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A CLASS OF $\mathbb{Z}^d$ SHIFTS OF FINITE TYPE WHICH FACTORS ONTO LOWER ENTROPY FULL SHIFTS

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ABSTRACT. We prove that if a $\mathbb{Z}^d$ shift of finite type with entropy greater than $\log N$ satisfies the corner gluing mixing condition of Johnson and Madden, then it must factor onto the full N-shift.

1. INTRODUCTION

A basic question in symbolic dynamics is the question of when one shift space can factor onto another. There are well-known results addressing this question for $\mathbb{Z}$ shifts of finite type (SFTs). In particular, any $\mathbb{Z}$ SFT with entropy at least $\log N$ factors onto the full N-shift.

The situation is more complicated for $d > 1$. Often, we must impose further requirements, such as mixing conditions, to achieve similar results. Robinson and Sahin [RS] extended Krieger's universal model results to $d > 1$ for SFTs with the uniform filling property. Lightwood [L1, L2] extended the Krieger Embedding Theorem [Kr] to $\mathbb{Z}^d$ subshifts with $d > 1$ for a class of SFTs called square-filling-mixing.

Introducing a new mixing condition called corner gluing, Johnson and Madden [JM] proved that any $\mathbb{Z}^d$ corner gluing SFT with entropy greater than $\log N$ has a finite extension which factors onto the full N-shift. They then posed the question of whether the extension is necessary. We prove that it is not.

Theorem 1.1. *Let X be a corner gluing $\mathbb{Z}^d$ SFT, and suppose $h(X) > \log N$. Then there exists a factor map $\varphi : X \to X_{[N]}$.*

2. DEFINITIONS AND NOTATION

Let $\mathcal{A} = \{0, 1, ..., N\}$, and let $X_{[N]} = \mathcal{A}^{\mathbb{Z}^d}$, $d \in \mathbb{N}$. Give $\mathcal{A}$ the discrete topology, and then give $X_{[N]}$ the product topology. A point $x \in X_{[N]}$ can be viewed as an infinite d-dimensional array of symbols: for $\mathbf{w} \in \mathbb{Z}^d$, let $x_{\mathbf{w}}$ be the symbol in location $\mathbf{w}$.

For each $\mathbf{v} \in \mathbb{Z}^d$, define a shift map $\sigma_{\mathbf{v}} : x \mapsto y$ by $y_{\mathbf{w}} = x_{\mathbf{v}+\mathbf{w}}$, and let σ be the $\mathbb{Z}^d$ action $\{\sigma_{\mathbf{v}}\}_{\mathbf{v}\in\mathbb{Z}^d}$. The system $(X_{[N]}, \sigma)$ is the *full* $\mathbb{Z}^d$ *N-shift.* For $R \subset \mathbb{Z}^d$, a *configuration* on R is some $\mathcal{M} \in \mathcal{A}^R$. For $x \in X_{[N]}$, denote the configuration occurring at R by x_R.

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If X is a closed, shift-invariant subset of $X_{[N]}$, then $(X, \sigma|_X)$ is called a $\mathbb{Z}^d$ shift space, or subshift. Let $\mathcal{A}_X$ be the symbol set of X. A configuration $\mathcal{M} \in \mathcal{A}^R$ is *allowed* in X if there is some $x \in X$ such that $x_R = \mathcal{M}$. Then we say that $\mathcal{M}$ occurs in x.

The most important subshifts are the shifts of finite type. A $\mathbb{Z}^d$ subshift X is a *shift of finite type* (SFT) if it can be defined by forbidding a finite set of configurations $\mathcal{F} = \{F_1, F_2, ..., F_m\}$ occurring in $(\mathcal{A}_X)^{\mathbb{Z}^d}$. X is a *one-step shift of finite type* if a point $x \in (\mathcal{A}_X)^{\mathbb{Z}^d}$ is allowed in X whenever $x_{\{\mathbf{m},\mathbf{n}\}}$ is not in $\mathcal{F}$ for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ with $\|\mathbf{m} - \mathbf{n}\| = 1$, where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^d$. Every SFT may be recoded to be a one-step shift of finite type. We will assume below that all shifts of finite type are one-step.

Let $\mathbf{c} = (1, 1, ..., 1) \in \mathbb{Z}^d$. Let $\Lambda(n) = \{\mathbf{v} = (v_1, ..., v_d) \in \mathbb{Z}^d : 0 \leq v_i < n\}$, the square of length n with lower left corner at the origin. Let $\overline{\Lambda}(2n-1) = \{\mathbf{v} = (v_1, ..., v_d) \in \mathbb{Z}^d : -n < v_i < n\}$, the square of length $2n-1$ centered at the origin. An *n-block* is a configuration on $\Lambda(n)$. Let $B_n(X)$ be the set of n-blocks allowed in X. Let $B(X) = \bigcup_n B_n(X)$.

If X and Y are subshifts, then a map $\phi : X \to Y$ is a *block code* if for $x \in X$, $\phi(x)_{\mathbf{v}}$ depends on some finite block configuration occurring in x, centered at $\mathbf{v}$ for all $\mathbf{v} \in \mathbb{Z}^{\mathbf{d}}$. That is, if there is a map $\Phi : B_{2n-1}(X) \to \mathcal{A}_Y$ such that $\phi(x)_{\mathbf{v}} = \Phi(x_{\overline{\Lambda}(2n-1)+\mathbf{v}})$ for all $\mathbf{v} \in \mathbb{Z}^{\mathbf{d}}$. The block codes from X to Y are exactly the continuous, shift-commuting maps. If ϕ is one-to-one, then it is called an embedding. If ϕ is onto, it is called a factor code, or a factor map. If ϕ is both one-to-one and onto, then it is a conjugacy. A subshift X is a *sofic shift* if there exists an SFT Y and a factor code $\pi : Y \to X$.

The *topological entropy* of a d-dimensional subshift X is defined to be

$$h(X) = \lim_{n \to \infty} \frac{1}{n^d} \log |B_n(X)|.$$

3. Proof of Theorem 1.1

For $\mathbf{k} = (k_1, k_2, ..., k_d) \in \mathbb{N}^d$, let $R_{\mathbf{k}} = \{(a_1, a_2, ..., a_d) \in \mathbb{Z}^d : 0 \leq a_i < k_i$ for $1 \leq i \leq d\}$.

Definition 3.1 ([JM])**.** A $\mathbb{Z}^d$ SFT X is corner gluing if there exists a gluing constant $g > 0$ such that given any two finite subsets $E_1, E_2 \subset \mathbb{Z}^d$ as defined below and any two allowable configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ on these subsets, there exists a point $x \in X$ with $x_{E_1} = \mathcal{C}_1$ and $x_{E_2} = \mathcal{C}_2$. Here $E_1 = R_{\mathbf{k}} + (\mathbf{k}' - \mathbf{k})$ for some $\mathbf{k} \in \mathbb{N}^d$ and some $\mathbf{k}' \in \mathbb{N}^d$ with $\mathbf{k}' > \mathbf{k} + g\mathbf{c}$, and $E_2 = R_{\mathbf{k}'} \backslash R_{\mathbf{k}+g\mathbf{c}}$ (see Figure 1 for the case where $d = 2$).

We can also think about this in terms of creating a larger rectangular configuration on $R_{\mathbf{k}'}$ containing $\mathcal{C}_1$ and $\mathcal{C}_2$ and some uncontrolled gluing symbols between them. Then we say we are gluing $\mathcal{C}_1$ to $\mathcal{C}_2$. We refer to the configurations used to glue them together as gluing strips.

In the proof of Theorem 1.1, we will need to make use of the following result:

Theorem 3.2 ([D])**.** *Let X be an SFT with $h(X) > 0$. Then there exists a family of SFT subsystems of X whose entropies are dense in* $[0, h(X)]$.

We also need the following lemma, which constructs a marker square M that is aperiodic for low periods. For $R \subset \mathbb{Z}^d$ and $\mathbf{v} \in \mathbb{Z}^d \setminus \{0\}$, a configuration $\mathcal{C}$ on R

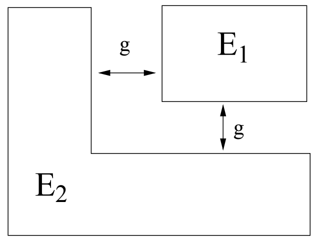

FIGURE 1. Corner gluing

is said to be $\mathbf{v}$-*periodic* if for every pair $\mathbf{w}, \mathbf{w}+\mathbf{v} \in R$ we have $\mathcal{C}_{\mathbf{w}} = \mathcal{C}_{\mathbf{w}+\mathbf{v}}$. For simplicity, throughout this section we will give arguments only for the case where $d = 2$. The proofs for $d \neq 2$ are similar.

Lemma 3.3. *Let X be a corner gluing $\mathbb{Z}^d$ SFT with $h(X) > 0$, and let g be the gluing constant. Then for $f, c \in \mathbb{N}$, if $F \in B_f(X)$, then there exists a square configuration $M \in B(X)$ as in Figure 2, such that M is not $\mathbf{v}$-periodic whenever $\|\mathbf{v}\|_\infty < c$.*

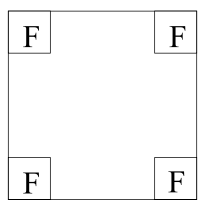

FIGURE 2. Marker square M

Proof. First we will construct a rectangular configuration Q such that Q is not $\mathbf{v}$-periodic whenever $\|\mathbf{v}\|_\infty < c$. Choose some $Q_0 \in B_c(X)$. Consider the $\mathbf{v} \in \mathbb{Z}^2$ such that $\|\mathbf{v}\|_\infty < c$ and Q_0 is $\mathbf{v}$-periodic. Enumerate these as $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_p}$.

Let $i \geq 1$. Assume that $Q_{i-1} \in B_l(X)$, for some $l \in \mathbb{N}$, is not $\mathbf{v_j}$-periodic for $j \leq i-1$ and occurs with lower left corner at the origin. Let $\mathbf{v_i} = (a, b)$. By symmetry, we may assume $a \geq 0$. The block Q_{i-1} will be the corner of the block Q_i, as pictured in Figure 3, according to the following cases: (i) $a, b > 0$, (ii) $a = 0, b > 0$, (iii) $a > 0, b = 0$, and (iv) $a > 0, b < 0$.

Consider case (i). Choose $k \in \mathbb{N}$ large enough that $ka, kb > l + g$ and suppose α is the symbol occurring at the lower left corner of Q_{i-1}. Since $h(X) > 0$, there is some $\beta \in \mathcal{A}_X$ with $\beta \neq \alpha$. Extend Q_{i-1} to an L-shape shown by the dashed lines in Figure 3(i), then glue the symbol β in at position $k\mathbf{v_i}$. Extend the resulting rectangle to a square Q_i. Q_i is not $\mathbf{v_i}$-periodic because $\mathbf{v_i}$-periodicity would imply $\alpha = \beta$. As Q_i has Q_{i-1} as a subblock, it is not $\mathbf{v_j}$-periodic for $j < i-1$ either. For the remaining three cases the argument is the same (see Figure 3 (ii),(iii),(iv)). The construction of Q_i is the same, based on the corresponding figures. End this process with $Q = Q_p$. Then Q will not be $\mathbf{v}$-periodic for $\|\mathbf{v}\|_\infty < c$.

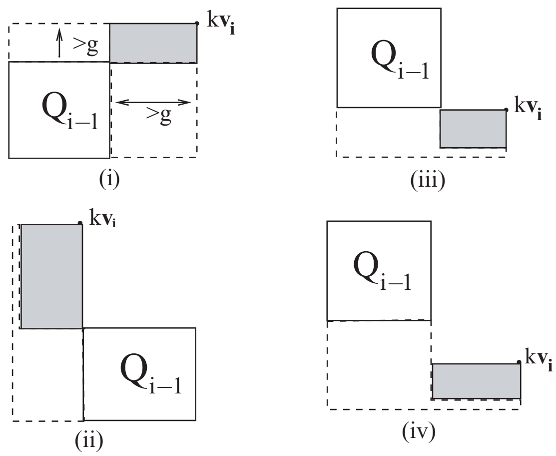

FIGURE 3. Construction of Q_i

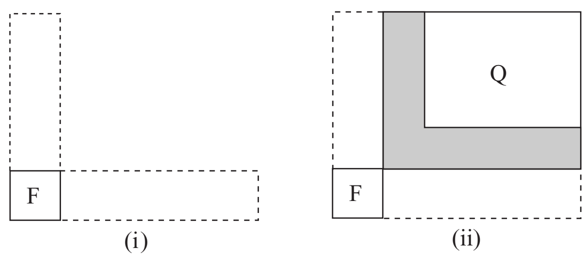

FIGURE 4. Construction of M, step 1

Now construct the marker square M as follows. Extend F to an L-shaped configuration as in Figure 4(i). Then glue in Q as in Figure 4(ii), where the shaded region is the gluing region of width g necessary in the definition of corner gluing.

Extend this configuration to another L-shaped configuration, represented in Figure 5(i) with dashed lines. Choose some rectangle extension of F of the form seen in Figure 5(ii). Glue this rectangle to the L-shaped configuration to form a configuration as in Figure 5(iii).

Next, extend this rectangle to another L-shape as in Figure 6(i), and choose some rectangle as in Figure 6(ii) with an F at the right and left ends. Note that such a configuration is allowed because there is a point which contains it in Figure 6(i). Glue these configurations together to form the square in Figure 6(iii). Take M to be the subblock with an F at each corner. $\square$

With this lemma, we are ready to prove Theorem 1.1, using methods similar to those used by Johnson and Madden in [JM].

Proof of Theorem 1.1. By Theorem 3.2, there is a proper subsystem Y of finite type in X with $h(Y) > \log N$. Choose some square configuration $F \in B(X)$ that

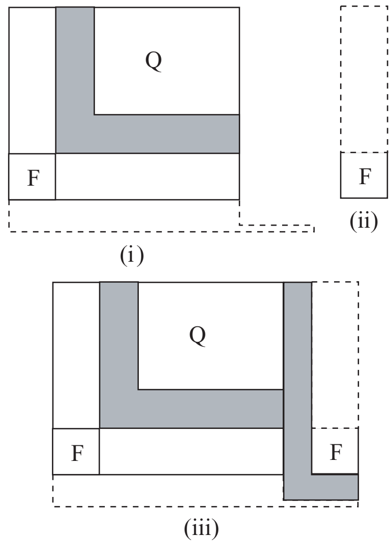

FIGURE 5. Construction of M, step 2

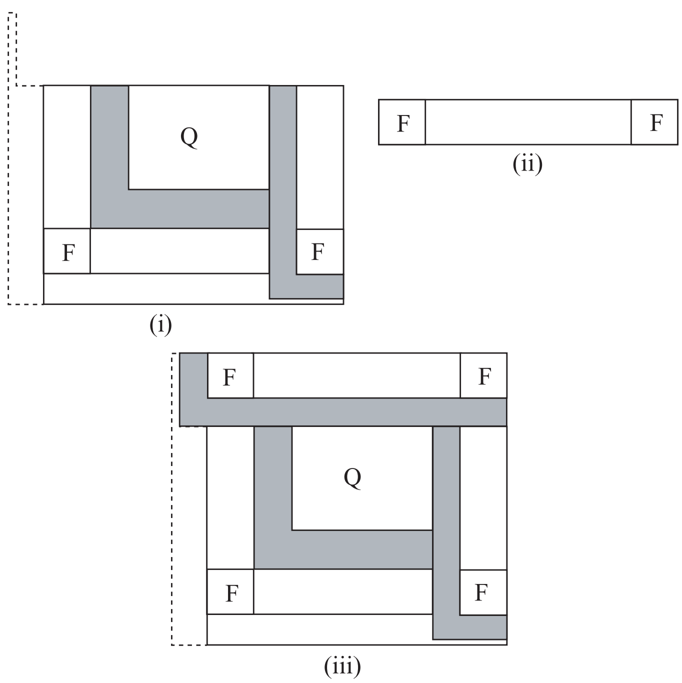

FIGURE 6. Construction of M, step 3

is forbidden in Y, and call its side length f. Construct a square configuration M using F as in Lemma 3.3 for $c = 2(f + g)$. Denote the side length of M by m.

FIGURE 7. Gluing M and G

Given the marker square M and any rectangular configuration G allowed in Y of height m and arbitrary length, first extend M to an L-shaped configuration as in Figure 7(i). Then glue this configuration to G to get the new configuration seen in Figure 7(ii).

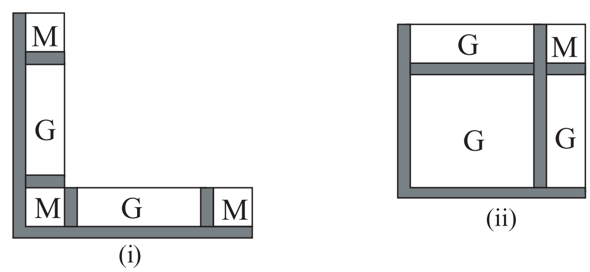

FIGURE 8. Configurations (i) $\mathcal{L}$ and (ii) $\mathcal{C}$

Continue this process to construct a configuration $\mathcal{L}$ of the form seen in Figure 8(i). The blocks labeled G can be filled in with any configuration of the appropriate size allowed in Y (we think of these as 'good' blocks), and the shaded regions are the necessary gluing strips (which may depend on the choice of G-blocks). By the *inside corner* of $\mathcal{L}$, we mean the upper right corner of the block M in the lower left corner of $\mathcal{L}$.

Glue a block of the type in Figure 8(ii) to $\mathcal{L}$ to get a legal block $\mathcal{D}$ of the type in Figure 9; $\mathcal{C}$, the complement of $\mathcal{L}$ in $\mathcal{D}$, will be called a follower of $\mathcal{L}$. We do not control the symbols in the gluing strips, but all G-configurations of the correct size will appear in follower blocks for some choice of gluing strip configuration. Let l be the side length of the central block G in $\mathcal{D}$, and $J = l + 2g + m$; then $\mathcal{C} \in B_J(X)$. Each $\mathcal{L}$ has at least $|B_l(Y)|$ followers and because $h(Y) > \log N$, we have $|B_l(Y)| > N^{J^2}$ for large enough l. For each $\mathcal{L}$, partition its followers into N^{J^2} nonempty sets, $P(\mathcal{L})_1, P(\mathcal{L})_2, ..., P(\mathcal{L})_{N^{J^2}}$, depending only on the follower's central G-block.

Claim. Let $x \in X$. If blocks $\mathcal{D}$ and $\mathcal{D}'$ of the form in Figure 9 occur at different places in x, then their follower portions, $\mathcal{C}$ and $\mathcal{C}'$, do not overlap.

Proof of claim. Without loss of generality, assume $\mathcal{D}$ occurs with lower left corner at the origin, and $\mathcal{D}'$ occurs with lower left corner at $\mathbf{v}$. Suppose $\mathcal{C}$ and $\mathcal{C}'$ do overlap. We know that $\mathbf{v}$ is not such that $\|\mathbf{v}\|_\infty < c$, as that would contradict the lack of small periodicity of M assured by Lemma 3.3. But the lower left corner M of $\mathcal{D}'$ cannot overlap too much with any other M in $\mathcal{D}$ either, and we are assuming

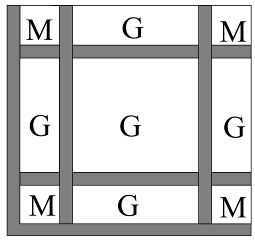

FIGURE 9. Configuration $\mathcal{D}$

that $\mathcal{C}$ and $\mathcal{C}'$ overlap. Therefore $\|\mathbf{v}\|_\infty \leq J - c$. Now since M was constructed with an F at each corner and $c = 2(f+g)$, at least one subblock F of $\mathcal{D}'$ must occur entirely in a 'good' block G of $\mathcal{D}$. However these blocks were chosen from the blocks allowed in Y and so cannot contain F as a subblock. Thus $\mathcal{C}$ and $\mathcal{C}'$ cannot overlap. □

Consider the J-blocks of $X_{[N]}$. Enumerate them as $E_1, E_2, ..., E_{N^{J^2}}$. Now we are ready to construct a factor map $\varphi : X \to X_{[N]}$. We will define φ so that it essentially maps blocks from $P(\mathcal{L})_i$ to E_i for each configuration $\mathcal{L}$ and $i = 1, 2, ..., N^{J^2}$.

We make this precise as follows. For $x \in X$, suppose a configuration $\mathcal{D}$ as in Figure 9 occurs in $x_{\Lambda(2J-1)+\mathbf{v}-J\mathbf{c}}$, the $(2J-1)$-block centered at $\mathbf{v}$, and $x_\mathbf{v}$ is in the follower portion of $\mathcal{D}$. By the claim, $x_\mathbf{v}$ occurs in the follower portion of no other such block $\mathcal{D}'$. Therefore, there exist unique $\mathbf{u}, \mathbf{w} \in \mathbb{Z}^2$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathcal{L}$ has its inside corner at $\mathbf{u}$, and $0 < w_i \leq J$ for $i = 1, 2$. If $x_\mathbf{v}$ occurs in $\mathcal{C} \in P(\mathcal{L})_j$, then we define $\varphi(x)_\mathbf{v}$ to be the symbol from coordinate $\mathbf{w}$ of E_j. If $x_\mathbf{v}$ is not in a follower, then $\varphi(x)_\mathbf{v} = 0$.

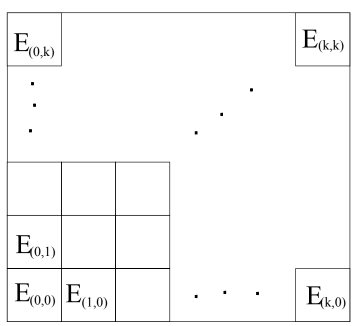

FIGURE 10. $E \in B_{kJ}(X_{[N]})$

Claim. φ is onto.

Proof of claim. Let $E \in B_{kJ}(X_{[N]})$ be as in Figure 10. Choose a configuration $\mathcal{R}$ of the form shown in Figure 11(i) whose height and width are both $kJ + m + g$. We will glue configurations to R that will result in a square configuration which maps to E. Consider the configuration $\mathcal{L}$ in the lower left corner of $\mathcal{R}$.

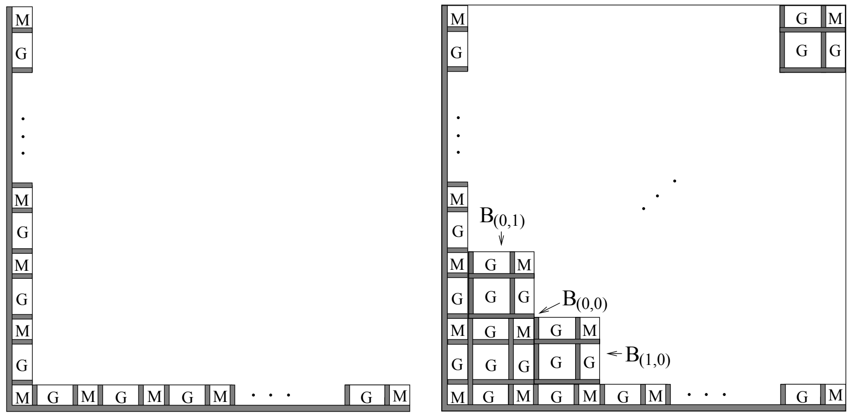

FIGURE 11. Configurations (i) $\mathcal{R}$ and (ii) $B \in B_{kJ}(X)$

If $E_{(0,0)} = E_i$, then choose a configuration $B_{(0,0)} \in P(\mathcal{L})_i$ as in Figure 8(ii) to glue to $\mathcal{L}$. This new block $B_{(0,0)}$ together with $\mathcal{R}$ forms two new $\mathcal{L}$-configurations as shown in 11(ii). One will be above $B_{(0,0)}$ and one will be to the right of it. Glue in followers of each $\mathcal{L}$ from the partition elements corresponding to $E_{(1,0)}$ and $E_{(0,1)}$. Continuing in this manner, complete a block $B \in B_{kJ}(X)$ that maps to E under the block map. $\square$

Johnson and Madden give the following example of a $\mathbb{Z}^2$ SFT X, defined by the matrices below, that is corner gluing with $h(X) > \log 2$ [JM]. Johnson and Madden's theorem tells us only that X is the finite-to-one factor of an SFT that factors onto the full shift, and they ask whether X itself can factor onto $X_{[2]}$. Theorem 1.1 tells us that it does.

$$\mathbf{A_h} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{A_v} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

It is still not known whether every $\mathbb{Z}^d$ SFT with $h(X) > \log N$ factors onto the full N-shift.

REFERENCES

[B] M. Boyle, *Lower entropy factors of sofic systems*, Erg. Th. & Dyn. Sys. **3** (1983), 541-557. MR753922 (85m:54014)

[D] A. Desai, *Subsystem entropy for $\mathbb{Z}^d$ sofic shifts*, Indag. Math. (N.S.) **17** (2006), 353-359. MR2321105

[DGS] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic Theory on Compact Spaces*, Springer Lec. Notes in Math. **527**, Springer-Verlag (1976). MR0457675 (56:15879)

[JM] A. Johnson and K. Madden, *Factoring higher-dimensional shifts of finite type onto the full shift*, Erg. Th. & Dyn. Syst. **25** (2005), 811-822. MR2142947 (2006c:37008)

[Kr] W. Krieger, *On the subsystems of topological Markov chains*, Erg. Th. & Dyn. Syst. **2** (1982), 195-202. MR693975 (85b:28020)

[L1] S. Lightwood, *Morphisms from non-periodic $\mathbb{Z}^2$ subshifts. I: Constructing embeddings from homomorphisms*, Erg. Th. & Dyn. Syst. **23** (2003), 587-609. MR1972240 (2004a:37022)

[L2] S. Lightwood, *Morphisms from non-periodic $\mathbb{Z}^2$ subshifts. II: Constructing homomorphisms to square-filling mixing shifts of finite type*, Erg. Th. & Dyn. Syst. **24** (2004), 1227-1260. MR2085910 (2005c:37021)

[LM] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press (1995). MR1369092 (97a:58050)

[M] B. Marcus, *Factors and extensions of full shifts*, Monatsh. Math. **88** (1979), 239-247. MR553733 (81g:28023)

[RS] E. A. Robinson, Jr., and A. Şahin, *Modeling ergodic, measure preserving actions on $\mathbb{Z}^d$ shifts of finite type*, Monatsh. Math. **132** (2001), 237-253. MR1844076 (2002e:37005)

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