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ON THE COMPLETENESS OF GRADIENT RICCI SOLITONS

ZHU-HONG ZHANG

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ABSTRACT. A gradient Ricci soliton is a triple (M, g, f) satisfying $R_{ij}+\nabla_i\nabla_j f = \lambda g_{ij}$ for some real number λ. In this paper, we will show that the completeness of the metric g implies that of the vector field ∇f.

1. Introduction

Definition 1.1. Let (M, g, X) be a smooth Riemannian manifold with X a smooth vector field. We call M a Ricci soliton if $Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g$ for some real number λ. It is called shrinking when $\lambda > 0$, steady when $\lambda = 0$, and expanding when $\lambda < 0$. If (M, g, f) is a smooth Riemannian manifold, where f is a smooth function such that $(M, g, \nabla f)$ is a Ricci soliton, i.e. $R_{ij} + \nabla_i\nabla_j f = \lambda g_{ij}$, we call (M, g, f) a gradient Ricci soliton and f the soliton function.

On the other hand, one has the following definition (see chapter 2 of [3]).

Definition 1.2. Let $(M, g(t), X)$ be a smooth Riemannian manifold with a solution $g(t)$ of the Ricci flow on a time interval (a, b) containing 0, where X is a smooth vector field. We call $(M, g(t), X)$ a self-similar solution if there exist scalars $\sigma(t)$ such that $g(t) = \sigma(t)\varphi_t^*(g_0)$, where the diffeomorphism φ_t is generated by X. If the vector field X comes from a gradient of a smooth function f, then we call $(M, g(t), f)$ a gradient self-similar solution.

It is easy to see that if $(M, g(t), f)$ is a complete gradient self-similar solution, then $(M, g(0), f)$ must be a complete gradient Ricci soliton. Conversely, when (M, g, f) is a complete gradient Ricci soliton and, in addition, the vector field ∇f is complete, it is well known (see for example Theorem 4.1 of [2]) that there is a complete gradient self-similar solution $(M, g(t), f)$, $t \in (a, b)$ (with $0 \in (a, b)$), such that $g(0) = g$. Here we say that a vector field ∇f is complete if it generates a family of diffeomorphisms φ_t of M for $t \in (a, b)$.

So when the vector field is complete, the definitions of gradient Ricci soliton and gradient self-similar solution are equivalent. In the literature, people sometimes confuse the gradient Ricci solitons with the gradient self-similar solutions. Indeed, if the gradient Ricci soliton has bounded curvature, then it is not hard to see that the vector field ∇f is complete. But, in general, the soliton does not have bounded curvature.

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The purpose of this paper is to show that the completeness of the metric g of a gradient Ricci soliton (M, g, f) implies that of the vector field ∇f, even though the soliton does not have bounded curvature. Our main result is the following.

Theorem 1.3. *Let (M, g, f) be a gradient Ricci soliton. Suppose the metric g is complete. Then we have:*
(i) ∇f is complete;
(ii) $R \geq 0$ if the soliton is steady or shrinking;
(iii) $\exists C \geq 0$, such that $R \geq -C$ if the soliton is expanding.

Indeed, we will show that the vector field ∇f grows at most linearly and so it is integrable. Hence Definitions 1.1 and 1.2 are equivalent when the metric is complete.

2. Gradient Ricci solitons

Let (M, g, f) be a gradient Ricci soliton, i.e., $R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}$. By using the contracted second Bianchi identity we get the equation $R + |\nabla f|^2 - 2\lambda f = const.$

Definition 2.1. Let (M, g, f) be a gradient shrinking or expanding soliton. By rescaling g and changing f by a constant we can assume $\lambda \in \{-\frac{1}{2}, \frac{1}{2}\}$ and $R + |\nabla f|^2 - 2\lambda f = 0$. We call such a soliton normalized, and we call f a normalized soliton function.

Proposition 2.2. *Let (M, g, f) be a gradient Ricci soliton. Fix p on M and define $d(x) \triangleq d(p, x)$. Then the following hold:*
(i) $\triangle R = \langle \nabla f, \nabla R \rangle + 2\lambda R - |Ric|^2$.
(ii) Suppose $Ric \leq (n-1)K$ on $B_{r_0}(p)$, for some positive numbers r_0 and K. Then for an arbitrary point x, outside $B_{r_0}(p)$, we have

$$\triangle d - \langle \nabla f, \nabla d \rangle \leq -\lambda d(x) + (n-1)\Big\{\frac{2}{3}K r_0 + r_0^{-1}\Big\} + |\nabla f|(p).$$

Proof. (i) By using the soliton equation and the contracted second Bianchi identity $\nabla_i R = 2g^{jk}\nabla_j R_{ik}$, we have

$$\begin{aligned}
\triangle R &= g^{ij}\nabla_i \nabla_j R = g^{ij}\nabla_i (2g^{kl}R_{jk}\nabla_l f) = 2g^{ij}g^{kl}\nabla_i (R_{jk}\nabla_l f) \\
&= 2g^{ij}g^{kl}\nabla_i (R_{jk})\nabla_l f + 2g^{ij}g^{kl}R_{jk}\nabla_i \nabla_l f \\
&= g^{kl}\nabla_k R \nabla_l f + 2g^{ij}g^{kl}R_{jk}(\lambda g_{il} - R_{il}) \\
&= \langle \nabla f, \nabla R \rangle + 2\lambda R - 2|Ric|^2.
\end{aligned}$$

(ii) Let $\gamma : [0, d(x)] \to M$ be a shortest normal geodesic from p to x. We may assume that x and p are not conjugate to each other; otherwise, we can understand the differential inequality in the barrier sense. Let $\{\dot\gamma(0), e_1, \cdots, e_{n-1}\}$ be an orthonormal basis of T_pM. Extend this basis parallel along γ to form a parallel orthonormal basis $\{\dot\gamma(t), e_1(t), \cdots, e_{n-1}(t)\}$ along γ.

Let $X_i(t)$, $i = 1, 2, \cdots, n-1$, be the Jacobian fields along γ with $X_i(0) = 0$ and $X_i(d(x)) = e_i(d(x))$. Then it is well-known that (see for example [4])

$$\triangle d(x) = \sum_{i=1}^{n-1} \int_0^{d(x)} [|\dot X_i|^2 - R(\dot\gamma, X_i, \dot\gamma, X_i)]dt.$$

Define vector fields Y_i, $i = 1, 2, \cdots, n-1$, along γ as follows:

$$Y_i(t) = \begin{cases} \frac{t}{r_0} e_i(t), & \text{if } t \in [0, r_0]; \\ e_i(t), & \text{if } t \in [r_0, d(x)]. \end{cases}$$

Then by using the standard index comparison theorem we have

$$\begin{aligned}
\triangle d(x) &= \sum_{i=1}^{n-1} \int_0^{d(x)} [|\dot{X}_i|^2 - R(\dot\gamma, X_i, \dot\gamma, X_i)] dt \\
&\le \sum_{i=1}^{n-1} \int_0^{d(x)} [|\dot{Y}_i|^2 - R(\dot\gamma, Y_i, \dot\gamma, Y_i)] dt \\
&= \int_0^{r_0} [\tfrac{n-1}{r_0^2} - \tfrac{t^2}{r_0^2} Ric(\dot\gamma, \dot\gamma)] dt + \int_{r_0}^{d(x)} [-Ric(\dot\gamma, \dot\gamma] dt \\
&= -\int_0^{d(x)} Ric(\dot\gamma, \dot\gamma) dt + \int_0^{r_0} [\tfrac{n-1}{r_0^2} + (1 - \tfrac{t^2}{r_0^2}) Ric(\dot\gamma, \dot\gamma)] dt \\
&\le -\int_\gamma Ric(\dot\gamma, \dot\gamma) dt + (n-1) \Big\{ \tfrac{2}{3} K r_0 + r_0^{-1} \Big\}.
\end{aligned}$$

On the other hand,

$$\langle \nabla f, \nabla d \rangle (x) = \nabla_{\dot\gamma} f(x) = \int_0^{d(x)} (\frac{d}{dt} \nabla_{\dot\gamma} f) dt + \nabla_{\dot\gamma} f(p) \ge \int_\gamma (\nabla_{\dot\gamma} \nabla_{\dot\gamma} f) dt - |\nabla f|(p).$$

Using the soliton equation, we have

$$\begin{aligned}
\triangle d - \langle \nabla f, \nabla d \rangle &\le -\int_\gamma \Big[Ric(\dot\gamma, \dot\gamma) + \nabla_{\dot\gamma} \nabla_{\dot\gamma} f \Big] dt + (n-1) \Big\{ \tfrac{2}{3} K r_0 + r_0^{-1} \Big\} + |\nabla f|(p) \\
&= -\lambda d(x) + (n-1) \Big\{ \tfrac{2}{3} K r_0 + r_0^{-1} \Big\} + |\nabla f|(p).
\end{aligned}$$

□

Now we are ready to prove Theorem 1.3.

Proof. Fix a point p on M, and define $d(x) \triangleq d(p, x)$. We divide the argument into three steps.

Step 1. We want to prove a curvature estimate in the following assertion.

Claim. For any gradient Ricci soliton, we have:
(i) If the soliton is shrinking or steady, then $R \ge 0$.
(ii) If the soliton is expanding, then there exists a nonnegative constant $C = C(n)$ such that $R \ge \lambda C$.

We only prove the case (i), $\lambda \ge 0$. Note that there is a positive constant r_0, such that $Ric \le (n-1) r_0^{-2}$ on $B_{r_0}(p)$, and $|\nabla f|(p) \le (n-1) r_0^{-1}$. Then by Proposition 2.2, we have

$$\triangle d - \langle \nabla f, \nabla d \rangle \le \frac{8}{3} (n-1) r_0^{-1},$$

for any $x \notin B_{r_0}(p)$.

For any fixed constant $A > 2$, we consider the function $u(x) = \varphi(\frac{d(x)}{A r_0}) R(x)$, where φ is a fixed smooth nonnegative decreasing function such that $\varphi = 1$ on $(-\infty, \frac{1}{2}]$, and $\varphi = 0$ on $[1, \infty)$.

Then by Proposition 2.2, we have

$$\begin{aligned}\triangle u &= R\triangle\varphi + \varphi\triangle R + 2\langle\nabla\varphi, \nabla R\rangle \\ &= R(\varphi''\tfrac{1}{(Ar_0)^2} + \varphi'\tfrac{1}{Ar_0}\triangle d) + \varphi(\langle\nabla f, \nabla R\rangle + 2\lambda R - |Ric|^2) + 2\langle\nabla\varphi, \nabla R\rangle .\end{aligned}$$

If $\min\limits_{x\in M} u \geq 0$, then $R \geq 0$ on $B_{\frac{1}{2}Ar_0}(p)$. Otherwise, $\min\limits_{x\in M} u < 0$. Then there exists some point $x_1 \in B_{Ar_0}(p)$ such that $u(x_1) = \varphi R(x_1) = \min\limits_{x\in M} u < 0$. Because $u(x_1)$ is the minimum of the function $u(x)$, we have $\varphi' R(x_1) > 0$, $\nabla u(x_1) = 0$, and $\triangle u(x_1) \geq 0$.

Let us first consider the case that $x_1 \notin B_{r_0}(p)$. Then by direct computation, we have

$$\begin{aligned}\triangle u(x_1) = {} & (\tfrac{\varphi''}{\varphi}\tfrac{1}{(Ar_0)^2} + \tfrac{\varphi'}{\varphi}\tfrac{1}{Ar_0}\triangle d)u(x_1) - \tfrac{\varphi'}{\varphi}\tfrac{1}{Ar_0}\langle\nabla f, \nabla d\rangle u(x_1) \\ & +2\lambda u(x_1) - \varphi|Ric|^2 - \tfrac{\varphi'^2}{\varphi^2}\tfrac{2}{(Ar_0)^2}u(x_1) \\ \leq {} & (\tfrac{\varphi''}{\varphi}\tfrac{1}{(Ar_0)^2} - \tfrac{\varphi'^2}{\varphi^2}\tfrac{2}{(Ar_0)^2})u(x_1) - \tfrac{2}{n}\varphi R^2 \\ & +\tfrac{\varphi'}{\varphi}\tfrac{1}{Ar_0}u(x_1)(\triangle d - \langle\nabla f, \nabla d\rangle). \\ \leq {} & (\tfrac{\varphi''}{\varphi}\tfrac{1}{(Ar_0)^2} - \tfrac{\varphi'^2}{\varphi^2}\tfrac{2}{(Ar_0)^2})u(x_1) - \tfrac{2}{n}\tfrac{1}{\varphi}u(x_1)^2 \\ & +\tfrac{8}{3}(n-1)\tfrac{\varphi'}{\varphi}\tfrac{1}{Ar_0^2}u(x_1) \\ = {} & \tfrac{u(x_1)}{\varphi}\Big\{(\varphi''\tfrac{1}{(Ar_0)^2} - \tfrac{\varphi'^2}{\varphi}\tfrac{2}{(Ar_0)^2}) + \tfrac{8}{3}(n-1)\varphi'\tfrac{1}{Ar_0^2} - \tfrac{2}{n}u(x_1)\Big\} \\ \leq {} & \tfrac{|u(x_1)|}{\varphi}\Big\{\tfrac{\varphi'^2}{\varphi}\tfrac{2}{Ar_0^2} + \tfrac{8(n-1)}{3}(-\varphi')\tfrac{1}{Ar_0^2} + |\varphi''|\tfrac{1}{Ar_0^2} - \tfrac{2}{n}|u(x_1)|\Big\}.\end{aligned}$$

Note that there exists a constant $\widetilde{C} = \widetilde{C}(\varphi)$ such that $|\varphi'| \leq \widetilde{C}$, $\frac{\varphi'^2}{\varphi} \leq \widetilde{C}$, and $|\varphi''| \leq \widetilde{C}$. So

$$|u(x_1)| \leq \frac{C}{Ar_0^2},$$

where the constant $C = C(\varphi, n)$, i.e., $R \geq -\frac{C}{Ar_0^2}$ on $B_{\frac{1}{2}Ar_0}(p)$.

We now consider the remaining case that $x_1 \in B_{r_0}(p)$. Then $\varphi'(x_1) = \varphi''(x_1) = 0$, and we have

$$\triangle u(x_1) = 2\lambda u(x_1) - \varphi|Ric|^2 \leq |u(x_1)|[-2\lambda - \frac{2}{n}|u(x_1)|].$$

Since $\lambda \geq 0$, we have $|u(x_1)| \leq 0$, i.e., $u(x_1) = 0$. This is a contradiction.

Combining the above two cases, we have $R \geq -\frac{C}{Ar_0^2}$ on $B_{\frac{1}{2}Ar_0}(p)$ for any $A > 2$, which implies that $R \geq 0$ on M.

The proof of (ii) is similar.

Step 2. We next want to show that the gradient field grows at most linearly.

Claim. For any gradient Ricci soliton, there exist constants a and b depending only on the soliton, such that

(i) $|\nabla f|(x) \leq |\lambda| d(x) + a$;

(ii) $|f|(x) \leq \frac{|\lambda|}{2} d(x)^2 + ad(x) + b$.

For any point x on M, we connect p and x by a shortest normal geodesic $\gamma(t), t \in [0, d(x)]$.

We first consider that the soliton is steady. Then $R \geq 0$ and $R + |\nabla f|^2 = C \geq 0$, so we have $|\nabla f| \leq \sqrt{C}$.

Secondly, we consider that the soliton is shrinking. Without loss of generality, we may assume the soliton is normalized. So $R \geq 0$ and $R + |\nabla f|^2 - f = 0$; these imply $f \geq |\nabla f|^2$. Let $h(t) = f(\gamma(t))$. Then

$$|h'|(t) = |\langle \nabla f, \dot{\gamma} \rangle|(t) \leq |\nabla f|(\gamma(t)) \leq \sqrt{f(\gamma(t))} = \sqrt{h(t)}.$$

By integrating the above inequality, we get $|\sqrt{h(d(x))} - \sqrt{h(0)}| \leq \frac{1}{2} d(x)$. Thus $|\nabla f|(x) \leq \frac{1}{2} d(x) + \sqrt{f(p)}$.

Finally, we consider that the soliton is expanding. Similarly we only need to show the normalized case. So $R \geq -\frac{C}{2}$ and $R + |\nabla f|^2 + f = 0$. We obtain $-f + \frac{C}{2} \geq |\nabla f|^2$. Let $h(t) = -f(\gamma(t)) + \frac{C}{2}$. Thus

$$|h'|(t) = |\langle \nabla f, \dot{\gamma} \rangle|(t) \leq |\nabla f|(\gamma(t)) \leq \sqrt{h(t)}.$$

By integrating the above inequality, we get $|\sqrt{h(d(x))} - \sqrt{h(0)}| \leq \frac{1}{2} d(x)$. Thus $|\nabla f|(x) \leq \frac{1}{2} d(x) + \sqrt{-f(p) + \frac{C}{2}}$.

Therefore we have proved (i).

The conclusion (ii) follows from (i) immediately.

Step 3. Since the gradient field ∇f grows at most linearly, it must be integrable. Thus we have proved Theorem 1.3.

□

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Department of Mathematics, Sun Yat-sen University, Guangzhou, People's Republic of China 510275

E-mail address: juhoncheung@sina.com