \cap R' \neq \emptyset$. Then $R \cap R'$ is a right ideal of S contained in R. Consequently, $R \cap R' = R$, as R is minimal, and so $R \subseteq R'$. Since minimal right ideals are disjoint, such an R' is unique. To see that the correspondence $R \mapsto R'$ is injective, let R_1 and R_2 be minimal right ideals of S and assume that $R_1' = R_2'$. Pick any minimal left ideal L of S and let p_1 and p_2 be the identities of the groups $R_1 \cap L$ and $R_2 \cap L$, respectively. Being idempotents of L, p_1 and p_2 belong to the same left zero semigroup, so $p_1 + p_2 = p_1$, and being idempotents of $R_1' = R_2'$, they belong to the same right zero semigroup, so $p_1 + p_2 = p_2$. Hence, $p_1 = p_2$, and consequently, $R_1 = R_2$. (In fact, this correspondence is bijective, since S contains all the idempotents of $K(\beta G)$.)

It follows that the number of minimal right ideals of βG is greater than or equal to that of S, and consequently, it is 2^{2^κ}. □

To prove Theorem 2, we need three lemmas. The first of them is an elementary fact on infinite Abelian groups.

Lemma 1. *Let G be an infinite Abelian group of cardinality κ. Then G admits a homomorphism onto one of the following groups:*

(1) $\mathbb{Z}$, $\bigoplus_\omega \mathbb{Z}(p)$, $\mathbb{Z}(p^\infty)$ *and* $\bigoplus_{p \in Q} \mathbb{Z}(p)$ *if* $\kappa = \omega$,
(2) $\bigoplus_\kappa \mathbb{Z}(p)$ *and* $\bigoplus_\kappa \mathbb{Z}(p^\infty)$ *if* $\kappa > \omega$ *and* $\mathrm{cf}(\kappa) > \omega$,
(3) $\bigoplus_\kappa \mathbb{Z}(p)$, $\bigoplus_\kappa \mathbb{Z}(p^\infty)$, $\bigoplus_{p \in Q} \bigoplus_{\kappa_p} \mathbb{Z}(p)$ *and* $\bigoplus_{p \in Q} \bigoplus_{\kappa_p} \mathbb{Z}(p^\infty)$ *if* $\kappa > \omega$ *and* $\mathrm{cf}(\kappa) = \omega$.

Here, p is a prime number and Q is an infinite subset of the primes. The symbols $\mathbb{Z}$, $\mathbb{Z}(p)$ and $\mathbb{Z}(p^\infty)$ denote the infinite cyclic group, the cyclic group of order p, and the quasi-cyclic group, respectively. The symbol $\mathrm{cf}(\kappa)$ denotes the cofinality of κ. If $\kappa > \omega$ and $\mathrm{cf}(\kappa) = \omega$, $(\kappa_p)_{p \in Q}$ is an infinite increasing sequence of uncountable cardinals cofinal in κ, that is, $\sup_{p \in Q} \kappa_p = \kappa$.

Proof. If G is finitely generated, then $\kappa = \omega$ and G admits a homomorphism onto $\mathbb{Z}$. Therefore, one may assume that G is not finitely generated. We first prove that G admits a homomorphism onto a periodic group of cardinality κ.

Let $\{a_i : i \in I\}$ be a maximal independent subset of G and let $A = \langle a_i : i \in I \rangle$ be the subgroup generated by $\{a_i : i \in I\}$. Then $A = \bigoplus_{i \in I} \langle a_i \rangle$, and for every nonzero $g \in G$, one has $\langle g \rangle \cap A \neq \{0\}$, so G/A is periodic. If $|G/A| = \kappa$, we are done. Suppose that $|G/A| < \kappa$. Then $|A| = \kappa$ and $|I| = \kappa$, because G is not finitely generated. We show that there is a subgroup H of G and a subset $I_1 \subseteq I$ with $|I_1| = \kappa$ such that $G = H \oplus \bigoplus_{i \in I_1} \langle a_i \rangle$.

To this end, choose a complete set S for representatives of the cosets of A in G, and let $H_0 = \langle S\rangle \cap A$. Define $I_0 \subset I$ by

$$I_0 = \{i \in I : x(i) \neq 0 \text{ for some } x \in H_0\},$$

where $x(i)$ is the i-th coordinate of x, and put $I_1 = I \setminus I_0$. If G/A is finite, I_0 is finite as well. If G/A is infinite, $|I_0| \leq |G/A|$, because $|\langle S\rangle| = |G/A|$ and then $|H_0| \leq |G/A|$. In any case, $|I_0| < \kappa$, and consequently $|I_1| = \kappa$. Let

$$A_0 = \langle a_i : i \in I_0\rangle, \quad A_1 = \langle a_i : i \in I_1\rangle \quad \text{and} \quad H = \langle S \cup A_0\rangle.$$

We claim that $G = H \oplus A_1$. Indeed, since $G = \langle S \cup A_0 \cup A_1\rangle$, one has $H + A_1 = G$. To see that $H \cap A_1 = \{0\}$, let $g \in H \cap A_1$. Then $g = d + c_0 = c_1$ for some $d \in \langle S\rangle$, $c_0 \in A_0$ and $c_1 \in A_1$. Consequently, $d = -c_0 + c_1 \in A$. But then $d \in H_0 \subseteq A_0$. Hence, $c_1 = 0$, and $g = 0$.

Having established that $G = H \oplus \bigoplus_{i\in I_1}\langle a_i\rangle$, we obtain that G admits a homomorphism onto $\bigoplus_{i\in I_1}\langle a_i\rangle$, and so onto a periodic group of cardinality κ.

Now let G be a p-group. Then there is a so-called basic subgroup B of G (see [3, Theorem 32.3]). We have that B is a direct sum of cyclic groups, say $B = \bigoplus_{j\in J}\langle b_j\rangle$, and G/B is divisible, that is, isomorphic to $\bigoplus_\lambda \mathbb{Z}(p^\infty)$, where $0 \leq \lambda \leq \kappa$. Suppose that $|G/B| = \kappa$. Then $\lambda > 0$, and $\lambda = \kappa$ if $\kappa > \omega$. It follows that G admits a homomorphism onto $\mathbb{Z}(p^\infty)$ if $\kappa = \omega$, and onto $\bigoplus_\kappa \mathbb{Z}(p^\infty)$ if $\kappa > \omega$. Now suppose that $|G/B| < \kappa$. Then $|B| = \kappa$, and consequently $|J| = \kappa$. It follows that $G = C \oplus \bigoplus_{j\in J_1}\langle b_j\rangle$ for some subgroup C of G and a subset $J_1 \subset J$ with $|J_1| = \kappa$ (see the first part of the proof). Hence, G admits a homomorphism onto $\bigoplus_{j\in J_1}\langle b_j\rangle$, and so onto $\bigoplus_\kappa \mathbb{Z}(p)$.

Finally, let G be periodic. Then $G = \bigoplus_{p\in M} G_p$, where M is the set of all primes p such that the p-primary component G_p of G is nontrivial. If $|G_p| = \kappa$ for some $p \in M$, we are done, because then G admits a homomorphism onto G_p, a p-group of cardinality κ. Suppose that $|G_p| < \kappa$ for each $p \in M$. Then M is infinite and $\mathrm{cf}(\kappa) = \omega$. If $\kappa = \omega$, all G_p are finite, and so G admits a homomorphism onto $\bigoplus_{p\in M}\mathbb{Z}(p)$. Suppose that $\kappa > \omega$. For each $p \in M$, put $\kappa_p = |G_p|$. Clearly $\sup_{p\in M}\kappa_p = \kappa$. Choose an infinite subset $N \subseteq M$ such that $(\kappa_p)_{p\in N}$ is an increasing sequence of uncountable cardinals cofinal in κ. By the previous paragraph, for each $p \in N$, G_p admits a homomorphism onto a group K_p of cardinality κ_p which is isomorphic to $\bigoplus_{\kappa_p}\mathbb{Z}(p)$ or $\bigoplus_{\kappa_p}\mathbb{Z}(p^\infty)$. It follows that there is an infinite subset $Q \subseteq N$ such that either K_p is isomorphic to $\bigoplus_{\kappa_p}\mathbb{Z}(p)$ for all $p \in Q$ or K_p is isomorphic to $\bigoplus_{\kappa_p}\mathbb{Z}(p^\infty)$ for all $p \in Q$. Then the group $K = \bigoplus_{p\in Q} K_p$ is isomorphic to $\bigoplus_{p\in Q}\bigoplus_{\kappa_p}\mathbb{Z}(p)$ or $\bigoplus_{p\in Q}\bigoplus_{\kappa_p}\mathbb{Z}(p^\infty)$, $|K| = \kappa$, and G admits a homomorphism onto K. $\square$

Now, using Lemma 1 and the Pontrjagin duality, we prove the following statement on bG.

Lemma 2. *For every infinite discrete Abelian group G of cardinality κ, bG admits a continuous homomorphism onto $\prod_{2^\kappa}\mathbb{T}$ or $\prod_{2^\kappa}\mathbb{Z}(p)$.*

Here, $\mathbb{T}$ is the circle group, and both products $\prod_{2^\kappa}\mathbb{T}$ and $\prod_{2^\kappa}\mathbb{Z}(p)$ are endowed with the product topology.

Proof. The dual groups of continuous homomorphic images of bG are the subgroups of $\hat{G}_d$, and the dual groups of homomorphic images of G are the closed subgroups of $\hat{G}$ (see [4, Theorems 23.25 and 24.8]). The dual groups of $\prod_{2^\kappa} \mathbb{T}$ and $\prod_{2^\kappa} \mathbb{Z}(p)$ are $\bigoplus_{2^\kappa} \mathbb{Z}$ and $\bigoplus_{2^\kappa} \mathbb{Z}(p)$, respectively. Consequently, in order to prove the lemma, it suffices to show that G admits a homomorphism onto a group whose dual group contains an isomorphic copy of $\bigoplus_{2^\kappa} \mathbb{Z}$ or $\bigoplus_{2^\kappa} \mathbb{Z}(p)$. We distinguish between two cases.

Case 1. $\kappa = \omega$. Then G admits a homomorphism onto one of the following groups: $\mathbb{Z}$, $\bigoplus_\omega \mathbb{Z}(p)$, $\mathbb{Z}(p^\infty)$ and $\bigoplus_{p\in Q} \mathbb{Z}(p)$. Their dual groups are $\mathbb{T}$, $\prod_\omega \mathbb{Z}(p)$, $\mathbb{Z}_p$ and $\prod_{p\in Q} \mathbb{Z}(p)$, respectively. (Here, $\mathbb{Z}_p$ is the group of p-adic integers.) The second group is algebraically isomorphic to $\bigoplus_{2^\omega} \mathbb{Z}(p)$. The others contain torsion-free subgroups of cardinality 2^ω and so contain an isomorphic copy of $\bigoplus_{2^\omega} \mathbb{Z}$.

Case 2. $\kappa > \omega$. Then G admits a homomorphism onto one of the following groups: $\bigoplus_\kappa \mathbb{Z}(p)$, $\bigoplus_\kappa \mathbb{Z}(p^\infty)$, $\bigoplus_{p\in Q} \bigoplus_{\kappa_p} \mathbb{Z}(p)$ and $\bigoplus_{p\in Q} \bigoplus_{\kappa_p} \mathbb{Z}(p^\infty)$ (the two latter groups appear if $\mathrm{cf}(\kappa) = \omega$). Their dual groups are $\prod_\kappa \mathbb{Z}(p)$, $\prod_\kappa \mathbb{Z}_p$, $\prod_{p\in Q} \prod_{\kappa_p} \mathbb{Z}(p)$ and $\prod_{p\in Q} \prod_{\kappa_p} \mathbb{Z}_p$, respectively. The first group is algebraically isomorphic to $\bigoplus_{2^\kappa} \mathbb{Z}(p)$. The others contain torsion-free subgroups of cardinality 2^κ and so contain an isomorphic copy of $\bigoplus_{2^\kappa} \mathbb{Z}$ □

The third lemma deals with products of topological spaces.

Lemma 3. *Let κ be an infinite cardinal. For each $\alpha < \kappa$, let X_α be a space having at least two disjoint nonempty open sets, and let $X = \prod_{\alpha<\kappa} X_\alpha$. Then there are at least 2^κ many pairwise incompatible open filters on X converging to the same point.*

The proof of Lemma 3 involves the notion of an *extremally disconnected* space, that is, a space in which the closures of disjoint open sets are disjoint. Notice that if each factor in an infinite product $X = \prod_{n<\omega} X_n$ has at least two disjoint nonempty open sets, then X is not extremally disconnected. Indeed, for each $n < \omega$, let U_n and V_n be disjoint nonempty open subsets of X_n, let $x_n \in V_n$, and let $x = (x_n)_{n<\omega} \in X$. For every $m < \omega$, define an open subset $W_m = \prod_{n<\omega} W_{m,n} \subset X$ by

$$W_{m,n} = \begin{cases} U_n & \text{if } n = m, \\ V_n & \text{if } n < m, \\ X_n & \text{if } n > m. \end{cases}$$

It follows that $U = \bigcup_{m<\omega} W_{2m}$ and $V = \bigcup_{m<\omega} W_{2m+1}$ are disjoint open subsets of X with $x \in c\ell\, U \cap c\ell\, V$.

Proof of Lemma 3. Let L be the set of limit ordinals $< \kappa$ including 0. Then

$$X = \prod_{\alpha\in L} \prod_{n<\omega} X_{\alpha+n},$$

$|L| = \kappa$, and for each $\alpha \in L$, $\prod_{n<\omega} X_{\alpha+n}$ is not extremally disconnected. Therefore, one may suppose that each X_α in the product $X = \prod_{\alpha<\kappa} X_\alpha$ is not extremally disconnected, so there are disjoint open subsets $U_\alpha, V_\alpha \subset X_\alpha$ and $x_\alpha \in X_\alpha$ such that $x_\alpha \in c\ell\, U_\alpha \cap c\ell\, V_\alpha$. For every $Y = (Y_\alpha)_{\alpha<\kappa} \in \prod_{\alpha<\kappa}\{U_\alpha, V_\alpha\}$, define the filter $\mathcal{F}(Y)$ on X by declaring as a base the subsets of the form $\prod_{\alpha<\kappa} Z_\alpha$, where

(i) for each $\alpha < \kappa$, Z_α is a nonempty open subset of X_α,
(ii) for all but finitely many $\alpha < \kappa$, $Z_\alpha = X_\alpha$, and
(iii) if $Z_\alpha \neq X_\alpha$, then $Z_\alpha = Y_\alpha \cap W$ for some neighborhood W of $x_\alpha \in X_\alpha$.

Then $\mathcal{F}(Y)$, where $Y \in \prod_{\alpha<\kappa}\{U_\alpha, V_\alpha\}$, are pairwise incompatible open filters on X converging to $x = (x_\alpha)_{\alpha<\kappa}$. $\square$

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. By Lemma 2, there is a continuous surjective homomorphism $f : bG \to K$, where K is $\prod_{2^\kappa} \mathbb{T}$ or $\prod_{2^\kappa} \mathbb{Z}(p)$. By Lemma 3, there are pairwise incompatible open filters $\mathcal{F}_\alpha$ $(\alpha < 2^{2^\kappa})$ on K converging to zero. For each $\alpha < 2^{2^\kappa}$, let $\mathcal{H}_\alpha$ be the filter on bG with a base consisting of subsets of the form $f^{-1}(A) \cap U$, where $A \in \mathcal{F}_\alpha$ and U runs over neighborhoods of zero. Then $\mathcal{H}_\alpha$ $(\alpha < 2^{2^\kappa})$ are pairwise incompatible open filters on bG converging to zero. $\square$

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