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# CONCENTRATION OF 1-LIPSCHITZ MAPS INTO AN INFINITE DIMENSIONAL $\ell^p$ -BALL WITH THE $\ell^q$ -DISTANCE FUNCTION

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ABSTRACT. In this paper, we study the Lévy-Milman concentration phenomenon of 1-Lipschitz maps into infinite dimensional metric spaces. Our main theorem asserts that the concentration to an infinite dimensional  $\ell^p$ -ball with the  $\ell^q$ -distance function for  $1 \leq p < q \leq +\infty$  is equivalent to the concentration to the real line.

# 1. INTRODUCTION

This paper is devoted to investigating the Lévy-Milman concentration phenomenon of 1-Lipschitz maps from mm-spaces (metric measure spaces) to infinite dimensional metric spaces. Here, an *mm-space* is a triple  $(X, d_X, \mu_X)$ , where  $d_X$  is a complete separable metric on a set X and  $\mu_X$  is a finite Borel measure on  $(X, d_X)$ . The theory of concentration of 1-Lipschitz functions was first introduced by V. D. Milman in his investigation of asymptotic geometric analysis ([17], [18], [19]). Nowadays, the theory blends with various areas of mathematics, such as geometry, functional analysis and infinite dimensional integration, discrete mathematics and complexity theory, probability theory, and so on (see [16], [21], [22], [24] and the references therein for further information).

The theory of concentration of maps into general metric spaces was first studied by M. Gromov ([11], [12], [13]). He established the theory by introducing the observable diameter ObsDiam<sub>Y</sub>(X;  $-\kappa$ ) for an mm-space X, a metric space Y, and  $\kappa > 0$  in [13] (see Section 2 for the definition of the observable diameter). Given a sequence  $\{X_n\}_{n=1}^{\infty}$  of mm-spaces and a metric space Y, we note that  $\lim_{n\to\infty} ObsDiam_Y(X_n; -\kappa) = 0$  for any  $\kappa > 0$  if and only if for any sequence  $\{f_n : X_n \to Y\}_{n=1}^{\infty}$  of 1-Lipschitz maps, there exists a sequence  $\{m_{f_n}\}_{n=1}^{\infty}$  of points in Y such that

$$\lim_{n \to \infty} \mu_{X_n}(\{x_n \in X_n \mid d_Y(f_n(x_n), m_{f_n}) \ge \varepsilon\}) = 0$$

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for any  $\varepsilon > 0$ . If  $\lim_{n\to\infty} \text{ObsDiam}_{\mathbb{R}}(X_n; -\kappa) = 0$  for any  $\kappa > 0$ , then the sequence  $\{X_n\}_{n=1}^{\infty}$  of mm-spaces is called a *Lévy family*. The Lévy families were first introduced and analyzed by Gromov and Milman in [10]. In previous works [2], [3], [4], [5], the author proved that if a metric space Y is either an  $\mathbb{R}$ -tree, a doubling space, a metric graph, or a Hadamard manifold, then  $\lim_{n\to\infty} \text{ObsDiam}_Y(X_n; -\kappa) = 0$  holds for any  $\kappa > 0$  and any Lévy family  $\{X_n\}_{n=1}^{\infty}$ . To prove these results, we needed to assume the finiteness of the dimension of the target metric spaces.

In this paper, we treat the case where the dimension of the target metric space Y is infinite. The author has proved in [1] that if the target space Y is so big that an mm-space X with some homogeneity property can isometrically be embedded into Y, then its observable diameter  $ObsDiam_Y(X; -\kappa)$  is not close to zero. It seems from this result that the concentration to an infinite dimensional metric space cannot happen easily.

A main theorem of this paper is the following. For  $1 \leq p \leq +\infty$ , we denote by  $B_{\ell^p}^{\infty}$  an infinite dimensional  $\ell^p$ -ball  $\{(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\infty} \mid \sum_{n=1}^{\infty} |x_n|^p \leq 1\}$  and by  $d_{\ell^p}$  the  $\ell^p$ -distance function.

**Theorem 1.1.** Let  $\{X_n\}_{n=1}^{\infty}$  be a sequence of mm-spaces and  $1 \le p < q \le +\infty$ . Then, the sequence  $\{X_n\}_{n=1}^{\infty}$  is a Lévy family if and only if

(1.1) 
$$\lim_{n \to \infty} \text{ObsDiam}_{(B^{\infty}_{\ell^p}, d_{\ell^q})}(X_n; -\kappa) = 0 \text{ for any } \kappa > 0.$$

As a result, we obtain the example of the infinite dimensional target metric space such that the concentration to the space happens as often as the concentration to the real line.

The proof of the sufficiency of Theorem 1.1 is easy. The observations of A. Gournay and M. Tsukamoto play important roles for the proof of the converse ([9], [28]). Answering a question of Gromov in [14, Section 1.1.4], Tsukamoto proved in [28] that the "macroscopic" dimension of the space  $(B_{\ell^p}^{\infty}, d_{\ell^q})$  for  $1 \leq p < q \leq +\infty$  is finite. Gournay independently proved it in [9] in the case of  $q = +\infty$ . For any  $(p,q) \in \{(k,l) \in \mathbb{N} \times \mathbb{N} \mid l \leq k\} \setminus \{(k,1) \mid k \geq 2\}$ , we have an example of a Lévy family which does not satisfy (1.1) (see Proposition 4.4).

As applications of Theorem 1.1, by virtue of [3, Propositions 4.3 and 4.4], we obtain the following corollaries of a Lévy group action. Lévy groups were first introduced by Gromov and Milman in [10]. Let a topological group G act on a metric space X. The action is called *bounded* if for any  $\varepsilon > 0$  there exists a neighborhood U of the identity element  $e_G \in G$  such that  $d_X(x,gx) < \varepsilon$  for any  $g \in U$  and  $x \in X$ . Note that every bounded action is continuous. We say that the topological group G acts on X by uniform isomorphisms if for each  $g \in G$ , the map  $X \ni x \mapsto gx \in X$  is uniformly continuous. The action is said to be uniformly equicontinuous if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $d_X(gx,gy) < \varepsilon$  for every  $g \in G$  and  $x, y \in X$  with  $d_X(x, y) < \delta$ . Given a subset  $S \subseteq G$  and  $x \in X$ , we put  $Sx := \{gx \mid g \in S\}$ .

**Corollary 1.2.** Let  $1 \leq p < q \leq +\infty$  and assume that a Lévy group G boundedly acts on the metric space  $(B_{\ell^p}^{\infty}, d_{\ell^q})$  by uniform isomorphisms. Then for any compact subset  $K \subseteq G$  and any  $\varepsilon > 0$ , there exists a point  $x_{\varepsilon,K} \in B_{\ell^p}^{\infty}$  such that diam $(Kx_{\varepsilon,K}) \leq \varepsilon$ .

**Corollary 1.3.** There are no non-trivial bounded uniformly equicontinuous actions of a Lévy group to the metric space  $(B_{\ell p}^{\infty}, d_{\ell q})$  for  $1 \leq p < q \leq +\infty$ .

Gromov and Milman pointed out in [10] that the unitary group  $U(\ell^2)$  of the separable Hilbert space  $\ell^2$  with the strong topology is a Lévy group. Many concrete examples of Lévy groups are known by the works of S. Glasner [8], H. Furstenberg and B. Weiss (unpublished), T. Giordano and V. Pestov [6], [7], and Pestov [25], [26]. For examples, groups of measurable maps from the standard Lebesgue measure space to compact groups, unitary groups of some von Neumann algebras, groups of measure and measure-class preserving automorphisms of the standard Lebesgue measure space, full groups of amenable equivalence relations, and the isometry groups of the universal Urysohn metric spaces are Lévy groups (see the recent monograph [24] for precise statements).

## 2. Preliminaries

Let Y be a metric space and  $\nu$  a Borel measure on Y such that  $m := \nu(Y) < +\infty$ . We define for any  $\kappa > 0$ ,

 $\operatorname{diam}(\nu, m - \kappa) := \inf\{\operatorname{diam} Y_0 \mid Y_0 \subseteq Y \text{ is a Borel subset with } \nu(Y_0) \ge m - \kappa\}$ 

and call it the *partial diameter* of  $\nu$ .

**Definition 2.1** (Observable diameter). Let  $(X, d_X, \mu_X)$  be an mm-space with  $m_X := \mu_X(X)$  and Y a metric space. For any  $\kappa > 0$  we define the *observable diameter* of X by

ObsDiam<sub>Y</sub>(X; 
$$-\kappa$$
) := sup{diam( $f_*(\mu_X), m_X - \kappa$ ) |  
 $f: X \to Y$  is a 1-Lipschitz map},

where  $f_*(\mu_X)$  stands for the push-forward measure of  $\mu_X$  by f.

The idea of an observable diameter comes from quantum and statistical mechanics; that is, we think of  $\mu_X$  as a state on a configuration space X and f is interpreted as an observable.

Let  $(X, d_X, \mu_X)$  be an mm-space. For any  $\kappa_1, \kappa_2 \ge 0$ , we define the *separation* distance  $\operatorname{Sep}(X; \kappa_1, \kappa_2) = \operatorname{Sep}(\mu_X; \kappa_1, \kappa_2)$  of X as the supremum of the distance  $d_X(A, B) := \inf\{d_X(a, b) \mid a \in A \text{ and } b \in B\}$ , where A and B are Borel subsets of X satisfying that  $\mu_X(A) \ge \kappa_1$  and  $\mu_X(B) \ge \kappa_2$ .

**Lemma 2.2** (cf. [13, Section  $3\frac{1}{2}.33$ ]). Let X and Y be two mm-spaces and  $\alpha > 0$ . Assume that an  $\alpha$ -Lipschitz map  $f: X \to Y$  satisfies  $f_*(\mu_X) = \mu_Y$ . Then we have

$$\operatorname{Sep}(Y; \kappa_1, \kappa_2) \leq \alpha \operatorname{Sep}(X; \kappa_1, \kappa_2).$$

Relationships between the observable diameter and the separation distance are the following. We refer to [4, Subsection 2.2] for precise proofs.

**Lemma 2.3** (cf. [13, Section  $3\frac{1}{2}.33$ ]). Let X be an mm-space and  $\kappa, \kappa' > 0$  with  $\kappa > \kappa'$ . Then we have

$$ObsDiam_{\mathbb{R}}(X; -\kappa') \ge Sep(X; \kappa, \kappa).$$

Remark 2.4. In [13, Section  $3\frac{1}{2}$ .33], Lemma 2.3 is stated as  $\kappa = \kappa'$ , but that is not true in general. For example, let  $X := \{x_1, x_2\}, d_X(x_1, x_2) := 1$ , and  $\mu_X(\{x_1\}) = \mu_X(\{x_2\}) := 1/2$ . Putting  $\kappa = \kappa' = 1/2$ , we have  $ObsDiam_{\mathbb{R}}(X; -1/2) = 0$  and Sep(X; 1/2, 1/2) = 1. This issue can be removed if in the definition of  $diam(\nu, m - \kappa)$ , we ask for  $\nu(Y_0) > m - \kappa$  instead of  $\nu(Y_0) \ge m - \kappa$ .

**Lemma 2.5** (cf. [13, Section  $3\frac{1}{2}$ .33]). Let  $\nu$  be a Borel measure on  $\mathbb{R}$  with m := $\nu(\mathbb{R}) < +\infty$ . Then, for any  $\kappa > 0$  we have

$$\operatorname{diam}(\nu, m - 2\kappa) \le \operatorname{Sep}(\nu; \kappa, \kappa).$$

In particular, for any  $\kappa > 0$  we have

$$ObsDiam_{\mathbb{R}}(X; -2\kappa) \leq Sep(X; \kappa, \kappa)$$

Combining Lemma 2.3 with Lemma 2.5, we obtain the following corollary:

**Corollary 2.6** (cf. [13, Section  $3\frac{1}{2}.33$ ]). A sequence  $\{X_n\}_{n=1}^{\infty}$  of mm-spaces is a Lévy family if and only if  $\lim_{n\to\infty} \operatorname{Sep}(X_n;\kappa,\kappa) = 0$  for any  $\kappa > 0$ .

**Lemma 2.7.** Let  $\nu$  be a finite Borel measure on  $(\mathbb{R}^k, d_{\ell^p})$  with  $m := \nu(\mathbb{R}^k)$ . Then for any  $\kappa > 0$  we have

diam
$$(\nu, m - \kappa) \le k^{1/p} \operatorname{Sep}\left(\nu; \frac{\kappa}{2k}, \frac{\kappa}{2k}\right).$$

*Proof.* For  $i = 1, 2, \dots, k$ , let  $\operatorname{proj}_i : \mathbb{R}^k \ni (x_i)_{i=1}^k \mapsto x_i \in \mathbb{R}$  be the projection. For any Borel subsets  $A_1, A_2, \dots, A_k \subseteq \mathbb{R}$  with  $(\text{proj}_i)_*(\nu)(A_i) \ge m - \kappa/k$ , we have

$$\nu(A_1 \times A_2 \times \cdots \times A_k) = \nu\Big(\bigcap_{i=1}^k (\operatorname{proj}_i)^{-1}(A_i)\Big) \ge m - \kappa,$$

which leads to

$$\operatorname{diam}(\nu, m-\kappa) \leq \operatorname{diam}(A_1 \times A_2 \times \dots \times A_k) \leq k^{1/p} \max_{1 \leq i \leq k} \operatorname{diam} A_i.$$

We therefore get

$$\operatorname{diam}(\nu, m - \kappa) \le k^{1/p} \max_{1 \le i \le k} \operatorname{diam}\left((\operatorname{proj}_i)_*(\nu), m - \frac{\kappa}{k}\right).$$

Combining this with Lemmas 2.2 and 2.5, we obtain

$$\operatorname{diam}(\nu, m - \kappa) \leq k^{1/p} \max_{1 \leq i \leq k} \operatorname{Sep}\left((\operatorname{proj}_i)_*(\nu); \frac{\kappa}{2k}, \frac{\kappa}{2k}\right) \leq k^{1/p} \operatorname{Sep}\left(\nu; \frac{\kappa}{2k}, \frac{\kappa}{2k}\right).$$
  
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**Lemma 2.8.** Let a, b be two real numbers with a < b. Then, a sequence  $\{X_n\}_{n=1}^{\infty}$ of mm-spaces is a Lévy family if and only if

(2.1) 
$$\lim_{n \to \infty} \text{ObsDiam}_{[a,b]}(X_n; -\kappa) = 0 \text{ for any } \kappa > 0.$$

Proof. The necessity is obvious. We shall prove the converse. Suppose that the sequence  $\{X_n\}_{n=1}^{\infty}$  with the property (2.1) is not a Lévy family. Then, by Corollary 2.6, there exists  $\kappa > 0$  and Borel subsets  $A_n, B_n \subseteq X_n$  such that  $\mu_{X_n}(A_n) \ge \kappa$ ,  $\mu_{X_n}(B_n) \ge \kappa$ , and  $\limsup_{n \to \infty} d_{X_n}(A_n, B_n) > 0$ . Define a function  $f_n : X_n \to \mathbb{R}$  by  $f_n(x) := \max\{d_{X_n}(x, A_n) + a, b\}$ . Since  $\mu_{X_n}(B_n) \ge \kappa$  and  $\limsup_{n \to \infty} d_{X_n}(A_n, B_n)$ > 0, we have

$$\limsup_{n \to \infty} \operatorname{diam}((f_n)_*(\mu_{X_n}), m_{X_n} - \kappa') > 0$$

for any  $0 < \kappa' < \kappa$ . Since each  $f_n$  is a 1-Lipschitz function, this contradicts the assumption (2.1). This completes the proof. 

## 3. Proof of the main theorem

To prove the main theorem, we extract arguments from Gournay's paper [9] and Tsukamoto's paper [28].

For  $k \in \mathbb{N}$ , we identify  $\mathbb{R}^k$  with the subset  $\{(x_1, x_2, \cdots, x_k, 0, 0, \cdots) \in \mathbb{R}^\infty \mid x_i \in \mathbb{R}$  for all  $i\}$  of  $\mathbb{R}^\infty$ . Given  $k \in \mathbb{N} \cup \{\infty\}$ , let  $\mathfrak{S}_k$  be the k-th symmetric group. We consider the group  $G_k := \{\pm 1\}^k \rtimes \mathfrak{S}_k$ . The multiplication in  $G_k$  is given by

$$((\varepsilon_n)_{n=1}^k,\sigma)\cdot((\varepsilon'_n)_{n=1}^k,\sigma'):=((\varepsilon_n\varepsilon'_{\sigma^{-1}(n)})_{n=1}^k,\sigma\sigma').$$

The group  $G_k$  acts on the space  $\mathbb{R}^k$  by

$$((\varepsilon_n)_{n=1}^k, \sigma) \cdot (x_n)_{n=1}^k := (\varepsilon_n x_{\sigma^{-1}(n)})_{n=1}^k$$

Note that this action preserves the k-dimensional  $\ell^p$ -ball  $B_{\ell^p}^k \subseteq B_{\ell^p}^\infty$  and the  $\ell^q$ -distance function  $d_{\ell^q}$ . Define a subset  $\Lambda_k \subseteq B_{\ell^p}^k$  by

$$\Lambda_k := \{ x \in B_{\ell^p}^k \mid x_{i-1} \ge x_i \ge 0 \text{ for all } i \}$$

Given an arbitrary  $\varepsilon > 0$ , we put  $k(\varepsilon) := \lceil (2/\varepsilon)^{pq/(q-p)} \rceil - 1$ , where  $\lceil (2/\varepsilon)^{pq/(q-p)} \rceil$ denotes the smallest integer which is not less than  $(2/\varepsilon)^{pq/(q-p)}$ . For  $k \ge k(\varepsilon) + 1$ , we define a continuous map  $f_{k,\varepsilon} : \Lambda_k \to \mathbb{R}^{k(\varepsilon)}$  by

$$f_{k,\varepsilon}(x) := (x_1 - x_{k(\varepsilon)+1}, x_2 - x_{k(\varepsilon)+1}, \cdots, x_{k(\varepsilon)} - x_{k(\varepsilon)+1}, 0, 0, \cdots).$$

For any  $x \in B_{\ell^p}^k$ , taking  $g \in G_k$  such that  $gx \in \Lambda_k$ , we define

$$F_{k,\varepsilon}(x) := g^{-1} f_{k,\varepsilon}(gx).$$

This definition of the map  $F_{k,\varepsilon}: B_{\ell^p}^k \to B_{\ell^p}^k$  is well-defined (see [28, Section 2] for details). Given  $k \in \mathbb{N}$ , we put  $A_k := \bigcup_{g \in G_{\infty}} g\mathbb{R}^k \subseteq \mathbb{R}^{\infty}$ .

**Theorem 3.1** (cf. [9, Proposition 1.3] and [28, Section 2]). The map  $F_{k,\varepsilon} : B_{\ell^p}^k \to B_{\ell^p}^k$  satisfies that  $F_{k,\varepsilon}(B_{\ell^p}^k) \subseteq A_{k(\varepsilon)}$  and

(3.1) 
$$d_{\ell^q}(x, F_{k,\varepsilon}(x)) \le \frac{\varepsilon}{2}$$

for any  $x \in B^k_{\ell^p}$ .

**Lemma 3.2.** The map  $F_{k,\varepsilon}: (B_{\ell^p}^k, d_{\ell^q}) \to (A_{k(\varepsilon)}, d_{\ell^q})$  is a  $(1 + k(\varepsilon)^{1/q})$ -Lipschitz map.

*Proof.* By the definition of the map  $F_{k,\varepsilon}$ , it suffices to prove that the map  $F := F_{2k(\varepsilon)+2,\varepsilon} : (B_{\ell^p}^{2k(\varepsilon)+2}, d_{\ell^q}) \to (B_{\ell^p}^{2k(\varepsilon)+2}, d_{\ell^q})$  is  $(1 + k(\varepsilon)^{1/q})$ -Lipschitz. Recall that

$$F(x) = (x_1 - x_{k(\varepsilon)+1}, x_2 - x_{k(\varepsilon)+1}, \cdots, x_{k(\varepsilon)} - x_{k(\varepsilon)+1}, 0, 0, \cdots, 0)$$

for any  $x \in \Lambda_{2k(\varepsilon)+2}$ . We hence get

$$d_{\ell^q}(F(x), F(y)) \le d_{\ell^q}(x, y) + k(\varepsilon)^{1/q} |x_{k(\varepsilon)+1} - y_{k(\varepsilon)+1}| \le (1 + k(\varepsilon)^{1/q}) d_{\ell^q}(x, y)$$

for any  $x, y \in \Lambda_{2k(\varepsilon)+2}$ . Since each  $g \in G_{2k(\varepsilon)+2}$  preserves the distance function  $d_{\ell^q}$ , the map F is  $(1 + k(\varepsilon)^{1/q})$ -Lipschitz on each  $g\Lambda_{2k(\varepsilon)+2}$ .

Let  $x, y \in B_{\ell^p}^{2k(\varepsilon)+2}$  be arbitrary points. Observe that there exist a division  $t_0 := 0 \le t_1 \le t_2 \le \cdots \le t_{i-1} \le 1 =: t_i$  and  $g_1, g_2, \cdots, g_i \in G_{2k(\varepsilon)+2}$  such that

 $(1-t)x + ty \in g_j \Lambda_{2k(\varepsilon)+2}$  for any  $t \in [t_{j-1}, t_j]$ . We therefore obtain

$$d_{\ell^q}(F(x), F(y)) \leq \sum_{j=1}^i d_{\ell^q}(F((1-t_{j-1})x+t_{j-1}y), F((1-t_j)x+t_jy))$$
  
$$\leq (1+k(\varepsilon)^{1/q}) \sum_{j=1}^i d_{\ell^q}((1-t_{j-1})x+t_{j-1}y, (1-t_j)x+t_jy)$$
  
$$= (1+k(\varepsilon)^{1/q}) d_{\ell^q}(x, y).$$

This completes the proof.

The following lemma is a key to proving Theorem 1.1.

**Lemma 3.3.** Let  $k \in \mathbb{N}$  and  $\{\nu_{n,k}\}_{n=1}^{\infty}$  be a sequence of finite Borel measures on  $(A_k, d_{\ell^q})$  satisfying that

(3.2) 
$$\lim_{n \to \infty} \operatorname{Sep}(\nu_{n,k}; \kappa_1, \kappa_2) = 0$$

for any  $\kappa_1, \kappa_2 > 0$ . Then, putting  $m_n := \nu_{n,k}(A_k)$ , we have

(3.3) 
$$\lim_{n \to \infty} \operatorname{diam}(\nu_{n,k}, m_n - \kappa) = 0$$

for any  $\kappa > 0$ .

*Proof.* It suffices to prove (3.3) by choosing a subsequence. We shall prove it by induction on k.

For k = 0, since  $A_0 = \{(0, 0, \dots)\}$ , we have diam $(\nu_{n,0}, m_n - \kappa) = 0$ .

Assume that (3.3) holds for any sequence  $\{\nu_{n,k-1}\}_{n=1}^{\infty}$  of finite Borel measures on  $(A_{k-1}, d_{\ell^q})$  having the property (3.2). Let  $\{\nu_{n,k}\}_{n=1}^{\infty}$  be any sequence of finite Borel measures on  $(A_k, d_{\ell^q})$  having the property (3.2). Since  $\lim_{n\to\infty} m_n = 0$ implies (3.3), we assume that  $\inf_{n\in\mathbb{N}} m_n > 0$ . Putting

$$a_n := \max\left\{\operatorname{Sep}\left(\nu_{n,k}; \frac{m_n}{6}, \frac{\kappa}{2}\right), \operatorname{Sep}\left(\nu_{n,k}; \frac{m_n}{6}, \frac{m_n}{6}\right)\right\},\$$

we get  $\lim_{n\to\infty} a_n = 0$  by the assumption (3.2) and  $\inf_{n\in\mathbb{N}} m_n > 0$ . Define subsets  $B_{n,1}$  and  $B_{n,2}$  of the set  $A_k$  by  $B_{n,1} := (A_{k-1})_{a_n} \cap A_k$  and  $B_{n,2} := A_k \setminus B_{n,1}$ , where  $(A_{k-1})_{a_n}$  denotes the closed  $a_n$ -neighborhood of  $A_{k-1}$ . Since  $A_k = B_{n,1} \cup B_{n,2}$ , either (1) or (2) holds:

(1)  $\nu_{n,k}(B_{n,1}) \ge m_n/2$  for any sufficiently large  $n \in \mathbb{N}$ .

(2)  $\nu_{n,k}(B_{n,2}) \ge m_n/2$  for infinitely many  $n \in \mathbb{N}$ .

We first consider the case (2). We denote by  $C_n$  the set of all connected components of the set  $B_{n,2}$ . The proof of the following claim is easy, so we omit the proof.

Claim 3.4. For any  $C \in \mathcal{C}_n$ , there exists  $g \in G_\infty$  such that

$$C = g\{(x_1, \cdots, x_k) \in \mathbb{R}^k \mid x_i \ge a_n \text{ for each } i\}.$$

Claim 3.5. There exists  $C_n \in \mathcal{C}_n$  such that  $\nu_{n,k}(C_n) \ge m_n/6$ .

*Proof.* If  $\nu_{n,k}(C) < m_n/6$  for all  $C \in \mathcal{C}_n$ , then there exists  $\mathcal{C}'_n \subseteq \mathcal{C}_n$  such that

$$\frac{m_n}{6} \le \nu_{n,k} \Big(\bigcup_{C' \in \mathcal{C}'_n} C'\Big) < \frac{m_n}{3}$$

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since  $\nu_{n,k}(B_{n,2}) \ge m_n/2$ . Putting  $\mathcal{C}''_n := \mathcal{C}_n \setminus \mathcal{C}'_n$ , by Claim 3.4, we therefore obtain

$$2^{1/q}a_n \le d_{\ell^q} \left(\bigcup_{C' \in \mathcal{C}'_n} C', \bigcup_{C'' \in \mathcal{C}''_n} C''\right) \le \operatorname{Sep}\left(\nu_{n,k}; \frac{m_n}{6}, \frac{m_n}{6}\right) < a_n,$$

which is a contradiction. This completes the proof of the claim.

Claim 3.6. Putting  $D_n := A_k \cap (C_n)_{\text{Sep}(\nu_{n,k};m_n/6,\kappa/2)}$ , we have  $\nu_{n,k}(D_n) \ge m_n - \kappa/2$ . *Proof.* Take any  $\delta > 0$ . Supposing that  $\nu_{n,k}((D_n)_{\delta}) < m_n - \kappa/2$ , by Claim 3.5, we get

$$\operatorname{Sep}\left(\nu_{n,k};\frac{m_n}{6},\frac{\kappa}{2}\right) < d_{\ell^q}(C_n,A_k \setminus (D_n)_{\delta}) \leq \operatorname{Sep}\left(\nu_{n,k};\frac{m_n}{6},\frac{\kappa}{2}\right),$$

which is a contradiction. This proves that  $\nu_{n,k}((D_n)_{\delta}) \ge m_n - \kappa$  for any  $\delta > 0$ . Tending  $\delta$  to zero, we obtain the claim.

By using Claim 3.4, there exists  $g_n \in G_\infty$  such that

$$C_n = g_n\{(x_1, \cdots, x_k) \in \mathbb{R}^k \mid x_i \ge a_n \text{ for each } i\}.$$

Since  $a_n \geq \text{Sep}(\nu_{n,k}, m_n/6, \kappa/2)$ , we observe that

$$D_n \subseteq A_k \cap g_n(\{(x_1, \cdots, x_k) \in \mathbb{R}^k \mid x_i \ge a_n \text{ for each } i\})_{a_n}$$
  
=  $A_k \cap g_n\Big(\bigcup_{j=1}^k \{(x_1, \cdots, x_k) \in \mathbb{R}^k \mid x_j \ge 0 \text{ and } x_i \ge a_n \text{ for each } i \ne j\}$   
 $\cup \bigcup_{j=k+1}^\infty \{(x_1, \cdots, x_k, 0, \cdots, 0, x_j, 0, 0, \cdots) \in \mathbb{R}^\infty \mid |x_j| \le a_n$   
and  $x_i \ge a_n \text{ for each } i = 1, \cdots, k\}\Big)$ 

 $\subseteq g_n \mathbb{R}^k$ .

Hence  $D_n$  is isometrically embedded into the  $\ell^q$ -space  $(\mathbb{R}^k, d_{\ell^q})$ . Combining Lemma 2.7 and Claim 3.6, we therefore obtain

$$\begin{aligned} \operatorname{diam}(\nu_{n,k}, m_n - \kappa) &\leq \operatorname{diam}(\nu_{n,k}|_{D_n}, m_n - \kappa) \\ &\leq \operatorname{diam}\left(\nu_{n,k}|_{D_n}, \nu_{n,k}(D_n) - \frac{\kappa}{2}\right) \\ &\leq k^{1/q} \operatorname{Sep}\left(\nu_{n,k}|_{D_n}; \frac{\kappa}{4k}, \frac{\kappa}{4k}\right) \\ &\leq k^{1/q} \operatorname{Sep}\left(\nu_{n,k}; \frac{\kappa}{4k}, \frac{\kappa}{4k}\right) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies (3.2).

We next consider the case (1). Putting  $b_n := a_n + \text{Sep}(\nu_{n,k}; m_n/2, \kappa/2)$ , as in the proof of Claim 3.6, we get

$$\nu_{n,k}((A_{k-1})_{b_n} \cap A_k) = \nu_{n,k}((B_{n,1})_{\operatorname{Sep}(\nu_{n,k};m_n/2,\kappa/2)} \cap A_k) \ge m_n - \frac{\kappa}{2}$$

Note that there exists a Borel measurable map  $f_n: (A_{k-1})_{b_n} \cap A_k \to A_{k-1}$  such that

(3.4) 
$$d_{\ell^q}(x, f_n(x)) = \min\{d_{\ell^q}(x, y) \mid y \in A_{k-1}\} \le b_n$$

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for any  $x \in (A_{k-1})_{b_n} \cap A_k$ . Put  $\nu_{n,k-1} := (f_n)_* (\nu_{n,k}|_{(A_{k-1})_{b_n} \cap A_k})$ . An easy calculation proves that

$$\operatorname{Sep}(\nu_{n,k-1};\kappa_1,\kappa_2) \le \operatorname{Sep}(\nu_{n,k};\kappa_1,\kappa_2) + 2b_n$$

for any  $\kappa_1, \kappa_2 > 0$ . By this and the property (3.2) for  $\nu_{n,k}$ , the measures  $\nu_{n,k-1}$  on  $A_{k-1}$  satisfy that

$$\lim_{n \to \infty} \operatorname{Sep}(\nu_{n,k-1};\kappa_1,\kappa_2) = 0$$

for any  $\kappa_1, \kappa_2 > 0$ . By the assumption of the induction, we therefore get

$$\lim_{n \to \infty} \operatorname{diam}\left(\nu_{n,k-1}, \nu_{n,k-1}(A_{k-1}) - \frac{\kappa}{2}\right) = 0$$

for any  $\kappa > 0$ . By using (3.4), we finally obtain

$$\begin{aligned} \operatorname{diam}(\nu_{n,k}, m_n - \kappa) &\leq \operatorname{diam}(\nu_{n,k-1}, m_n - \kappa) + 2b_n \\ &\leq \operatorname{diam}(\nu_{n,k-1}, \nu_{n,k-1}(A_{k-1}) - \kappa/2) + 2b_n \to 0 \text{ as } n \to \infty. \end{aligned}$$
  
This completes the proof of the lemma.

This completes the proof of the lemma.

*Proof of Theorem* 1.1. Lemma 2.8 directly implies the sufficiency of Theorem 1.1. We shall prove the converse. Let  $\{f_n : X_n \to (B^{\infty}_{\ell^p}, d_{\ell^q})\}_{n=1}^{\infty}$  be any sequence of 1-Lipschitz maps. Given an arbitrary  $\varepsilon > 0$ , we shall prove that

$$\operatorname{diam}((f_n)_*(\mu_{X_n}), m_{X_n} - \kappa) \le 2\varepsilon$$

for any  $\kappa > 0$  and any sufficiently large  $n \in \mathbb{N}$ . Put  $k := k(\varepsilon)$  and  $\nu_{n,k} :=$  $(F_{\infty,\varepsilon} \circ f_n)_*(\mu_{X_n})$ . Since

$$\operatorname{diam}((f_n)_*(\mu_{X_n}), m_{X_n} - \kappa) \le \operatorname{diam}(\nu_{n,k}, m_{X_n} - \kappa) + \varepsilon$$

by (3.1), it suffices to prove that

(3.5) 
$$\lim_{k \to \infty} \operatorname{diam}(\nu_{n,k}, m_{X_n} - \kappa) = 0.$$

Since Lemma 2.2 together with Corollary 2.6 and Lemma 3.2 implies that

$$\operatorname{Sep}(\nu_{n,k};\kappa_1,\kappa_2) \le (1+k(\varepsilon)^{1/q})\operatorname{Sep}(X_n;\kappa_1,\kappa_2) \to 0 \text{ as } n \to \infty$$

for any  $\kappa_1, \kappa_2 > 0$ , by virtue of Lemma 3.3, we obtain (3.5). This completes the proof. 

4. Case of 
$$1 \le q \le p \le +\infty$$

For an mm-space X, we define the concentration function  $\alpha_X : (0, +\infty) \to \mathbb{R}$ as the supremum of  $\mu_X(X \setminus A_{+r})$ , where A runs over all Borel subsets of X with  $\mu_X(A) \ge m_X/2$  and  $A_{+r}$  is an open *r*-neighborhood of *A*.

**Lemma 4.1** (cf. [3, Corollary 2.6]). A sequence  $\{X_n\}_{n=1}^{\infty}$  of mm-spaces is a Lévy family if and only if  $\lim_{n\to\infty} \alpha_{X_n}(r) = 0$  for any r > 0.

Let  $p \ge 1$ . We shall consider the  $\ell_p^n$ -sphere  $\mathbb{S}_{\ell^p}^n := \{(x_i)_{i=1}^n \in \mathbb{R}^n \mid \sum_{i=1}^\infty |x_i|^p = 1\}$ 1}. We denote by  $\mu_{n,p}$  the cone measure and by  $\nu_{n,p}$  the surface measure on  $\mathbb{S}^n_{\ell p}$ normalized as  $\mu_{n,p}(\mathbb{S}^n_{\ell^p}) = \nu_{n,p}(\mathbb{S}^n_{\ell^p}) = 1$ . In other words, for any Borel subset  $A \subseteq \mathbb{S}^n_{\ell^p}$ , we put

$$\mu_{n,p}(A) := \frac{1}{\mathcal{L}(B^n_{\ell^p})} \cdot \mathcal{L}(\{tx \mid x \in A \text{ and } 0 \le t \le 1\}),$$

where  $\mathcal{L}$  is the Lebesgue measure on  $\mathbb{R}^n$ .

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By the works of G. Schechtman and J. Zinn [27, Theorems 3.1 and 4.1] and R. Latała and J. O. Wojtaszczyk [15, Theorem 5.31], we obtain

(4.1) 
$$\alpha_{(\mathbb{S}_{\ell^p}^n, d_{\ell^2}, \mu_{n,p})}(r) \le C \exp(-cnr^{\min\{2,p\}}).$$

This inequality for  $p \ge 2$  is also mentioned by A. Naor in [23, Introduction] (see also [15, Proposition 5.21]).

**Lemma 4.2.** Let  $1 \le q \le p \le +\infty$ . Then, we have

$$\alpha_{(\mathbb{S}^n_{\ell^p}, d_{\ell^q}, \mu_{n,p})}(r) \le C \exp(-cn^{1+(1/2-1/q)\min\{2,p\}}r^{\min\{2,p\}}) \text{ if } q < 2$$

and

$$\alpha_{(\mathbb{S}_{\ell^p}^n, d_{\ell^q}, \mu_{n,p})}(r) \le C \exp(-cnr^{\min\{2,p\}}) \text{ if } q \ge 2.$$

*Proof.* If q < 2, by  $d_{\ell^q}(x, y) \leq n^{1/q-1/2} d_{\ell^2}(x, y)$ , we then have

$$\begin{aligned} \alpha_{(\mathbb{S}_{\ell^p}^n, d_{\ell^q}, \mu_{n,p})}(r) &\leq \alpha_{(\mathbb{S}_{\ell^p}^n, d_{\ell^2}, \mu_{n,p})}(n^{1/2 - 1/q}r) \\ &\leq C \exp(-cn^{1 + (1/2 - 1/q)\min\{2, p\}}r^{\min\{2, p\}}). \end{aligned}$$

If  $q \ge 2$ , by  $d_{\ell^q}(x, y) \le d_{\ell^2}(x, y)$ , we then obtain

$$\alpha_{(\mathbb{S}_{\ell^p}^n, d_{\ell^q}, \mu_{n,p})}(r) \le \alpha_{(\mathbb{S}_{\ell^p}^n, d_{\ell^2}, \mu_{n,p})}(r) \le C \exp(-cnr^{\min\{2,p\}}).$$

This completes the proof.

**Corollary 4.3.** The sequences  $\{(\mathbb{S}_{\ell^p}^n, d_{\ell^q}, \mu_{n,p})\}_{n=1}^{\infty}$  and  $\{(\mathbb{S}_{\ell^p}^n, d_{\ell^q}, \nu_{n,p})\}_{n=1}^{\infty}$  are both Lévy families for  $(p,q) \in \mathbb{N} \times \mathbb{N} \setminus \{(k,1) \mid k \geq 2\}$ .

*Proof.* Since  $1 + (1/2 - 1/q) \min\{2, p\} > 0$ , by Lemmas 4.1 and 4.2, the sequence  $\{(\mathbb{S}_{\ell^p}^n, d_{\ell^q}, \mu_{n,p})\}_{n=1}^{\infty}$  is a Lévy family. By virtue of [23, Theorem 6], the sequence  $\{\mathbb{S}_{\ell^p}^n, d_{\ell^q}, \nu_{n,p}\}_{n=1}^{\infty}$  is also a Lévy family. This completes the proof.  $\Box$ 

**Proposition 4.4.** Let  $1 \le q \le p \le +\infty$  and let  $\mu$  be one of the two measures  $\mu_{n,p}$  and  $\nu_{n,p}$ . Then, for any  $\kappa$  with  $0 < \kappa < 1/2$ , we have

$$ObsDiam_{(B^{\infty}_{\ell p}, d_{\ell q})}((\mathbb{S}^{n}_{\ell^{p}}, d_{\ell^{q}}, \mu); -\kappa) \geq 2.$$

Proof. Let  $A \subseteq \mathbb{S}_{\ell^p}^n$  be a Borel subset such that  $\mu(A) \ge 1 - \kappa$ . Since  $\mu(A) = \mu(-A) > 1/2$ , we have  $\mu(A \cap (-A)) > 0$ . Hence, there exists  $x \in A$  such that  $-x \in A$ . Since diam  $A \ge d_{\ell^q}(x, -x) \ge d_{\ell^p}(x, -x) = 2$ , we obtain diam $(\mu, 1 - \kappa) = 2$ . Since the inclusion map from the space  $(\mathbb{S}_{\ell^p}^n, d_{\ell^q})$  to the space  $(B_{\ell^p}^\infty, d_{\ell^q})$  is 1-Lipschitz, we obtain the conclusion. This completes the proof.  $\Box$ 

Combining Corollary 4.3 with Proposition 4.4, we obtain an example of a Lévy family which does not satisfy (1.1) in the case of  $(p,q) \in \{(k,l) \in \mathbb{N} \times \mathbb{N} \mid l \leq k\} \setminus \{(k,1) \mid k \geq 2\}.$ 

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