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MIYAOKA-YAU INEQUALITY FOR MINIMAL PROJECTIVE MANIFOLDS OF GENERAL TYPE

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ABSTRACT. In this short paper, we prove the Miyaoka-Yau inequality for minimal projective n-manifolds of general type by using Kähler-Ricci flow.

1. INTRODUCTION

If M is a projective n-manifold with ample canonical bundle $\mathcal{K}_M$, there exists a Kähler-Einstein metric ω with negative scalar curvature by Yau's theorem on the Calabi conjecture ([14]), which was obtained by Aubin independently ([1]). As a consequence, there is an inequality for Chern numbers, the Miyaoka-Yau inequality,

$$(\frac{2(n+1)}{n}c_2(M) - c_1^2(M)) \cdot (-c_1(M))^{n-2} \geq 0, \tag{1.1}$$

where $c_1(M)$ and $c_2(M)$ are the first and the second Chern classes of M (cf. [13]). Furthermore, if the equality in (1.1) holds, the Kähler-Einstein metric ω is a complex hyperbolic metric; i.e. the holomorphic sectional curvature of ω is a negative constant. If $n = 2$, (1.1) even holds for algebraic surfaces of general type (cf. [4], [8], [9]), which may not admit any Kähler-Einstein metric. In [12], the inequality (1.1) is proved for any dimensional minimal projective manifold of general type by using conic Kähler-Einstein metrics. In this short paper, we give a different proof of (1.1) for minimal projective n-manifolds of general type by using Kähler-Ricci flow and study the extremal case of (1.1).

Let M be a minimal projective manifold of general type with $\dim_{\mathbb{C}} M = n \geq 2$. The canonical bundle $\mathcal{K}_M$ of M is big, and semi-ample, i.e. $\mathcal{K}_M^n > 0$, and, for a positive integer $m \gg 1$, the linear system $|m\mathcal{K}_M|$ is base point free (as quoted in [11]). For $m \gg 1$, the complete linear system $|m\mathcal{K}_M|$ defines a holomorphic map $\Phi : M \longrightarrow \mathbb{CP}^N$, which is birational onto its image M_{can}. M_{can} is called the canonical model of M, and Φ is called the contraction map. Note that M may not admit any Kähler-Einstein metric. The Kähler-Ricci flow is an evolution equation of a family of Kähler metrics ω_t, $t \in [0, T)$, on M,

$$\partial_t \omega_t = -Ric(\omega_t) - \omega_t, \tag{1.2}$$

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where $Ric(\omega_t)$ is the Ricci form of ω_t. By [11], [10], [3], and [15], for any Kähler metric as initial metric, the solution ω_t of the Kähler-Ricci flow equation exists for all time $t \in [0, \infty)$, and the scalar curvature of ω_t is uniformly bounded. Thus we can prove (1.1) by using the technique developed in [6], where a Hitchin-Thorpe type inequality was proved for 4-manifolds which admit a long time solution to a normalized Ricci flow equation with bounded scalar curvature. Before proving the Miyaoka-Yau inequality, we show that the L^2-norm of the Einstein tensor tends to zero along a subsequence of a solution of the Kähler-Ricci flow equation (1.2).

Theorem 1.1. *Let M be a minimal projective manifold of general type with $\dim_{\mathbb{C}} M = n \geq 2$, and let ω_t, $t \in [0, \infty)$, be a solution of the Kähler-Ricci flow equation (1.2). Then there exists a sequence of times $t_k \longrightarrow \infty$, when $k \longrightarrow \infty$, such that*

$$\lim_{k \longrightarrow \infty} \int_M |\rho_{t_k}|^2 \omega_{t_k}^n = 0,$$

where $\rho_{t_k} = Ric_{t_k} - \frac{R_{t_k}}{n}\omega_{t_k}$ denotes the Einstein tensor of ω_{t_k} and R_{t_k} denotes the scalar curvature of ω_{t_k}.

As a corollary of this theorem, we obtain the Miyaoka-Yau inequality for minimal projective manifolds of general type.

Corollary 1.2. *If M is a minimal projective manifold of general type with $\dim_{\mathbb{C}} M = n \geq 2$, then*

$$\left(\frac{2(n+1)}{n} c_2(M) - c_1^2(M)\right) \cdot (-c_1(M))^{n-2} \geq 0.$$

Furthermore, if the equality holds, there is a complex hyperbolic metric on the smooth part M_0 of the canonical model M_{can} of M.

2. Proof of Theorem 1.1

Let M be a minimal projective manifold of general type with $\dim_{\mathbb{C}} M = n \geq 2$, M_{can} be the canonical model of M, and $\Phi : M \longrightarrow M_{can}$ be the contraction map. Consider the Kähler-Ricci flow equation on M,

$$\partial_t \omega_t = -Ric(\omega_t) - \omega_t, \tag{2.1}$$

with initial metric ω_0. In [7], the short time existence of the solution of (2.1) is proved. Then, in [11], [10], and [3], it is proved that the solution ω_t of (2.1) exists for all time, i.e. $t \in [0, +\infty)$, and there exists a unique semi-positive current ω_∞ on M which satisfies that:

1. ω_∞ represents $-2\pi c_1(M)$.
2. ω_∞ is a smooth Kähler-Einstein metric with negative scalar curvature on $\Phi^{-1}(M_0)$, where M_0 is the smooth part of M_{can}.
3. On any compact subset $K \subset \Phi^{-1}(M_0)$, ω_t C^∞-converges to ω_∞ when $t \longrightarrow \infty$.

In [15], it is shown that there is a constant $C > 0$ depending only on ω_0 such that

$$|R_t| < C, \tag{2.2}$$

where R_t is the scalar curvature of ω_t.

First, we need evolution equations for volume forms and scalar curvatures as follows:

$$\partial_t \omega_t^n = -(R_t + n)\omega_t^n \tag{2.3}$$

and

$$\partial_t R_t = \triangle_t R_t + |Ric_t|^2 + R_t = \triangle_t R_t + |Ric_t{}^{\mathrm{o}}|^2 - (R_t + n), \tag{2.4}$$

where $Ric_t{}^{\mathrm{o}} = Ric_t + \omega_t$ and $|Ric_t{}^{\mathrm{o}}|^2 = |Ric_t|^2 + 2R_t + n$ (cf. Lemma 2.38 in [5]).

Lemma 2.1. *There are two constants $t_0 > 0$ and $c > 0$ independent of t such that, for $t > t_0$,*

$$\breve{R}_t = \inf_{x \in M} R_t(x) \le -n + e^{-t}c < -\frac{n}{2} < 0.$$

Proof. If we define $\alpha_t = [\omega_t] \in H^{1,1}(M, \mathbb{R})$, from (2.1) we have

$$\partial_t \alpha_t = -2\pi c_1(M) - \alpha_t$$

and

$$\alpha_t = -2\pi c_1(M) + e^{-t}(2\pi c_1(M) + \alpha_0). \tag{2.5}$$

Thus

$$[\omega_\infty] = \alpha_\infty = \lim_{t \longrightarrow \infty} \alpha_t = -2\pi c_1(M). \tag{2.6}$$

Since

$$\breve{R}_t \int_M \omega_t^n \le \int_M R_t \omega_t^n = n \int_M Ric_t \wedge \omega_t^{n-1} = n 2\pi c_1(M) \cdot \alpha_t^{n-1},$$

we obtain

$$\begin{aligned} \breve{R}_t \le n \frac{2\pi c_1(M) \cdot \alpha_t^{n-1}}{\alpha_t^n} &= n \frac{2\pi c_1(M) \cdot \alpha_t^{n-1}}{-2\pi c_1(M) \cdot \alpha_t^{n-1} + e^{-t}(2\pi c_1(M) + \alpha_0) \cdot \alpha_t^{n-1}} \\ &= \frac{-n}{1 + e^{-t} A_t}, \end{aligned}$$

where $A_t = -\frac{(2\pi c_1(M) + \alpha_0) \cdot \alpha_t^{n-1}}{2\pi c_1(M) \cdot \alpha_t^{n-1}}$. Note that $(-c_1(M))^n > 0$. Thus there is a $t_1 > 0$ such that if $t > t_1$, $A_t < |\frac{(\alpha_\infty + \alpha_0) \cdot \alpha_\infty^{n-1}}{\alpha_\infty^n}| + 1 = A$, and we obtain that

$$\breve{R}_t \le \frac{-n}{1 + e^{-t} A} < -n + e^{-t} c,$$

where $c = -n(\frac{A}{1 + e^{-t_1} A})$. By taking $t_0 > t_1$ such that $e^{-t_0} c < \frac{n}{2}$, we obtain the conclusion. $\square$

Lemma 2.2.

$$\int_0^\infty \int_M |R_t + n| \omega_t^n dt < \infty.$$

Proof. By (2.4) and the maximal principle, $\partial_t \breve{R}_t \ge -(\breve{R}_t + n)$, and so

$$n + \breve{R}_t \ge C e^{-t}, \tag{2.7}$$

for a constant C independent of t. Note that by Lemma 2.1, (2.7) and (2.5), when $t > t_0$,

$$\begin{aligned}\int_M |R_t + n|\omega_t^n &\leq \int_M (R_t - \breve{R}_t)\omega_t^n + \int_M |n + \breve{R}_t|\omega_t^n \\ &\leq \int_M (R_t + n)\omega_t^n + 2\int_M |n + \breve{R}_t|\omega_t^n \\ &\leq \int_M (R_t + n)\omega_t^n + C_3 e^{-t} \\ &= n(2\pi c_1 \cdot \alpha_t^{n-1} + \alpha_t^n) + C_3 e^{-t} \\ &= ne^{-t}(2\pi c_1 + \alpha_0) \cdot \alpha_t^{n-1} + C_3 e^{-t} \\ &\leq C_4 e^{-t}\end{aligned}$$

for two constants C_3 and C_4 independent of t. Thus

$$\int_0^\infty \int_M |R_t + n|\omega_t^n dt = \int_0^{t_0} \int_M |R_t + n|\omega_t^n dt + \int_{t_0}^\infty \int_M |R_t + n|\omega_t^n dt < \infty.$$

□

Proof of Theorem 1.1. From (2.4), (2.3), (2.6), (2.2), and Lemma 2.2, we obtain

$$\begin{aligned}\int_0^\infty \int_M |Ric^{\circ}{}_t|^2 \omega_t^n dt &= \int_0^\infty \int_M (\frac{\partial}{\partial t} R_t)\omega_t^n dt + \int_0^\infty \int_M (R_t + n)\omega_t^n dt \\ &= \int_0^\infty \frac{\partial}{\partial t}(\int_M R_t \omega_t^n) dt + \int_0^\infty \int_M (R_t + 1)(R_t + n)\omega_t^n dt \\ &\leq n\alpha_\infty^n - \int_M R_0 \omega_0^n + C\int_0^\infty \int_M |R_t + n|\omega_t^n dt \\ &< \infty.\end{aligned}$$

If $\rho_t = Ric_t - \frac{R_t}{n}\omega_t$ is the Einstein tensor of ω_t, then $|\rho_t|^2 = |Ric^{\circ}{}_t|^2 - \frac{1}{n}(R_t + n)^2$, and from the above estimation,

$$\int_0^\infty \int_M |\rho_t|^2 \omega_t^n dt \leq \int_0^\infty \int_M |Ric^{\circ}{}_t|^2 \omega_t^n dt < \infty.$$

Thus there is a sequence $t_k \longrightarrow \infty$ such that

$$\lim_{k \longrightarrow \infty} \int_M |\rho_{t_k}|^2 \omega_{t_k}^n = 0.$$

□

Proof of Corollary 1.2. Note that the Kähler curvature tensor has a decomposition

$$Rm_t = \frac{R_t}{2n^2}\omega_t \otimes \omega_t + \frac{1}{n}\omega_t \otimes \rho_t + \frac{1}{n}\rho_t \otimes \omega_t + B_t$$

(cf. (2.63) and (2.38) in [2]). By Chern-Weil theory,

$$(\frac{2(n+1)}{n}c_2(M) - c_1^2(M)) \cdot [\omega_t]^{n-2} = \frac{(n-2)!}{4\pi^2 n!}\int_M (\frac{n+1}{n}|B_{0,t}|^2 - \frac{(n^2-2)}{n^2}|\rho_t|^2)\omega_t^n$$

(cf. (2.82a) and (2.67) in [2]), where $B_{0,t} = B_t - \frac{\mathrm{tr} B_t}{n^2-1}\mathrm{Id}$ is the tensor given by (2.64) in [2] corresponding to ω_t. By Theorem 1.1, there is a sequence $t_k \longrightarrow \infty$ such that

$$\lim_{k \longrightarrow \infty} \int_M |\rho_{t_k}|^2 \omega_{t_k}^n = 0.$$

Hence

$$
\begin{aligned}
&(\frac{2(n+1)}{n}c_2(M)-c_1^2(M))\cdot(-2\pi c_1(M))^{n-2}\\
&\quad=(\frac{2(n+1)}{n}c_2(M)-c_1^2(M))\cdot[\omega_\infty]^{n-2}\\
&\quad=\lim_{k\longrightarrow\infty}(\frac{2(n+1)}{n}c_2(M)-c_1^2(M))\cdot[\omega_{t_k}]^{n-2}\\
&\quad=\lim_{k\longrightarrow\infty}\frac{(n-2)!}{4\pi^2 n!}\int_M(\frac{n+1}{n}|B_{0,t_k}|^2)\omega_{t_k}^n\\
&\quad\geq 0.
\end{aligned}
$$

If the equality holds, on any compact subset $K\subset\Phi^{-1}(M_0)$,

$$\int_K |B_{0,\infty}|^2\omega_\infty^n\leq\lim_{k\longrightarrow\infty}\int_M|B_{0,t_k}|^2\omega_{t_k}^n=0,$$

by the smooth convergence of ω_t to ω_∞. Thus $B_{0,\infty}\equiv 0$. Since ω_∞ is a Kähler-Einstein metric with negative scalar curvature on $\Phi^{-1}(M_0)$, the holomorphic sectional curvature is a negative constant by Section 2.66 in [2]; i.e. ω_∞ is a complex hyperbolic metric. □

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References

1. T. Aubin, *Équations du type Monge-Ampère sur les variétés kähleriennes compactes.* C. R. Math. Acad. Sci. Paris Sér. A-B, **283** (1976), A119-A121. MR0433520 (55:6496)
2. A. L. Besse, *Einstein manifolds.* Ergebnisse der Math., Springer-Verlag, Berlin-New York, 1987. MR867684 (88f:53087)
3. P. Cascini, P. La Nave, *Kähler-Ricci flow and the minimal model program for projective varieties.* arXiv:math/0603064.
4. S.Y. Cheng, S.-T. Yau, *Inequality between Chern numbers of singular Kähler surfaces and characterization of orbit space of discrete group of* SU(2, 1). Contemporary Math., **49**, Amer. Math. Soc., Providence, RI, 1986, 31-44. MR833802 (87m:53078)
5. B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, L. Ni, *The Ricci Flow: Techniques and Applications, Part I: Geometric Aspects.* Mathematical Surveys and Monographs, Vol. 135, Amer. Math. Soc., Providence, RI, 2007. MR2302600 (2008f:53088)
6. F. Fang, Y. Zhang, Z. Zhang, *Non-singular solutions to the normalized Ricci flow equation.* Math. Ann., **340** (2008), 647-674. MR2357999 (2009a:53112)
7. R. Hamilton, *Three-manifolds with positive Ricci curvature.* J. Diff. Geom., **17** (1982), 255-306. MR664497 (84a:53050)
8. R. Kobayashi, *Einstein-Kähler V-metrics on open Satake V-surfaces with isolated quotient singularities.* Math. Ann., **272**, no. 3 (1985), 385-398. MR799669 (87b:32050)
9. Y. Miyaoka, *On the Chern numbers of surfaces of general type.* Invent. Math., **42** (1977), 225-237. MR0460343 (57:337)
10. G. Tian, Z. Zhang, *On the Kähler-Ricci flow on projective manifolds of general type.* Chinese Annals of Mathematics Ser. B, **27** (2006), 179-192. MR2243679 (2007c:32029)
11. H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type.* Math. Ann., **281** (1988), 123-133. MR944606 (89e:53075)
12. H. Tsuji, *Stability of tangent bundles of minimal algebraic varieties*, Topology, **27**, no. 4 (1988), 429-442. MR0976585 (90d:14021)

13. S.T. Yau, *Calabi's conjecture and some new results in algebraic geometry.* Proc. Natl. Acad. Sci. USA, **74** (1977), 1798-1799. MR0451180 (56:9467)
14. S.T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.* Comm. Pure Appl. Math., **31** (1978), 339-411. MR480350 (81d:53045)
15. Z. Zhang, *Scalar Curvature Bound for Kähler-Ricci Flows over Minimal Manifolds of General Type.* arXiv:math/0801.32481.

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