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ERRATUM TO: COMPACTNESS PROPERTIES FOR OPERATORS DOMINATED BY AM-COMPACT OPERATORS

BELMESNAOUI AQZZOUZ AND AZIZ ELBOUR

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In [3, Theorem 2.10] it was proposed to characterize Banach lattices such that operators dominated by AM-compact operators are AM-compact. But there was an error in the proof of the above-mentioned theorem. The purpose of this erratum is to give a new and correct proof of Theorem 2.10 of [3]. Let us recall that if E is a Banach lattice, E' is its topological dual and $\varphi \in E'$, the null ideal of φ is defined by $N_\varphi = \{x \in E : |\varphi|(|x|) = 0\}$ and the carrier C_φ of $\varphi \in E'$ is defined by $C_\varphi = (N_\varphi)^d = \{u \in E : |u| \wedge |v| = 0 \text{ for all } v \in N_\varphi\}$.

To give our proof, we will need the following lemma:

Lemma 1. *Let E be a Banach lattice. If the norm of E is not order continuous, then there exist $y \in E^+$ and a disjoint sequence $(y_n) \subset [0, y]$ such that $\|y_n\| = 1$ for all n. Moreover, there exists a positive disjoint sequence (g_n) of E' with $\|g_n\| \leq 1$ such that $g_n(y_n) = 1$ for all n and $g_n(y_m) = 0$ for $n \neq m$.*

Proof. If the norm of E is not order continuous, then Theorem 4.14 of [2] implies the existence of some $u \in E^+$ and a disjoint sequence (u_n) in $[0, u]$ which does not converge to zero in norm. By choosing a subsequence we may suppose that $\|u_n\| > \varepsilon$ for all n and some $\varepsilon > 0$. If we take $y_n = \frac{u_n}{\|u_n\|}$ and $y = \frac{u}{\varepsilon}$, we obtain a disjoint sequence (y_n) in $[0, y]$ satisfying $\|y_n\| = 1$ for all n.

On the other hand, by Theorem 39.3 of [6], for each n there exists $f_n \in (E')^+$ such that $\|f_n\| = 1$ and $f_n(y_n) = \|y_n\| = 1$. Under the natural embedding of E into its topological bidual E'', the space E becomes a sublattice of $(E')'_n$. This implies that (y_n) is a disjoint sequence of positive order continuous functionals on E'. Now, it follows from Nakano's Theorem [2, Theorem 1.67] that the carriers C_{y_n} are mutually disjoint bands in E'. If g_n is the projection of f_n onto C_{y_n}, then it is easy to verify that the sequence (g_n) satisfies the desired properties.

Also, we shall need the following characterisation, which follows from Theorem 3.27 of [2].

Lemma 2. *Let E be a Banach lattice and X a Banach space, and let $T : E \to X$ be an operator. Then T is AM-compact if and only if $T'(B_{X'})$ is precompact for the topology $|\sigma|(E', E)$, where $B_{X'}$ is the closed unit ball of X'.*

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Let E be a Banach lattice and let $u \in E^+$. Then the order ideal E_u generated by u with the norm $\|y\|_\infty = \inf\{\lambda > 0 : |y| \leq \lambda.u\}$ is an AM-space having u as a unit and $[-u, u]$ as a closed unit ball, and the embedding $i_u : (E_u, \|.\|_\infty) \to E$ is continuous. Moreover, for every $f \in E'$ we have $f \circ i_u \in (E_u)'$ and

$$\begin{aligned} \|f \circ i_u\|_{(E_u)'} &= \sup\{|(f \circ i_u)(y)| : y \in [-u, u]\} \\ &= \sup\{|f(y)| : |y| \leq u\} \\ &= |f|(u). \end{aligned}$$

Now we are in a position to give the correct proof of Theorem 2.10 of [3].

Theorem 1. *Let E and F be two Banach lattices. Then the following statements are equivalent:*

(1) *For all operators $S, T : E \to F$ such that $0 \leq S \leq T$ and T is AM-compact, the operator S is AM-compact.*
(2) *One of the following conditions holds:*
  (a) *the norm of F is order continuous;*
  (b) *E' is discrete.*

Proof. $(2.a) \Rightarrow (1)$ This is just a result of Fremlin (see [5, Proposition 3.7.2] for the proof).
$(2.b) \Rightarrow (1)$ By Lemma 2, it suffices to show that $S'(B_{F'})$ is precompact for $|\sigma|(E', E)$. Let V be a solid neighborhood of zero for $|\sigma|(E', E)$. Since T is AM-compact, it follows from Lemma 2 that $T'(B_{F'})$ is precompact for $|\sigma|(E', E)$. Then there exists a finite subset K of E' such that $T'(B_{F'}) \subset K + V$. Choose $0 \leq f \in E'$ such that $K \subset [-f, f]$ and note that

$$|S'(g)| \leq S'(|g|) \leq T'(|g|) \in [-f, f] + V \quad \forall g \in B_{F'}.$$

Then

$$(*) \qquad S'(B_{F'}) \subset [-f, f] + V.$$

Since E' is discrete and order complete and the topology $|\sigma|(E', E)$ is Lebesgue, it follows from Corollary 6.57 of [1] that $[-f, f]$ is compact for $|\sigma|(E', E)$. Finally, by (*) we see that $S'(B_{F'})$ is precompact for $|\sigma|(E', E)$.

$(1) \Rightarrow (2)$ Assume by way of contradiction that the condititons (a) and (b) fail. To finish the proof, we have to construct two operators $S, T : E \to F$ such that T is AM-compact, S is not AM-compact and $0 \leq S \leq T$.

Since F does not have an order continuous norm, it follows from Lemma 1 that there exists $y \in F^+$ and there is a disjoint sequence $(y_n) \subset [0, y]$ such that $\|y_n\| = 1$ for each n and there exists a positive disjoint sequence (g_n) of E' with $\|g_n\| \leq 1$ such that $g_n(y_n) = 1$ for all n and $g_n(y_m) = 0$ for $n \neq m$. (**)

On the other hand, as E' is not discrete, Theorem 3.1 of [4] implies the existence of a sequence $(f_n) \subset E'$ such that $f_n \to 0$ for $\sigma(E', E)$ as $n \to \infty$ and $|f_n| = f > 0$ for all n and some $f \in E'$.

Now, we consider the operators $S, T : E \to F$ defined by

$$S(x) = \left(\sum_{n=1}^{\infty} f_n(x) y_n \right) + f(x) y \quad \text{and} \quad T(x) = 2f(x) y \quad \forall x \in E.$$

The sum in the definition of S is norm convergent for each $x \in E$, because $f_n(x) \to 0$ and the sequence (y_n) is disjoint and order bounded.

Clearly, $0 \le S \le T$ holds. (In fact, for each $x \in E^+$ and each $n \ge 1$, we have

$$\left|\sum_{k=1}^{n} f_k(x) y_k\right| \le \sum_{k=1}^{n} f(x) y_k \le f(x) y.$$

Then $|\sum_{n=1}^{\infty} f_n(x) y_n| \le f(x)y$ for each $x \in E^+$. Hence $0 \le S(x) \le T(x)$ for each $x \in E^+$.) Also, it is clear that T is compact (it has rank one) and hence T is AM-compact.

To end the proof, we need to prove that S is not AM-compact. Choose $u \in E^+$ such that $f(u) > 0$, and note that $(f_n \circ i_u)_n$ has no norm convergent subsequence in $(E_u)'$. In fact, for each $y \in E_u$ we have $f_n \circ i_u(y) = f_n(y) \to 0$ as $n \to \infty$. Then $f_n \circ i_u \to 0$ for $\sigma((E_u)', E_u)$. As $\|f_n \circ i_u\|_{(E_u)'} = |f_n|(u) = f(u) > 0$ for all n, we conclude that $(f_n \circ i_u)_n$ has no norm convergent subsequence in $(E_u)'$.

If S is AM-compact, then $S \circ i_u : E_u \to E \to F$ is compact and so is $(S \circ i_u)'$. We have $(S \circ i_u)'(g) = (\sum_{n=1}^{\infty} g(y_n).(f_n \circ i_u)) + g(y).(f \circ i_u)$ for all $g \in F'$. Then, by (**), $(S \circ i_u)'(g_k) = (f_k \circ i_u) + g_k(y).(f \circ i_u)$ for all k. Hence $\left((S \circ i_u)'(g_k)\right)_k$ has a norm convergent subsequence in $(E_u)'$. Since $(g_k(y))_k \subset [-\|y\|, \|y\|] \subset \mathbb{R}$ has a convergent subsequence (because it is a bounded sequence in $\mathbb{R}$), we conclude that $(f_k \circ i_u)_k$ has a convergent subsequence in $(E_u)'$. This is a contradiction, and then S is not AM-compact.

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Département d'Économie, Faculté des Sciences Économiques, Juridiques et Sociales, Université Mohammed V-Souissi, B.P. 5295, Sala Eljadida, Morocco

E-mail address: baqzzouz@hotmail.com

Équipe d'Analyse Fonctionnelle, Département de Mathématiques, Faculté des Sciences, Université Ibn Tofail, B.P. 133, Kénitra, Morocco

E-mail address: azizelbour@hotmail.com