PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 8, August 2009, Pages 2809–2812
S 0002-9939(09)09657-9
Article electronically published on March 16, 2009

ADDENDUM: LACK OF UNIFORMLY EXPONENTIAL STABILIZATION FOR ISOMETRIC C_0-SEMIGROUPS UNDER COMPACT PERTURBATION OF THE GENERATORS IN BANACH SPACES

LI-XIN LIU AND FAMING GUO

(Communicated by Marius Junge)

Let X be an infinite-dimensional Banach space, $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup with the generator $(A, D(A))$ on X, K be a compact operator on X, and $\{S_K(t)\}_{t\geq 0}$ be the C_0-semigroup generated by $(A+K,\ D(A))$.

In [3] we discussed the non-exponential stabilization for isometric C_0-semigroups under compact perturbation, but a slip has occurred in the proof of Theorem 3. More precisely, we have used the isometricity of the adjoint semigroup $\{S^*(t)\}_{t\geq 0}$ while the isometricity of the C_0-semigroup $\{S(t)\}_{t\geq 0}$ is assumed. However, there are some isometric C_0-semigroups whose adjoint semigroups are not isometric; for a concrete example, we refer to [1] and [2]. Therefore, the statement of Theorem 3 in [3] must be changed as follows.

Theorem 1. *Let $\{S(t)\}_{t\geq 0}$ be an isometric C_0-semigroup on an infinite-dimensional reflexive Banach space X. Then, the C_0-semigroup $\{S_K(t)\}_{t\geq 0}$ cannot be uniformly exponentially stable.*

Proof. Let $U := \{x \in X :\ \|x\| \leq 1\}$ be the unit ball in X. Since X is an infinite-dimensional Banach space, then U is not a compact set in X. Thus, there exist $\epsilon_0 > 0$ and $y_j \in U, j = 1, 2, ...,$ such that

$$\|y_i - y_j\| \geq \epsilon_0 > 0$$

for all $i \neq j$.

For the compactness of operator K, we deduce that K^* is compact and $R(K^*)$, the range of operator K^*, is separable, and consequently, $\{S^*(\tau)K^*x^* :\ x^* \in X^*\}$ is separable for each $\tau \geq 0$. Letting Q^+ be the set of non-negative rational numbers, we have that Q^+ is denumerable, and hence

$$\bigcup_{\tau\in Q^+} \{S^*(\tau)K^*x^* :\ x^* \in X^*\}$$

is also separable. Therefore, $V^* := span\{S^*(\tau)K^*x^* :\ \ x^* \in X^*,\ \tau \geq 0\}$ is separable, or equivalently, there exists a countable subset $\{v_1^*, v_2^*, ..., v_m^*, ...\}$ which is dense in V^*. From the uniform boundedness of $\{y_k\}_{k=1}^{\infty}$ and the diagonal method, there is a subsequence $\{y_{j_n}\}_{n=1}^{\infty}$ of $\{y_j\}_{j=1}^{\infty}$ such that $\{(v_m^*(y_{j_n})\}_{n=1}^{\infty}$ is convergent

Received by the editors September 14, 2007.
2000 *Mathematics Subject Classification.* Primary 47A50, 47A55.
The original paper appeared in Proc. Amer. Math. Soc. 135(2007), 1881-1887.

for any v_m^*. Therefore, from $\|y_{j_n}\| \le 1$ and the denseness of $\{v_m^*\}_{m=1}^{\infty}$ in V^*, it follows that $\{x^*(y_{j_n})\}_{n=1}^{\infty}$ is convergent for any $x^* \in \overline{V^*}$.

Defining a map $f_0 : \overline{V^*} \to C$ by

$$f_0(x^*) := \lim_{n\to\infty} x^*(y_{j_n})$$

for $x^* \in \overline{V^*}$, we can easily see that f_0 is a continuous and linear functional on $\overline{V^*}$. From the Hahn-Banach Theorem and the reflexivity of X, there exists a bounded linear functional $f \in X^{**}(= X)$ such that

$$f|_{\overline{V^*}} = f_0 \ \text{ and } \ \|f\| = \|f_0\|.$$

Furthermore, we can deduce that

$$KS(\tau)y_{j_n} \longrightarrow KS(\tau)f \ \text{ as } n \to \infty \tag{1}$$

for each $\tau \ge 0$. In fact, assume that there exist $\eta > 0$, $\tau_0 > 0$ and a subsequence $\{y_{j_{n_k}}\}_{k=1}^{\infty}$ of $\{y_{j_n}\}_{n=1}^{\infty}$ such that

$$\|KS(\tau_0)y_{j_{n_k}} - KS(\tau_0)f\| > \eta, \ \ k = 1, 2, ...$$

Hence, there exists $x_k^* \in X^*$ with $\|x_k^*\| = 1$ such that

$$|x_k^*(KS(\tau_0)y_{j_{n_k}}) - x_k^*(KS(\tau_0)f)| > \frac{\eta}{2}, \ \ k = 1, 2, ... \tag{2}$$

Since K^* is compact, without loss of generality, there exists $y_0^* \in X^*$ such that $K^*x_k^* \longrightarrow y_0^*$ as $k \to \infty$, and then $S^*(\tau_0)y_0^* \in \overline{V^*}$. Therefore, we deduce that

$$\begin{aligned}
& |x_k^*(KS(\tau_0)y_{j_{n_k}}) - x_k^*(KS(\tau_0)f)| \\
= \ & |(S^*(\tau_0)K^*x_k^*)(y_{j_{n_k}}) - (S^*(\tau_0)K^*x_k^*)(f)| \\
\le \ & |(S^*(\tau_0)K^*x_k^*)(y_{j_{n_k}}) - (S^*(\tau_0)y_0^*)(y_{j_{n_k}})| + |(S^*(\tau_0)y_0^*)(y_{j_{n_k}}) - (S^*(\tau_0)y_0^*)(f_0)| \\
& +|(S^*(\tau_0)y_0^*)(f) - (S^*(\tau_0)K^*x_k^*)(f)| \\
\le \ & (1 + \|f_0\|)\|K^*x_k^* - y_0^*\| + |(S^*(\tau_0)y_0^*)(y_{j_{n_k}}) - (S^*(\tau_0)y_0^*)(f_0)| \\
\longrightarrow \ & 0 \ \ (k \to \infty),
\end{aligned}$$

where we have used $f|_{\overline{V^*}} = f_0$, the isometricity of $\{S^*(t)\}_{t\ge0}$ and the reflexivity of X. There is a contradiction with (2) which shows the validity of (1).

Noting that $\|y_i - y_j\| \ge \epsilon_0$ for $i \ne j$, we may assume that $\|y_{j_n} - f\| > \frac{\epsilon_0}{2}$; and letting $x_n = \frac{y_{j_n} - f}{\|y_{j_n} - f\|}$, we deduce from (1) and the definition of x_n that

$$\|x_n\| = 1 \ \text{ and } \ \|KS(\tau)x_n\| \to 0 \ (n \to \infty) \tag{3}$$

for each $\tau \ge 0$.

Let $f_{n,m}(\tau) : \ [0, \ \infty) \mapsto X$ be defined by

$$f_{n,m}(\tau) = \begin{cases} S_K(\tau)KS(m-\tau)x_n, & \tau \in [0, \ m], \\ 0, & \tau \in (m, \ \infty); \end{cases}$$

thus, from (3), we have

$$\|f_{n,m}(\tau)\| \to 0 \ \ (n \to \infty)$$

for $m = 1, 2, ...$ and $\tau \in [0, \infty)$. Therefore, by the diagonal method, there exists a subsequence $\{x_{n_m}\}_{m=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$$g_m(\tau) := f_{n_m,m}(\tau) \to 0 \ \text{ as } m \to \infty$$

for $\tau \in [0, \infty)$.

Suppose on the contrary that $\{S_K(t)\}_{t\geq 0}$ is uniformly exponentially stable. Then there exist constants $M_K \geq 1$ and $w_K > 0$ such that

$$\|S_K(t)\| \leq M_K e^{-w_K t}$$

for all $t \geq 0$. Thus,

$$\|g_m(\tau)\| \quad \leq \quad \begin{cases} M_K\|K\|e^{-w_K\tau}, & \tau \in [0,\ m], \\ 0, & \tau \in (m,\ \infty). \end{cases}$$

Hence, we conclude that

$$\lim_{m\to\infty} \int_0^\infty \|g_m(\tau)\| d\tau = \lim_{m\to\infty} \int_0^m \|S_K(\tau)KS(m-\tau)x_{n_m}\| d\tau = 0$$

by Lebesgue's dominated convergence theorem.

From the bounded perturbation theorem of C_0-semigroups, it follows that

$$S_K(t)x = S(t)x + \int_0^t S_K(\tau)KS(t-\tau)x d\tau,\ x \in X,\ t \geq 0,$$

which yields that

$$\begin{aligned} & \lim_{m\to\infty} \|S(m)x_{n_m}\| \\ \leq \quad & \lim_{m\to\infty} \left(\|S_K(m)x_{n_m}\| + \| \int_0^m S_K(\tau)KS(m-\tau)x_{n_m} d\tau\| \right) \\ = \quad & 0. \end{aligned} \tag{4}$$

On the other hand, $\{S(t)\}_{t\geq 0}$ is an isometric C_0-semigroup. Then we have $\|S(m)x_{n_m}\| = \|x_{n_m}\| = 1$ for all m. There is a contradiction with (4), which ends the proof of Theorem 1. □

Finally, we recall the following result in Luo, Weng, and Feng [4].

Theorem 2. *Let X be a reflexive Banach space, and let the adjoint semigroup $\{S^*(t)\}_{t\geq 0}$ be strongly stable. If $\{S_K(t)\}_{t\geq 0}$ is uniformly exponentially stable, then $\{S(t)\}_{t\geq 0}$ is also uniformly exponentially stable.*

It is well known that a C_0-semigroup $\{S(t)\}_{t\geq 0}$ is uniformly exponentially stable if and only if $\|S(t_0)\| < 1$ for some $t_0 > 0$. Therefore, the following natural question arises: Can a C_0-semigroup whose adjoint semigroup is isometric be uniformly exponentially stabilized by a compact operator? Actually, the following results have been proved in [3].

Theorem 3. *If $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup on an infinite-dimensional Banach space X and its adjoint semigroup $\{S^*(t)\}_{t\geq 0}$ is an isometric semigroup, then the C_0-semigroup $\{S_K(t)\}_{t\geq 0}$ cannot be uniformly exponentially stable for any compact operator K on X.*

Proof. Suppose that $\{S_K(t)\}_{t\geq 0}$ is uniformly exponentially stable. Then there exist constants $M_K \geq 1$ and $w_K > 0$ such that

$$\|S_K(t)\| \leq M_K e^{-w_K t}, \quad t \geq 0.$$

□

From the proof of Theorem 3 in [3], as the notation in [3], there exists a sequence $\{x^*_{n_m}\}_{m=1}^{\infty} \subset X^*$ with $\|x^*_{n_m}\| = 1$ such that

$$\lim_{m\to\infty} \int_0^m \|S^*_K(\tau)K^*S^*(m-\tau)x^*_{n_m}\| d\tau = 0.$$

From the bounded perturbation theorem of C_0-semigroups, we deduce that

$$\begin{aligned} & \lim_{m\to\infty} \|S^*(m)x^*_{n_m}\| \\ \leq\ & \lim_{m\to\infty} \left(\|S^*_K(m)x^*_{n_m}\| + \| \int_0^m S^*_K(\tau)K^*S^*(m-\tau)x^*_{n_m} d\tau\| \right) \\ \leq\ & \lim_{m\to\infty} \left(M_K e^{-w_K m} + \int_0^m \|S^*_K(\tau)K^*S^*(m-\tau)x^*_{n_m}\| d\tau \right) \\ =\ & 0. \end{aligned}$$

There is a contradiction with the isometricity of $\{S^*(t)\}_{t\geq 0}$.

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Department of Mathematics and Physics, Panzhihua University, Panzhihua, Sichuan 617000, People's Republic of China

E-mail address: pzhliulixin@yahoo.com.cn

Department of Applied Mathematics, Chengdu University of Technology, Chengdu, Sichuan 610059, People's Republic of China

E-mail address: guofm@cdut.edu.cn