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# OFF-DIAGONAL MATRIX COEFFICIENTS ARE TANGENTS TO STATE SPACE: ORIENTATION AND C*-ALGEBRAS

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ABSTRACT. Any non-commutative C*-algebra $\mathcal{A}$, e.g., two by two complex matrices, has at least two associative multiplications for which the collection of positive linear functionals is the same. Alfsen and Shultz have shown that by selecting an orientation for the state space $K$ of $\mathcal{A}$, i.e., the convex set of positive linear functionals of norm one, a unique associative multiplication for $\mathcal{A}$ is determined. We give a simple method for describing this orientation.

## 1. INTRODUCTION

In [1] and [2], Alfsen and Shultz give a complete description of state spaces of operator algebras, Jordan as well as associative, such as C*-algebras and von Neumann algebras, plus necessary prerequisites. In particular they show that the state space of a C*-algebra $\mathcal{A}$ together with an orientation of that state space determine the associative multiplication of $\mathcal{A}$, hence completely determine the C*-algebra structure of $\mathcal{A}$. Let us define some terminology.

We will follow the notation from [1]; in particular, $H$ will denote a Hilbert space, with inner product (sesquilinear form) $(\xi, \eta) \in H \times H \mapsto (\xi|\eta) \in \mathbb{C}$, conjugate linear in $\eta$, where $\mathbb{C}$ is the complex numbers; and $\mathcal{B}(H)$ will denote the (Banach $*$-algebra of) bounded, i.e., continuous, linear maps of $H$ to itself with sup norm, i.e., if $T \in \mathcal{B}(H)$, $||T|| = \sup\{|(T\xi|\eta)| : \xi, \eta \in H,\ ||\xi|| \leq 1,\ ||\eta|| \leq 1\}$. The adjoint operation is defined by $(T\xi|\eta) = (\xi|T^*\eta)\ \forall \xi, \eta \in H$. Since $||T^*T|| = ||T||^2\ \forall T \in \mathcal{B}(H)$, $\mathcal{B}(H)$ is a C*-algebra.

The linear functional $\omega_{\xi,\eta} : \mathcal{B}(H) \to \mathbb{C}$ is defined by $\omega_{\xi,\eta}(T) = (T\xi|\eta)$ for $\xi, \eta \in H$. If $\xi = \eta$ we often write $\omega_\xi$ in place of $\omega_{\xi,\xi}$. A (linear) functional $f : \mathcal{B}(H) \to \mathbb{C}$ is positive iff $f(T^*T) \geq 0\ \forall T \in \mathcal{B}(H)$. The functional $\omega_\xi$ is positive, and $||\omega_\xi|| = ||\xi||^2$. In general, if $f$ is a positive linear functional, $||f|| = \sup\{|f(T)| : ||T|| \leq 1,\ T \in \mathcal{B}(H)\} = f(I)$, where $I$ is the identity operator in $B(H)$. We say $f$ is a state on $\mathcal{B}(H)$ if $f$ is a positive linear functional with $||f|| = 1$. We let $K$ be the set of all states on $\mathcal{B}(H)$. Observe that $K$ is a convex set and $\partial_e K$ denotes the extreme points of this convex set. Note for specialists: If $H$ is infinite dimensional, then $K$ in [1] denotes the normal states, i.e., states continuous in a certain topology on $\mathcal{B}(H)$; cf. [1, Chapter 2]. The proofs in this paper will mostly be for the case

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$\dim H = 2$, since this case essentially contains all our new ideas. If $\dim H < \infty$, then all topologies of interest on $\mathcal{B}(H)$ are equivalent (to the norm topology) and the normal state space of $\mathcal{B}(H)$ is identical with the state space of $\mathcal{B}(H)$. Thus we note from [1, Proposition 4.1] that $\partial_e K$, the extreme points of the (normal) state space $K$ of $\mathcal{B}(H)$, equals $\{\omega_\xi : \xi \in H, ||\xi|| = 1\}$. The interested reader can refer to [1, 2] as an excellent, detailed guide for generalizing our results to arbitrary C* and von Neumann algebras.

Now from [1, p. 197] we read:

"In this chapter we have seen that the canonical map $\xi \mapsto \omega_\xi$ from a Hilbert space $H$ to the normal state space $K$ of $\mathcal{B}(H)$ relates the "flat complex" geometry of $H$ to the "curved real" geometry of $\partial_e K$. More specifically, we have seen how it connects important geometric constructs in the two spaces, .... But there is one concept in $H$ which has no obvious counterpart in $K$, namely the complex structure. This structure is relevant in our context because it serves to distinguish between left and right multiplication in $\mathcal{B}(H)\ldots$ ."

In the next section we show in detail, for the case $\mathcal{A} = M_2(C)$, the $2 \times 2$ complex matrices, how the complex structure of $H = \mathbb{C} \oplus \mathbb{C}$ transfers to $K$, the state space of $\mathcal{A}$, and to the (complex) tangent space(s) of $K$. We then show how an orientation of $K$ can be defined and specified which uniquely determines the C*-product on $M_2(\mathbb{C})$. Observe that $M_2(\mathbb{C})$ is a C*-algebra with respect to two possible products. The first is the usual matrix product of $X, Y \in M_2(\mathbb{C}) : (XY)_{ij} = X_{i1}Y_{1j} + X_{i2}Y_{2j}$, $1 \le i, j \le 2$, where $(\ )_{ij}$ refers to the component in the i-th row and j-th column. The second is the "opposed product" defined thusly: $X \circ Y = YX$. The transpose operation $(X^t)_{ij} = X_{ji}$ for $X \in M_2(\mathbb{C})$ defines a linear isomorphism between these two multiplicative structures, i.e., $(X \circ Y)^t = (YX)^t = X^t Y^t$.

## 2. The case $\mathcal{A} = M_2(\mathbb{C})$

We begin by stating a result which clearly shows in our context the geometric consequences of non-commutativity. From [1, Corollary 4.8] we have: The face generated by two distinct extreme points of the normal state space of $\mathcal{B}(H)$ is a Euclidean 3-ball.

Recall that in a convex set such as $K$, $F \subset K$ is a face of $K$ if $\lambda f_1 + (1-\lambda) f_2 \in F$, for $0 < \lambda < 1$ and $f_i \in K$, $i = 1, 2$, imply that $f_1$ and $f_2$ are both in $F$. Extreme points of $K$ are faces which are a single point.

This result, which those who deal mainly with "commutative objects" initially find startling, is a consequence of the (partial) order structure on $K$. To understand why the face generated by two extreme points of $K$ is <u>not</u> just the line segment joining them, we refer the reader to [1].

We now follow [1, pp. 179–180] and describe in detail the state space, $K$, of $M_2(\mathbb{C})$. Let $B^3$ denote the Euclidean 3-ball, i.e., $B^3 = \{(\beta_1, \beta_2, \beta_3) : \beta_i \in \mathbb{R}$, $\sum_{i=1}^3 \beta_i^2 \le 1\}$, where $\mathbb{R}$ denotes the real numbers.

Now $K$ identifies precisely with the set of positive (definite) $2 \times 2$ matrices with trace equal to 1. In [1] it is shown that

$$(\beta_1, \beta_2, \beta_2) \mapsto \frac{1}{2} \begin{pmatrix} 1 + \beta_1 & \beta_2 + i\beta_3 \\ \beta_2 - i\beta_3 & 1 - \beta_1 \end{pmatrix}$$

is an affine isomorphism of $B^3$ with $K$. Thus one can think of $K$ as a solid Euclidean 3-ball sitting inside $M_2(\mathbb{C})$, and the extreme points are precisely the points on the surface of this unit ball. Observe that the tangent space to a point on the surface of $K$, viewed as a subset in $M_2(\mathbb{C})$, is not just a copy of $\mathbb{R}^2$, but rather a complex vector space, as we will see!

Moreover if $\omega = (\rho_{ij})$ is a $2 \times 2$ positive (definite) matrix with trace 1, we have from [1]: $\beta_1 = \rho_{11} - \rho_{22}$, $\beta_2 = \rho_{12} + \rho_{21}$, $\beta_3 = -i(\rho_{12} - \rho_{21})$, where $i = \sqrt{-1}$. Recall that a matrix $(\rho_{ij})$ is positive (definite), by definition, if $\sum_{i,j} \lambda_i \bar{\lambda_j} \rho_{ij} \geq 0$ whenever the $\lambda_i, \lambda_j$ are in $\mathbb{C}$ . We have denoted by $\bar{\lambda_j}$ the complex conjugate of $\lambda_j$.

Now if $X \in M_2(\mathbb{C})$ and $\{\xi_1, \xi_2\}$ is an orthonormal basis of $H = \mathbb{C} \oplus \mathbb{C}$, then without loss of generality $\xi_1 = (1, 0)$, $\xi_2 = (0, 1)$, and

$$X = (\omega_{\xi_i,\xi_j}(X))_{1 \leq i,j \leq 2}.$$

Thus $\omega_{\xi_1,\xi_1}$ and $\omega_{\xi_2,\xi_2}$ are (extreme) states and $\omega_{\xi_1,\xi_2}(X)$ and $\omega_{\xi_2,\xi_1}(X)$ are called *off-diagonal matrix coefficients* of $X$. Using the affine isomorphism of $K$ with $B^3$ we see that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \omega_{\xi_1} \in \partial_e K \mapsto (1, 0, 0) \in B^3$ and that $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \omega_{\xi_2} \in \partial_e K \mapsto (-1, 0, 0) \in B^3$. Thus $\omega_{\xi_1}$ and $\omega_{\xi_2}$ are antipodal points of $K$.

We now ask the fundamental question: Does $K$ determine $\omega_{\xi_1,\xi_2}$ and $\omega_{\xi_2,\xi_1}$? If yes, how?

A trivial answer to this question is given by polarization of the inner product on $H$, viz., $\omega_{\xi_1,\xi_2} = \frac{1}{4} \sum_{n=0}^{3} i^n \omega_{\xi_1 + i^n \xi_2}$.

The following observation, however, leads to a geometric determination by $K$ of $\omega_{\xi_1,\xi_2}$ and $\omega_{\xi_2,\xi_1}$ in the spirit of the work of Alfsen and Shultz.

The curve $p_1(\theta) = \omega_{\xi_1 \cos\theta + \xi_2 \sin\theta} \in \partial_e K$, $0 \leq \theta \leq \pi/2$, is a great semicircle of $B^3$, starting at $p_1(0) = \omega_{\xi_1}$ and ending at $p_1(\pi/2) = \omega_{\xi_2}$.

In fact the four terms in the polarization identity lead to four (pairwise) distinct great semicircles joining $\omega_{\xi_1}$ to $\omega_{\xi_2}$. Thus for $0 \leq \theta \leq \pi/2$ we have:

$$\begin{aligned} p_{-1}(\theta) &= \omega_{\xi_1 \cos\theta - \xi_2 \sin\theta}, \\ p_i(\theta) &= \omega_{\xi_1 \cos\theta + i\xi_2 \sin\theta}, \\ p_{-i}(\theta) &= \omega_{\xi_1 \cos\theta - i\xi_2 \sin\theta}. \end{aligned}$$

Note that $\omega_{\xi_2} = \omega_{-\xi_2} = \omega_{i\xi_2} = \omega_{-i\xi_2}$.

Consider the curves $p_1(\theta)$ and $p_i(\theta)$, $0 \leq \theta \leq \pi/2$, bounding one "quadrant" of $B^3$. If $X \in M_2(\mathbb{C})$, let $p_1(\theta, X) = \omega_{\xi_1 \cos\theta + \xi_2 \sin\theta}(X)$. For a fixed $X$, let us compute the derivative with respect to $\theta$ at $\theta = 0$. Leibnitz's rule applies to the inner product; thus $\left.\frac{dp_1(\theta,X)}{d\theta}\right|_{\theta=0} = \dot{p}_1(0, X) = \omega_{\xi_1,\xi_2}(X) + \omega_{\xi_2,\xi_1}(X)$.

*Remark.* We can think of $p_1(\theta, X)$ as $\widehat{X}(p_1(\theta))$, where $\widehat{X}(f) = f(X)$, by definition, for any state $f$. Thus $\widehat{X}$ is an affine function on the state space $K$, analogous to a "Fourier transform." Then $\dot{p}_1(0, X)$ is the derivative of $\widehat{X}$ at $\omega_{\xi_1,\xi_1}$ in the direction defined by $p_1(\theta)$.

Similarly, consider $p_i(\theta, X)$ and calculate its derivative with respect to $\theta$ at $\theta = 0$ and get $\dot{p}_i(0, X) = i[\omega_{\xi_2,\xi_1}(X) - \omega_{\xi_1,\xi_2}(X)]$. We can now solve for $\omega_{\xi_1,\xi_2}$ and $\omega_{\xi_2,\xi_1}$ in terms of $\dot{p}_1$ and $\dot{p}_i$, and we get:

$$\begin{aligned} \omega_{\xi_1,\xi_2} &= \frac{1}{2}[\dot{p}_1(0) + i\dot{p}_i(0)], \\ \omega_{\xi_2,\xi_1} &= \frac{1}{2}[\dot{p}_1(0) - i\dot{p}_i(0)]. \end{aligned}$$

Some calculation shows that in the "$B^3$-picture",

$$\omega_{\xi_1,\xi_2} = (0,1,i),\ \text{tangent at } \omega_{\xi_1,\xi_1} \in B^3;$$
$$\omega_{\xi_2,\xi_1} = (0,1,-i),\ \text{tangent at } \omega_{\xi_1,\xi_1} \in B^3.$$

Thus $K$ and in particular its tangent spaces are equipped in a natural way with a structure involving $i = \sqrt{-1}$, i.e., a complex structure. In this structure replacing $i$ by $-i$ has the effect of interchanging the matrix coefficients $\omega_{\xi_1,\xi_2}(X)$ and $\omega_{\xi_2,\xi_1}(X)$ for any $X \in M_2(\mathbb{C})$. This is equivalent to taking the transpose of the matrix for $X$ when written in terms of the $\omega_{\xi_i,\xi_j}(X)$.

In fact, in the above setup, replacing $i$ by $-i$ causes a reflection of $K = B^3$ in the great circle (disk) determined by great semicircles $p_1$ and $p_{-1}$. Thus the act of choosing $i$ or $-i$ in the above discussion is equivalent, in the sense of Alfsen and Shultz, to choosing an orientation of $K$. Note that if $U(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, a one-parameter group of unitaries in $M_2(\mathbb{C})$, then $p_1(\theta) = \omega_{U(\theta)\xi_1}$ and $p_{-1}(\theta) = \omega_{U(-\theta)\xi_1}$. In the "$B^3$ picture" these $U(\theta)$ implement rotations of $B^3$ about the axis through the antipodal states $\frac{1}{2}\omega_{\xi_1+i\xi_2}$ and $\frac{1}{2}\omega_{\xi_1-i\xi_2}$.

## 3. The case of $M_n(\mathbb{C})$

Given an $n \times n$ matrix with respect to an orthonormal basis $\{\xi_i\}_{1\le i\le n}$, we can consider $X = (\omega_{\xi_i,\xi_j}(X))_{1\le i,j\le n}$ and examine as in the previous section the $2\times 2$ submatrix $\begin{pmatrix} \omega_{\xi_{i_0},\xi_{i_0}} & \omega_{\xi_{i_0},\xi_{j_0}} \\ \omega_{\xi_{j_0},\xi_{i_0}} & \omega_{\xi_{j_0},\xi_{j_0}} \end{pmatrix}$ for each choice of $i_0 \neq j_0$, since the face of the state space of $M_n(\mathbb{C})$ generated by $\omega_{\xi_{i_0},\xi_{i_0}}$ and $\omega_{\xi_{j_0},\xi_{j_0}}$ is a Euclidean 3-ball.

How one consistently pastes together the orientations of these 3-balls, and more generally how one treats the case of a general C*-algebra or von Neumann algebra, can be understood from reading [1, 2].

*Remarks.* We have benefited from discussions long ago with Professor Alfsen and thank Professors Alfsen and Shultz for writing [1] [2]. The path that led us to this note was inspired not only by [1, 2] but also by [3, 4, 5] and differentiation on the duals of the quaternion and dihedral groups of order 8. This path is quite distinct from the logical presentation above. Finally, we observe that the present note can be used to see a connection between off-diagonal matrix coefficients and order derivations; cf. [2, Chapter 6].

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