

PURIFICATION AND SATURATION

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ABSTRACT. This paper illustrates the general technique established in 1984 by Hoover and Keisler for extending certain types of results from atomless Loeb measure spaces to measure spaces that we shall call “nowhere countably generated”. The Hoover-Keisler technique is applied here to further extend the authors’ 2006 generalization of a theorem of Dvoretzky, Wald and Wolfowitz on the purification of measure-valued maps. The authors’ 2006 result was first extended to these more general spaces by K. Podczeck in 2007; he used new results in functional analysis produced for that purpose. This paper demonstrates that, in general, such extensions follow from the Hoover-Keisler technique. Moreover, adaptations of counterexamples from earlier papers show that the extension obtained here holds only for nowhere countably generated spaces.

1. INTRODUCTION

In 1975 [6], the first author showed how to modify a measure formed from infinitesimal weights in a nonstandard model to obtain a standard real-valued measure. The simplest examples start with uniform infinitesimal weights on hyperfinite sets, that is, sets that have the formal properties of finite sets. Since the publication of [6], various results not valid for any Lebesgue space have been established in the literature using the construction from [6], now called the Loeb measure construction; see [2], [7] and [8] for some recent references for these kind of results. This paper illustrates the general technique established in 1984 [3] by Hoover and Keisler for extending such results from atomless Loeb measure spaces to measure spaces that we shall call “nowhere countably generated”.

An example of a result that does not hold for any Lebesgue space is the authors’ 2006 [7] generalization of a theorem of Dvoretzky, Wald and Wolfowitz [1, Theorem 4]. In [7], the authors showed that when given a measurable mapping f from a nonatomic Loeb probability space $(\Omega, \mathcal{A}, \lambda)$ to the space of Borel probability measures on a compact metric space A , there then exists a measurable mapping g from $(\Omega, \mathcal{A}, \lambda)$ to A such that f and g yield the same values for the integrals associated with a countable class of functions on $\Omega \times A$. The map g is called a “purification” of the measure-valued map f . The original paper of Dvoretzky, Wald

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and Wolfowitz used a general nonatomic probability space $(\Omega, \mathcal{A}, \lambda)$ and therefore needed the target space A to be finite.

The authors' restriction in [7] on the measure space $(\Omega, \mathcal{A}, \lambda)$ was replaced by Konrad Podczeck in 2007 [9] with the more liberal restriction that for any set of positive measure in \mathcal{A} , the relative σ -algebra of measurable subsets (modulo sets of measure 0) cannot be countably generated. We shall call such a space **nowhere countably generated**, replacing Podczeck's term "superatomless".

Podczeck's extension of the author's result used new results in functional analysis produced for that purpose. This note demonstrates that, in general, such extensions follow from a technique established in 1984 [3] by D. N. Hoover and H. J. Keisler. (In [3], the property of a space being "saturated" was formulated and shown (implicitly) to be equivalent to a space being nowhere countably generated; see Fajardo-Keisler [2, Theorem 3B.7, page 47] for an explicit equivalence result.)

Our demonstration here of the Hoover-Keisler technique shows that with the choice of appropriate mappings, results such as the purification result in [7] can easily be validated for nowhere countably generated measure spaces once they have been established for Loeb measure spaces, or even just such a space formed on a hyperfinite set. This generalization technique is analogous to the extension of certain results established for the Lebesgue unit interval to more general atomless spaces. Just as the Lebesgue unit interval forms the prototype and simplest such space, hyperfinite Loeb counting spaces play a similar role in the class of nowhere countably generated measure spaces.

Konrad Podczeck shows in [9] that purification results for a compact metric target space A can only hold for nowhere countably generated spaces. By the method used in the proof of Theorem 3.7 of [4], this fact also follows from counterexamples in our earlier papers when adapted using appropriate mappings for that purpose.

2. THE RESULTS

Given a Polish space Z (i.e., a complete, separable, metrizable space), we will use $\mathcal{M}(Z)$ to denote the space of Borel probability measures on Z with the topology of weak convergence. Let $(\Omega, \mathcal{A}, \lambda)$ be an atomless probability space. Given a measurable mapping h from (Ω, \mathcal{A}) to Z , we write λh^{-1} for the Borel probability measure on Z taking the value $\lambda(h^{-1}[B])$ at each Borel set $B \subseteq Z$; this measure is also called the distribution of h . Any Borel probability measure ν on Z is the distribution of some measurable mapping h from $(\Omega, \mathcal{A}, \lambda)$.

Following [3], we say that a probability space $(\Omega, \mathcal{A}, \lambda)$ is saturated if given any two Polish spaces X and Y , any $\tau \in \mathcal{M}(X \times Y)$ (with marginal probability measure τ_X on X), and any measurable mapping Φ from (Ω, \mathcal{A}) to X with distribution τ_X , there is a measurable mapping Ψ from (Ω, \mathcal{A}) to Y such that the measurable mapping (Φ, Ψ) from (Ω, \mathcal{A}) to $X \times Y$ has distribution τ . It is easy to see that a saturated probability space must be atomless. It is shown in [3] that $(\Omega, \mathcal{A}, \lambda)$ is saturated if and only if for any $S \in \mathcal{A}$ with $\lambda(S) > 0$, the restriction of \mathcal{A} to the measurable subsets of S cannot be countably generated; we say that $(\Omega, \mathcal{A}, \lambda)$ is nowhere countably generated.

Let A be a compact metric space with the Borel σ -algebra \mathcal{B} . For any mapping f from Ω to $\mathcal{M}(A)$, the \mathcal{A} -measurability of f with respect to the topology on $\mathcal{M}(A)$ is equivalent to the \mathcal{A} -measurability of $f(\cdot)(B)$ for each $B \in \mathcal{B}$. The space $C(A)$ of continuous real-valued functions on A is supplied with the sup-norm topology. For

any $\gamma \in \mathcal{M}(A)$, $\text{supp } \gamma$ is the support of γ , i.e., the complement of the union of all open γ -null subsets of A .

Let \mathcal{F} be the collection of all functions ψ from $\Omega \times A$ to \mathbb{R} such that (1) $\psi(\cdot, a)$ is \mathcal{A} -measurable on Ω for each $a \in A$, (2) $\psi(\omega, \cdot)$ is continuous on A for each $\omega \in \Omega$, and (3) for each $\psi \in \mathcal{F}$ there is a nonnegative, λ -integrable function α_ψ on Ω with $|\psi(\omega, a)| \leq \alpha_\psi(\omega)$ for all $(\omega, a) \in \Omega \times A$. In what follows, we shall call a set countable if it is finite or countably infinite.

Lemma 2.1. *Any $\psi \in \mathcal{F}$ can be viewed as a measurable mapping from Ω to $C(A)$, so $\|\psi(\omega, \cdot)\|$ is a λ -integrable function of ω .*

Proof. Since $C(A)$ is separable, we need show only that for $\varphi \in C(A)$ and $\varepsilon > 0$,

$$S := \{\omega \in \Omega : \sup_{a \in A} |\psi(\omega, a) - \varphi(a)| \leq \varepsilon\} \in \mathcal{A}.$$

Let $\{a_i : i \in I\}$ be a countable dense set in A . For each $i \in I$,

$$S_i := \{\omega \in \Omega : |\psi(\omega, a_i) - \varphi(a_i)| \leq \varepsilon\} \in \mathcal{A},$$

so $S = \bigcap_{i \in I} S_i \in \mathcal{A}$. □

Theorem 2.2. *Assume $(\Omega, \mathcal{A}, \lambda)$ is saturated. Let J be a countable set, and ψ_j , $j \in J$, a countable subcollection of \mathcal{F} . Given an \mathcal{A} -measurable mapping f from Ω to $\mathcal{M}(A)$, there is an \mathcal{A} -measurable mapping g from Ω to A such that for each $j \in J$,*

$$(2.1) \quad \int_{\Omega} \int_A \psi_j(\omega, a) df(\omega)(a) d\lambda(\omega) = \int_{\Omega} \psi_j(\omega, g(\omega)) d\lambda(\omega).$$

Proof. By Lemma 2.1, each ψ_j can be viewed as a measurable mapping from Ω to $C(A)$. Define a measurable mapping F from Ω to $\mathcal{M}(A) \times C(A)^J$ by letting $F(\omega) = (f(\omega), \{\psi_j(\omega)\}_{j \in J})$.

Let (T, \mathcal{T}, P) be an atomless Loeb probability space. Since random variables on an atomless probability space can represent all distributions (see, for example, Chapter 1 of [2]), there is a measurable mapping G from T to $\mathcal{M}(A) \times C(A)^J$ such that F and G have the same distribution on $\mathcal{M}(A) \times C(A)^J$. The mapping G has the form $G(t) = (f'(t), \{\phi_j(t)\}_{j \in J})$, where f' is a \mathcal{T} -measurable mapping from T to $\mathcal{M}(A)$, and for each $j \in J$, ϕ_j is a \mathcal{T} -measurable mapping from T to $C(A)$ with the same distribution as ψ_j , so $\|\phi_j(t, \cdot)\|$ is a P -integrable function of t .

It now follows from Theorem 2.2 in [7] that there is a \mathcal{T} -measurable mapping g' from T to A such that for each $j \in J$,

$$(2.2) \quad \int_T \int_A \phi_j(t, a) df'(t)(a) dP(t) = \int_T \phi_j(t, g'(t)) dP(t).$$

Since $(\Omega, \mathcal{A}, \lambda)$ is saturated, there is an \mathcal{A} -measurable mapping g from Ω to A such that (F, g) and (G, g') have the same distribution, which we denote by τ , on $[\mathcal{M}(A) \times C(A)^J] \times A$. Fix any $j \in J$. Let τ_j^1 be the marginal measure of τ on the product of $\mathcal{M}(A)$ with the j -th factor of $C(A)^J$, and let τ_j^2 be the marginal measure of τ on the product of the j -th factor of $C(A)^J$ with A . Note that τ_j^1 is the joint distribution of (f, ψ_j) as well as of (f', ϕ_j) , and τ_j^2 is the joint distribution of (ψ_j, g) as well as of (ϕ_j, g') .

Consider the real-valued mapping H^1 on $\mathcal{M}(A) \times C(A)$ defined by setting $H^1(\nu, u) = \int_A u d\nu$ for each $\nu \in \mathcal{M}(A)$ and $u \in C(A)$. This function is jointly

continuous on $\mathcal{M}(A) \times C(A)$. Since τ_j^1 is the joint distribution of (f, ψ_j) , the left side of equation (2.1) equals

$$\int_{\Omega} H^1(f(\omega), \psi_j(\omega)) d\lambda(\omega) = \int_{\mathcal{M}(A) \times C(A)} H^1(\nu, u) d\tau_j^1.$$

On the other hand, since τ_j^2 is the joint distribution of (ψ_j, g) , it follows from the joint continuity of the mapping $H^2(u, a) = u(a)$ on $C(A) \times A$ that the right side of equation (2.1) equals

$$\int_{\Omega} H^2(\psi_j(\omega), g(\omega)) d\lambda(\omega) = \int_{C(A) \times A} H^2(u, a) d\tau_j^2.$$

Similarly the left and right sides of equation (2.2) equal $\int_{\mathcal{M}(A) \times C(A)} H^1(\nu, u) d\tau_j^1$ and $\int_{C(A) \times A} H^2(u, a) d\tau_j^2$ respectively. Hence, Equation (2.1) follows from Equation (2.2). \square

Corollary 2.3. *Given the saturated probability space $(\Omega, \mathcal{A}, \lambda)$, for each k in a countable set K , let μ_k be a finite signed measure on (Ω, \mathcal{A}) that is absolutely continuous with respect to λ . For each j in a countable set J , let ψ_j be an element of \mathcal{F} . If f is an \mathcal{A} -measurable mapping from Ω to $\mathcal{M}(A)$, then there is an \mathcal{A} -measurable mapping g from Ω to A such that $g(\omega) \in \text{supp } f(\omega)$ for λ -almost all $\omega \in \Omega$, and for all $k \in K$, $j \in J$, $B \in \mathcal{B}$, and all bounded Borel measurable functions θ on A ,*

- (1) $\int_A \psi_j(\omega, a) f(\omega)(da) d\lambda(\omega) = \int_{\Omega} \psi_j(\omega, g(\omega)) d\lambda(\omega),$
- (2) $\int_{\Omega} f(\omega)(B) d\mu_k(\omega) = \mu_k(g^{-1}[B]),$
- (3) $\int_{\Omega} \int_A \theta(a) f(\omega)(da) d\mu_k(\omega) = \int_{\Omega} \theta(g(\omega)) d\mu_k(\omega).$

Proof. The corollary follows from Theorem 2.2 in the same way that Corollary 2.4 of [7] follows from Theorem 2.2 of [7]. \square

Remark 2.4. Suppose $(\Omega, \mathcal{A}, \lambda)$ is atomless but not saturated. Then, as noted, there is a set $S \in \mathcal{A}$ such that $\lambda(S) > 0$ and $\mathcal{A}^S = \{B \in \mathcal{A} : B \subseteq S\}$ is countably generated. We follow the method used in the proof of Theorem 3.7 of [4] to generate a counterexample. Let P be the probability measure on (S, \mathcal{A}^S) rescaled from λ . There is a measurable mapping h from S to $[0, 1]$ such that h induces an isomorphism between the corresponding measure algebras of (S, \mathcal{A}^S, P) and the Lebesgue unit interval $([0, 1], \mathcal{C}, \mu)$. Let \mathcal{B} denote the Borel subsets of $[-1, 1]$. Let f be an \mathcal{A} -measurable mapping from Ω to $\mathcal{M}([-1, 1])$ by setting $f(\omega) = G(h(\omega))$ for $\omega \in S$ and $f(\omega) = \delta_0$ for $\omega \notin S$, where $G(x) = (\delta_x + \delta_{-x})/2$, $x \in [0, 1]$, as in Example 2 of [10] and δ_x is the unit mass at x . We claim that there is no \mathcal{A} -measurable mapping g from Ω to $[-1, 1]$ such that $g(\omega) \in \text{supp } f(\omega)$ for λ -almost all $\omega \in \Omega$, and

$$\forall B \in \mathcal{B}, \int_{\Omega} f(\omega)(B) d\lambda(\omega) = \lambda g^{-1}[B].$$

Suppose such a g exists. Then, $g(\omega) = 0$ for $\omega \notin S$. Then, for any $B \in \mathcal{B}$, $\int_S f(\omega)(B) dP(\omega) = P g^{-1}[B]$. There is a Borel measurable mapping ϕ from $[0, 1]$ to $[-1, 1]$ such that $g = \phi(h)$, which means that $\phi(x) \in \text{supp } G(x)$ for μ -almost all $x \in [0, 1]$, and for any $B \in \mathcal{B}$, $\int_{[0, 1]} G(x)(B) d\mu = \mu \phi^{-1}[B]$; this is contrary to what is shown in Example 2 of [10]. By using a Borel bijection, we can modify this example to work for any uncountable compact metric space A instead of $[-1, 1]$.

This means that the results in Theorem 2.2 and Corollary 2.3 hold for an atomless probability space $(\Omega, \mathcal{A}, \lambda)$ and an uncountable compact metric space A if and only if the probability space is saturated.

Remark 2.5. One of the aims of [4] is to apply the saturation property itself to prove some results on saturated probability spaces directly without going through Loeb spaces. This is also possible in the context of [7] and this note. In fact, nonstandard analysis is only used in the proof of Lemma 2.1 of [7] using Loeb spaces. That result, however, follows directly from the saturation property with the following proof. Let $\{g_n, n \in \mathbb{N}\}$ be a sequence of measurable mappings from a saturated probability space $(\Omega, \mathcal{A}, \lambda)$ to A such that for each $j \in J$, the sequence $\int_{\Omega} \psi_j(\omega, g_n(\omega)) d\lambda(\omega)$ converges. We claim that there is an \mathcal{A} -measurable mapping g from Ω to A such that for each $j \in J$,

$$(2.3) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \psi_j(\omega, g_n(\omega)) d\lambda(\omega) = \int_{\Omega} \psi_j(\omega, g(\omega)) d\lambda(\omega).$$

For each n , define a measurable mapping G^n from Ω to $A \times C(A)^J$ by letting $G^n(\omega) = (g_n(\omega), \{\psi_j(\omega)\}_{j \in J})$; let ρ^n be the distribution of G^n on $A \times C(A)^J$. Since A is compact, some subsequence of $\{\rho^n, n \in \mathbb{N}\}$ weakly converges to a measure ρ on $A \times C(A)^J$; without loss of generality, we assume the whole sequence weakly converges to ρ . By the saturation property, there is a measurable mapping g from Ω to A such that the distribution of $(g, \{\psi_j\}_{j \in J})$ is ρ . Let ρ_j^n (ρ_j) be the marginal measure of ρ^n (ρ) on the product of A with the j -th factor of $C(A)^J$. Then, $\{\rho_j^n, n \in \mathbb{N}\}$ weakly converges to ρ_j . It follows from the joint continuity of the mapping $H^2(u, a) = u(a)$ on $C(A) \times A$ that

$$\lim_{n \rightarrow \infty} \int_{C(A) \times A} H^2(u, a) d\rho_j^n = \int_{C(A) \times A} H^2(u, a) d\rho_j,$$

which implies equation (2.3). The rest of the results in [7] follow with exactly the same standard measure-theoretic arguments as in [7]. Thus, all the results in [7] on Loeb spaces generalize to saturated probability spaces without using nonstandard analysis.

Remark 2.6. In a private communication, Jerry Keisler has suggested another approach for the proof of Theorem 2.2 here (and thus Theorem 2.2 in [7] as well). His approach is based on another general principle in [3]: If a property is approximately true in every atomless probability space, then the property is true in every saturated space. Thus, Theorem 2.2 for a compact metric target space A can be proved from the classical result in [1], which is valid for a finite target space, by using approximations and saturation.

Here is a sketch of the approach. An \mathcal{A} -measurable mapping h from Ω to $\mathcal{M}(A)$ is said to be simple if h has finite range and each element of the range of h is supported by a finite subset of A . Then, an \mathcal{A} -measurable mapping f from Ω to $\mathcal{M}(A)$ can be approximated by a sequence $\{f_n\}_{n=1}^{\infty}$ of simple functions from Ω to $\mathcal{M}(A)$. An \mathcal{A} -measurable θ from Ω to $C(A)$ is simple if θ has finite range. Each ψ_j can be approximated by a sequence $\{\psi_{jn}\}_{n=1}^{\infty}$ of simple functions from Ω to $C(A)$. By the classical result in [1] valid for a finite target space, for each n there is an \mathcal{A} -measurable function g_n from Ω to A with finite range such that for each $j \leq n$,

$$(2.4) \quad \int_{\Omega} \int_A \psi_{jn}(\omega, a) df_n(\omega)(a) d\lambda(\omega) = \int_{\Omega} \psi_{jn}(\omega, g_n(\omega)) d\lambda(\omega).$$

Recall from the proof of Theorem 2.2 that F is a measurable mapping from Ω to $\mathcal{M}(A) \times C(A)^J$ such that $F(\omega) = (f(\omega), \{\psi_j(\omega)\}_{j \in J})$. For each n , define a measurable mapping F_n from Ω to $\mathcal{M}(A) \times C(A)^J$ by letting $F_n(\omega) = (f_n(\omega), \{\psi_{jn}(\omega)\}_{j \in J})$. The approximation property, the compactness of A and the saturation property of $(\Omega, \mathcal{A}, \lambda)$ imply that there is an \mathcal{A} -measurable function g from Ω to A such that a subsequence of $\{(F_n, g_n)\}_{n=1}^\infty$ converges to (F, g) in distribution. By using a limiting argument for Equation (2.4) and a similar argument as in the proof of Theorem 2.2, one can check that Equation (2.1) will be satisfied by g .

The proof sketched in this remark uses the main result in [1] to prove Theorem 2.2. On the other hand, the main result in [1] is given a new proof as a corollary of Theorem 2.2 as it is presented in [7].

Remark 2.7. For some recent applications to game theory of the classical result of Dvoretzky-Wald-Wolfowitz in [1, Theorem 4], see [5]. Finally, we note that with the choice of appropriate mappings, the results in [8] should also extend to nowhere countably generated measure spaces using the technique employed here.

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