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A MULTIPLICATION FORMULA FOR MODULE SUBCATEGORIES OF EXT-SYMMETRY

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Abstract. We define evaluation forms associated to objects in a module subcategory of Ext-symmetry generated by finitely many simple modules over a path algebra with relations and prove a multiplication formula for the product of two evaluation forms. It is analogous to a multiplication formula for the product of two evaluation forms associated to modules over a preprojective algebra given by Geiss, Leclerc and Schröer in Compositio Math. **143** (2007), 1313–1334.

Introduction

Let Λ be the preprojective algebra associated to a connected quiver without loops (see e.g. [12]) and let $\mathrm{mod}(\Lambda)$ be the category of finite-dimensional nilpotent left Λ-modules. We denote by $\Lambda_{\underline{e}}$ the variety of finite-dimensional nilpotent left Λ-modules with dimension vector $\underline{e}$. For any $x \in \Lambda_{\underline{e}}$, there is an evaluation form δ_x associated to x satisfying that there is a finite subset $R(\underline{e})$ of $\Lambda_{\underline{e}}$ such that $\Lambda_{\underline{e}} = \bigsqcup_{x\in R(\underline{e})}\langle x\rangle$, where $\langle x\rangle := \{y \in \Lambda_{\underline{e}} \mid \delta_x = \delta_y\}$ [7, Section 1.2]. Inspired by the Caldero-Keller cluster multiplication theorem for finite type [4], Geiss, Leclerc and Schröer [7] proved a multiplication formula (the Geiss-Leclerc-Schröer multiplication formula) as follows:

$$\chi(\mathbb{P}\mathrm{Ext}^{\,1}_{\Lambda}(x',x''))\,\delta_{x'\oplus x''} = \sum_{x\in R(\underline{e})} \big(\chi(\mathbb{P}\mathrm{Ext}^{\,1}_{\Lambda}(x',x'')_{\langle x\rangle}) + \chi(\mathbb{P}\mathrm{Ext}^{\,1}_{\Lambda}(x'',x')_{\langle x\rangle})\big)\,\delta_x,$$

where $x' \in \Lambda_{\underline{e}'}$, $x'' \in \Lambda_{\underline{e}''}$, $\underline{e} = \underline{e}' + \underline{e}''$, $\mathbb{P}\mathrm{Ext}^{\,1}_{\Lambda}(x',x'')_{\langle x\rangle}$ is the constructible subset of $\mathbb{P}\mathrm{Ext}^{\,1}_{\Lambda}(x',x'')$ with the middle terms belonging to $\langle x\rangle$, and $\mathbb{P}\mathrm{Ext}^{\,1}_{\Lambda}(x'',x')_{\langle x\rangle}$ is defined similarly.

The proof of the formula depends heavily on the fact that the category $\mathrm{mod}(\Lambda)$ is of Ext-symmetry. A category $\mathcal{C}$ is of Ext-symmetry if there is a bifunctorial isomorphism: $\mathrm{Ext}^1_{\mathcal{C}}(M,N) \cong \mathrm{DExt}^1_{\mathcal{C}}(N,M)$ for any objects $M, N \in \mathcal{C}$.

Let Q be a finite quiver and A be a quotient algebra $\mathbb{C}Q/\mathcal{I}$ by an ideal $\mathcal{I}$. We denote by $\mathrm{mod}(A)$ the category of finite-dimensional left A-modules. We call A an algebra of Ext-symmetry if $\mathrm{mod}(A)$ is of Ext-symmetry. It is proved that preprojective algebras and deformed preprojective algebras are of Ext-symmetry (see [7, Theorem 3] and Section 3 in this paper).

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In this paper, we focus on the module subcategories of Ext-symmetry of $\mathrm{mod}(A)$. Let $\mathcal{S} = \{S_1, \cdots, S_n\}$ be a finite subset of finite-dimensional simple A-modules. We denote by $\mathcal{C}(\mathcal{S})$ the full subcategory of $\mathrm{mod}(A)$ consisting of modules M satisfying that the isomorphism classes of the composition factors of M belong to $\mathcal{S}$. We associate to modules in $\mathcal{C}(\mathcal{S})$ some evaluation forms and prove that if $\mathcal{C}(\mathcal{S})$ is of Ext-symmetry, then the product of two evaluation forms satisfies an identity (Theorem 2.3). The identity is analogous to the Geiss-Leclerc-Schröer multiplication formula. There are no known examples of algebras of Ext-symmetry, apart from preprojective and deformed preprojective algebras (see Section 3), and it is an open question whether further examples exist. However, other examples of module subcategories of Ext-symmetry can be easily constructed, and we give an example in Section 3.

1. The product of two evaluation forms

1.1. **Module varieties.** Let $Q = (Q_0, Q_1, s, t)$ be a finite connected quiver, where Q_0 and Q_1 are the sets of vertices and arrows, respectively, and $s, t : Q_1 \to Q_0$ are maps such that any arrow α starts at $s(\alpha)$ and terminates at $t(\alpha)$. The space spanned by all paths of nonzero length is a graded ideal of $\mathbb{C}Q$, and we will denote it by $\mathcal{J}$. A relation for Q is a linear combination $\sum_{i=1}^{r} \lambda_i p_i$, where $\lambda_i \in \mathbb{C}$ and the p_i are paths with $s(p_i) = s(p_j)$ and $t(p_i) = t(p_j)$ for any $1 \le i, j \le r$. Here if p_i is a vertex in Q_0, then $s(p_i) = t(p_i) = p_i$. Let $A = \mathbb{C}Q/\mathcal{I}$, where $\mathcal{I}$ is an ideal generated by a finite set of relations. We don't assume that $\mathcal{I}$ is admissible, i.e. $\mathcal{I} \subset \mathcal{J}^2$.

A dimension vector for A is a map $\underline{d} : Q_0 \to \mathbb{N}$. We write d_i instead of $d(i)$ for any $i \in Q_0$. For any dimension vector $\underline{d} = (d_i)_{i \in Q_0}$, we consider the affine space over $\mathbb{C}$,

$$\mathbb{E}_{\underline{d}}(Q) = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{s(\alpha)}}, \mathbb{C}^{d_{t(\alpha)}}).$$

Any element $x = (x_\alpha)_{\alpha \in Q_1}$ in $\mathbb{E}_{\underline{d}}(Q)$ defines a representation $(\mathbb{C}^{\underline{d}}, x)$, where $\mathbb{C}^{\underline{d}} = \bigoplus_{i \in Q_0} \mathbb{C}^{d_i}$. For any $x = (x_\alpha)_{\alpha \in Q_1} \in \mathbb{E}_{\underline{d}}(Q)$ and any path $p = \alpha_1 \alpha_2 \cdots \alpha_m$ in Q, we set $x_p = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_m}$. Then x satisfies a relation $\sum_{i=1}^{r} \lambda_i p_i$ if $\sum_{i=1}^{r} \lambda_i x_{p_i} = 0$. Here if p_i is a vertex in Q_0, then x_{p_i} is the identity matrix. Let R be a finite set of relations generating the ideal $\mathcal{I}$. Then we denote by $\mathbb{E}_{\underline{d}}(A)$ the closed subvariety of $\mathbb{E}_{\underline{d}}(Q)$ which consists of elements satisfying all relations in R.

Let $\mathcal{S} = \{S_1, \cdots, S_n\}$ be a finite subset of finite-dimensional simple A-modules and $\mathcal{C}(\mathcal{S})$ be a module subcategory of Ext-symmetry of $\mathrm{mod}(A)$. We denote by $A_{\underline{d}}(\mathcal{S})$ the constructible subset of $\mathbb{E}_{\underline{d}}(A)$ consisting of modules in $\mathcal{C}(\mathcal{S})$. In the sequel, we will fix the finite set $\mathcal{S}$ and write $A_{\underline{d}}$ instead of $A_{\underline{d}}(\mathcal{S})$. The algebraic group $G_{\underline{d}} := G_{\underline{d}}(Q) = \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\mathbb{C})$ acts on $\mathbb{E}_{\underline{d}}(Q)$ by $(x_\alpha)^g_{\alpha \in Q_1} = (g_{t(\alpha)} x_\alpha g^{-1}_{s(\alpha)})_{\alpha \in Q_1}$ for $g \in G_{\underline{d}}$ and $(x_\alpha)_{\alpha \in Q_1} \in \mathbb{E}_{\underline{d}}(Q)$. It naturally induces the action of $G_{\underline{d}}$ on $A_{\underline{d}}(\mathcal{S})$. The orbit space is denoted by $\overline{A}_{\underline{d}}(\mathcal{S})$. A constructible function over $\mathbb{E}_{\underline{d}}(A)$ is a function $f : \mathbb{E}_{\underline{d}}(A) \to \mathbb{C}$ such that $f(\mathbb{E}_{\underline{d}}(A))$ is a finite subset of $\mathbb{C}$ and $f^{-1}(c)$ is a constructible subset of $\mathbb{E}_{\underline{d}}(A)$ for any $c \in Q$.

Throughout this paper, we always assume that $\mathcal{C}(\mathcal{S})$ is of Ext-symmetry and that constructible functions over $\mathbb{E}_{\underline{d}}(A)$ are $G_{\underline{d}}$-invariant for any dimension vector $\underline{d}$ unless particularly stated.

1.2. **Euler characteristics.** Let χ denote the Euler characteristic in compactly supported cohomology. Let X be a complex algebraic variety and $\mathcal{O}$ a constructible subset as the disjoint union of finitely many locally closed subsets X_i for $i = 1, \cdots, m$. Define $\chi(\mathcal{O}) = \sum_{i=1}^{m} \chi(X_i)$. We note that it is well-defined. The following properties will be applied to compute Euler characteristics.

Proposition 1.1 ([11] and [9]). *Let X, Y be algebraic varieties over $\mathbb{C}$. Then*

(1) *If an algebraic variety X is the disjoint union of finitely many constructible sets $X_1, \cdots, X_r$, then*

$$\chi(X) = \sum_{i=1}^{r} \chi(X_i).$$

(2) *If $\varphi : X \longrightarrow Y$ is a morphism with the property that all fibers have the same Euler characteristic χ, then $\chi(X) = \chi \cdot \chi(Y)$. In particular, if φ is a locally trivial fibration in the analytic topology with fibre F, then $\chi(Z) = \chi(F) \cdot \chi(Y)$.*

(3) *$\chi(\mathbb{C}^n) = 1$ and $\chi(\mathbb{P}^n) = n + 1$ for all $n \geq 0$.*

We recall the *pushforward* functor from the category of algebraic varieties over $\mathbb{C}$ and the category of $\mathbb{C}$-vector spaces (see [10] and [9]). Let $\phi : X \to Y$ be a morphism of varieties. Write $M(X)$ for the $\mathbb{C}$-vector space of constructible functions on X. For $f \in M(X)$ and $y \in Y$, define

$$\phi_*(f)(y) = \sum_{c \neq 0} c\chi(f^{-1}(c) \cap \phi^{-1}(y)).$$

Theorem 1.2 ([5],[9]). *Let X, Y and Z be algebraic varieties over $\mathbb{C}$, $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms of varieties, and $f \in M(X)$. Then $\phi_*(f)$ is constructible, $\phi_* : M(X) \to M(Y)$ is a $\mathbb{C}$-linear map and $(\psi \circ \phi)_* = (\psi)_* \circ (\phi)_*$ as $\mathbb{C}$-linear maps from $M(X)$ to $M(Z)$.*

1.3. **The actions of $\mathbb{C}^*$ on the extensions and flags.** Let $A = \mathbb{C}Q/\langle R \rangle$ be an algebra as in Section 1.1. For any A-modules X, Y, let $D(X, Y)$ be the vector space over $\mathbb{C}$ of all tuples $d = (d(\alpha))_{\alpha \in Q_1}$ such that linear maps $d(\alpha) \in \mathrm{Hom}_{\mathbb{C}}(X_{s(\alpha)}, Y_{t(\alpha)})$ and the matrices $L(d)_\alpha = \begin{pmatrix} Y_\alpha & d(\alpha) \\ 0 & X_\alpha \end{pmatrix}$ satisfy the relations in R. Define $\pi : D(X, Y) \to \mathrm{Ext}^1(X, Y)$ by sending d to the equivalence class of the following short exact sequence:

$$\varepsilon : \quad 0 \longrightarrow Y \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} L(d) \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} X \longrightarrow 0\,,$$

where, as a vector space, $L(d) = (L(d)_\alpha)_{\alpha \in Q_1}$ is the direct sum of Y and X. The direct computation shows that $\mathrm{Ker}\pi$ is the subspace of $D(X, Y)$ consisting of all tuples $d = (d(\alpha))_{\alpha \in Q_1}$ such that there exist $(\phi_i)_{i \in Q_0} \in \bigoplus_{i \in Q_0} \mathrm{Hom}_{\mathbb{C}}(X_i, Y_i)$ satisfying $d(\alpha) = \phi_{t(\alpha)} X_\alpha - Y_\alpha \phi_{s(\alpha)}$ for all $\alpha \in Q_1$ (see [7, Section 5.1] for a similar discussion).

Fix a vector space decomposition $D(X, Y) = \mathrm{Ker}\pi \oplus E(X, Y)$. We can identify $\mathrm{Ext}_A^1(X, Y)$ with $E(X, Y)$ ([11], [6], [7]). Let $\mathrm{Ext}_A^1(X, Y)_L$ be the subset of $\mathrm{Ext}_A^1(X, Y)$ with the middle term isomorphic to L. Then $\mathrm{Ext}^1(X, Y)_L$ can be viewed as a constructible subset of $\mathrm{Ext}_A^1(X, Y)$ by the identification between

$\mathrm{Ext}^1_A(X,Y)$ and $E(X,Y)$. There is a natural $\mathbb{C}^*$-action on $E(X,Y)\setminus\{0\}$ by $t.d = (td(\alpha))$ for any $t\in\mathbb{C}^*$. This induces the action of $\mathbb{C}^*$ on $\mathrm{Ext}^1_A(X,Y)\setminus\{0\}$. For any $t\in\mathbb{C}^*$, we have that $t.\varepsilon$ is the following short exact sequence:

$$0 \longrightarrow Y \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} L(t.d) \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} X \longrightarrow 0,$$

where $L(t.d)_\alpha = \begin{pmatrix} Y_\alpha & td(\alpha)\\ 0 & X_\alpha\end{pmatrix}$ for any $\alpha\in Q_1$. The orbit space is denoted by $\mathbb{P}\mathrm{Ext}^1_A(X,Y)$ and the orbit of $\varepsilon\in\mathrm{Ext}^1_A(X,Y)$ is denoted by $\mathbb{P}\varepsilon$. For a $G_{\underline{d}}$-invariant constructible subset $\mathcal{O}$ of $\mathbb{E}_{\underline{d}}(A)$, we set $\mathrm{Ext}^1_A(X,Y)_{\mathcal{O}}$ to be the subset of $\mathrm{Ext}^1_A(X,Y)$ consisting of the equivalence classes of extensions with middle terms belonging to $\mathcal{O}$.

The above $\mathbb{C}^*$-action on the extensions induces an action on the middle terms. As a vector space, $L = Y\oplus X$. So we can define $t.(y,x) = (ty,x)$ for any $t\in\mathbb{C}^*$ and $x\in X, y\in Y$ [7, Section 5.4] or [11, Lemma 1]. For any $L_1\subseteq L$, this action yields a submodule $t.L_1$ of L isomorphic to L_1. In general, if $\mathfrak{f}_L = (L\supseteq L_1\supseteq L_2\supseteq\cdots\supseteq L_m = 0)$ is a flag of submodules of L, then $t.\mathfrak{f}_L = (L\supseteq t.L_1\supseteq t.L_2\supseteq\cdots\supseteq t.L_m = 0)$. Hence, we obtain an action of $\mathbb{C}^*$ on the flag of L.

1.4. **The product of two evaluation forms.** Let $A_{\underline{d}} := A_{\underline{d}}(\mathcal{S})$ be the constructible subset of $\mathbb{E}_{\underline{d}}(A)$ as in Section 1.1. For any module $M\in\mathbb{E}_{\underline{d}}(A)$, let $Gr_{\underline{e}}(M)$ be the subvariety of $Gr_{\underline{e}}(\mathbb{C}^{\underline{d}}) := \prod_{i\in Q_0} Gr_{e_i}(\mathbb{C}^{d_i})$ consisting of submodules of M with dimension vector $\underline{e} = (e_i)_{i\in Q_0}$, and let $Gr_{\underline{e}}(\mathbb{E}_{\underline{d}}(A))$ be the constructible subset of $\mathbb{E}_{\underline{d}}(A)\times Gr_{\underline{e}}(\mathbb{C}^{\underline{d}})$ consisting of pairs (M,M_1) such that $M_1\in Gr_{\underline{e}}(M)$.

Proposition 1.3. *Let $\underline{d}$ and $\underline{e}$ be two dimension vectors. Then the function $gr(\underline{e},\underline{d}) : \mathbb{E}_{\underline{d}}(A)\to\mathbb{C}$ sending M to $\chi(Gr_{\underline{e}}(M))$ is a $G_{\underline{d}}$-invariant constructible function.*

Proof. Consider the projection: $\phi : Gr_{\underline{e}}(\mathbb{E}_{\underline{d}}(A))\to\mathbb{E}_{\underline{d}}(A)$ mapping (M,M_1) to M. It is clear that ϕ is a morphism of varieties. By Theorem 1.2, $gr(\underline{e},\underline{d}) = \phi_*(1_{Gr_{\underline{e}}(\mathbb{E}_{\underline{d}}(A))})$ is constructible. □

For fixed $\underline{d}$, we can make finitely many choices of $\underline{e}$ such that $Gr_{\underline{e}}(\mathbb{E}_{\underline{d}}(A))$ is nonempty. This implies the following corollary.

Corollary 1.4. *There is a finite subset $S(\underline{d})$ of $A_{\underline{d}}$ such that $A_{\underline{d}} = \bigcup_{i\in S(\underline{d})}\mathcal{O}(\underline{d})_i$, where all $\mathcal{O}(\underline{d})_i$ are constructible subsets of $A_{\underline{d}}$ satisfying that for any $M,M'\in\mathcal{O}(\underline{d})_i$, $\chi(Gr_{\underline{e}}(M)) = \chi(Gr_{\underline{e}}(M'))$ for any $\underline{e}$.*

Let $\mathcal{M}(\underline{d})$ be the vector space over $\mathbb{C}$ spanned by the constructible functions $gr(\underline{e},\underline{d})$ for any dimension vector $\underline{e}$. For any $M\in A_{\underline{d}}$, we define the evaluation form $\delta_M : \mathcal{M}(\underline{d})\to\mathbb{C}$ which maps the constructible function $gr(\underline{e})$ to $\chi(Gr_{\underline{e}}(M)) = gr(\underline{e})(M)$. Using the notation in [7], we set $\langle L\rangle := \mathcal{O}(\underline{d})_i$ for arbitrary $L\in\mathcal{O}(\underline{d})_i$. Indeed, $\delta_L = \delta_{L'}$ for any $L,L'\in\mathcal{O}(\underline{d})_i$. By abuse of notation, we have $A_{\underline{d}} = \bigcup_{L\in S(\underline{d})}\langle L\rangle$.

Let M,N be A-modules and $\underline{e}_1,\underline{e}_2$ be dimension vectors. Fixing $M_1\in Gr_{\underline{e}_1}(M)$, $N_1\in Gr_{\underline{e}_2}(N)$, we consider the natural map

$$\beta_{N_1,M_1} : \mathrm{Ext}^1_A(N,M_1)\to\mathrm{Ext}^1_A(N,M)\oplus\mathrm{Ext}^1_A(N_1,M_1)$$

mapping $\varepsilon_* \in \operatorname{Ext}^1_A(N, M_1)$ to $(\varepsilon, \varepsilon')$ such that the following diagram commutes:

$$\begin{tikzcd}
\varepsilon' : & 0 \arrow[r] & M_1 \arrow[r] \arrow[d, equal] & L'' \arrow[r] \arrow[d] & N_1 \arrow[r] \arrow[d] & 0 \\
\varepsilon_* : & 0 \arrow[r] & M_1 \arrow[r] \arrow[d] & L' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
\varepsilon : & 0 \arrow[r] & M \arrow[r, "i"] & L \arrow[r, "\pi"] & N \arrow[r] & 0
\end{tikzcd}$$

where L and L'' are the pushout and pullback, respectively. Define

$$EF^g_{\underline{e}_1,\underline{e}_2}(N, M) = \{(M_1, N_1, \varepsilon, L_1) \mid M_1 \in Gr_{\underline{e}_1}(M), N_1 \in Gr_{\underline{e}_2}(N),\ \varepsilon \neq 0 \in \operatorname{Ext}^1_A(N, M)_L \cap \operatorname{Im}\beta_{N_1,M_1}, L_1 \in Gr_{\underline{e}_1+\underline{e}_2}(L), L_1 \cap i(M) = i(M_1), \pi(L_1) = N_1\}$$

and $EF^g_{\underline{e}}(N, M) = \bigcup_{\underline{e}_1+\underline{e}_2=\underline{e}} EF^g_{\underline{e}_1,\underline{e}_2}(N, M)$. By the discussion in Section 1.3, the action of $\mathbb{C}^*$ on $\operatorname{Ext}^1_A(N, M) \setminus \{0\}$ naturally induces the action on $EF^g_{\underline{e}}(N, M)$ by setting

$$t.(M_1, N_1, \varepsilon, L_1) = (M_1, N_1, t.\varepsilon, t.L_1)$$

for $(M_1, N_1, \varepsilon, L_1) \in EF^g_{\underline{e}}(N, M)$ and $t \in \mathbb{C}^*$. We denote its orbit space by $\mathbb{P}EF^g_{\underline{e}}(N, M)$. We also set the evaluation form $\delta : \mathcal{M} \to \mathbb{C}$ mapping $gr(\underline{e})$ to $\chi(\mathbb{P}EF^g_{\underline{e}}(N, M))$.

Theorem 1.5. *Let $M, N \in \mathcal{C}(\mathcal{S})$. We have*

$$\chi(\mathbb{P}\mathrm{Ext}^1_A(M, N))\delta_{M\oplus N} = \sum_{L\in S(\underline{d})} \chi(\mathbb{P}\mathrm{Ext}^1_A(M, N)_{\langle L\rangle})\delta_L + \delta.$$

Proof. Since (for example, see [1] or [6])

$$\chi(Gr_{\underline{e}}(M \oplus N)) = \sum_{\underline{e}_1+\underline{e}_2=\underline{e}} \chi(Gr_{\underline{e}_1}(M)) \cdot \chi(Gr_{\underline{e}_2}(N)),$$

the above formula has the following reformulation:

$$\begin{aligned}
&\chi(\mathbb{P}\operatorname{Ext}^1_A(M, N)) \sum_{\underline{e}_1+\underline{e}_2=\underline{e}} \chi(Gr_{\underline{e}_1}(M)) \cdot \chi(Gr_{\underline{e}_2}(N)) \\
&\quad = \sum_{L\in S(\underline{d})} \chi(\mathbb{P}\operatorname{Ext}^1_A(M, N)_{\langle L\rangle})\chi(Gr_{\underline{e}}(L)) + \chi(\mathbb{P}EF^g_{\underline{e}}(N, M)).
\end{aligned}$$

Now we prove the above reformulation. Define

$$EF(M, N) = \{(\varepsilon, L_1) \mid \varepsilon \in \operatorname{Ext}^1_A(M, N)_L \setminus \{0\}, L_1 \in Gr_{\underline{e}}(L)\}.$$

The action of $\mathbb{C}^*$ on $\operatorname{Ext}^1_A(M, N)$ naturally induces the action on $EF(M, N)$ [7, section 5.4]. Under the action of $\mathbb{C}^*$, it has the geometric quotient:

$$\pi : EF(M, N) \to \mathbb{P}EF(M, N).$$

We have the natural projection:

$$p : \mathbb{P}EF(M, N) \to \mathbb{P}\operatorname{Ext}^1_A(M, N).$$

Using Proposition 1.1, we have

$$\chi(\mathbb{P}EF(M, N)) = \sum_{L\in S(\underline{d})} \chi(\mathbb{P}\operatorname{Ext}^1_A(M, N)_{\langle L\rangle})\chi(Gr_{\underline{e}}(L)).$$

Given $(\varepsilon, L_1) \in EF(M,N)$, let ε be the equivalence class of the following short exact sequence:

$$\varepsilon: \quad 0 \longrightarrow N \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} L \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} M \longrightarrow 0 .$$

As a vector space, $L = N \oplus M$ and L_1 is the subspace of L. We put $M_1 = (0,1)(L_1)$ and $N_1 = (1,0)(L_1)$. It is clear that M_1 and N_1 are the submodules of M and N, respectively. Then there is a natural morphism

$$\phi_0 : EF(M,N) \to \bigcup_{\underline{e}_1+\underline{e}_2=\underline{e}} Gr_{\underline{e}_1}(M) \times Gr_{\underline{e}_2}(N)$$

defined by mapping (ε, L_1) to (M_1, N_1). Furthermore, we have

$$\phi_0((\varepsilon, L_1)) = \phi_0(t.(\varepsilon, L_1))$$

for any $(\varepsilon, L_1) \in EF(M,N)$ and $t \in \mathbb{C}^*$. This induces the morphism

$$\phi : \mathbb{P}EF(M,N) \to \bigcup_{\underline{e}_1+\underline{e}_2=\underline{e}} Gr_{\underline{e}_1}(M) \times Gr_{\underline{e}_2}(N).$$

Now we compute the fibre of this morphism for $M_1 \in Gr_{\underline{e}_1}(M)$ and $N_1 \in Gr_{\underline{e}_2}(N)$. Consider the following linear map dual to β_{M_1,N_1}:

$$\beta'_{M_1,N_1} : \mathrm{Ext}\,^1_A(M,N) \oplus \mathrm{Ext}\,^1_A(M_1,N_1) \to \mathrm{Ext}\,^1(M_1,N)$$

mapping $(\varepsilon, \varepsilon')$ to $\varepsilon_{M_1} - \varepsilon'_N$, where ε_{M_1} and ε'_N are induced by the inclusions $M_1 \subseteq M$ and $N_1 \subseteq N$, respectively, as follows:

$$\begin{tikzcd}
\varepsilon_{M_1}: & 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & L_1 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
\varepsilon: & 0 \arrow[r] & N \arrow[r] & L \arrow[r, "\pi"] & M \arrow[r] & 0
\end{tikzcd}$$

where L_1 is the pullback, and

$$\begin{tikzcd}
\varepsilon': & 0 \arrow[r] & N_1 \arrow[r] \arrow[d] & L' \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d, equal] & 0 \\
\varepsilon'_N: & 0 \arrow[r] & N \arrow[r] & L'_1 \arrow[r] & M_1 \arrow[r] & 0
\end{tikzcd}$$

where L'_1 is the pushout. It is clear that $\varepsilon, \varepsilon'$ and M_1, N_1 induce the inclusions $L_1 \subseteq L$ and $L' \subseteq L'_1$ and

$$p_0 : \mathrm{Ext}\,^1_A(M,N) \oplus \mathrm{Ext}\,^1_A(M_1,N_1) \to \mathrm{Ext}\,^1_A(M,N)$$

is a projection. By a similar discussion as in [7, Lemma 2.4.2], we know that

$$p(\phi^{-1}((M_1,N_1))) = \mathbb{P}(p_0(\mathrm{Ker}(\beta'_{M_1,N_1}))).$$

Moreover, by [8, Lemma 7], for fixed $\varepsilon \in p_0(\mathrm{Ker}(\beta'_{M_1,N_1}))$, let $\mathbb{P}\varepsilon$ be its orbit in $\mathbb{P}(p_0(\mathrm{Ker}(\beta'_{M_1,N_1})))$. Then we have

$$p^{-1}(\mathbb{P}\varepsilon) \cap \phi^{-1}((M_1,N_1)) \cong \mathrm{Hom}(M_1, N/N_1).$$

Using Proposition 1.1, we get

$$\chi(\phi^{-1}((M_1,N_1))) = \chi(\mathbb{P}(p_0(\mathrm{Ker}(\beta'_{M_1,N_1}))) = \dim_{\mathbb{C}} p_0(\mathrm{Ker}(\beta'_{M_1,N_1})).$$

In the same way, we consider the projection

$$\varphi : \mathbb{P}EF^{g}_{\underline{e}}(N,M) \to \bigcup_{\underline{e}_1+\underline{e}_2=\underline{e}} Gr_{\underline{e}_1}(M)\times Gr_{\underline{e}_2}(N).$$

Then

$$\chi(\varphi^{-1}((M_1,N_1))) = \dim_{\mathbb{C}}\mathrm{Ext}^{\,1}_A(N,M)\cap \mathrm{Im}\beta_{N_1,M_1}.$$

Now, depending on the fact that $\mathcal{C}(\mathcal{S})$ is of Ext-symmetry, we have

$$\dim_{\mathbb{C}} p_0(\mathrm{Ker}(\beta'_{M_1,N_1})) + \dim_{\mathbb{C}}\mathrm{Ext}^{\,1}_A(N,M)\cap\mathrm{Im}\beta_{N_1,M_1} = \dim_{\mathbb{C}}\mathrm{Ext}^{\,1}_A(M,N).$$

Using Proposition 1.1 again, we complete the proof of the theorem. □

2. The multiplication formula

The formula in the last section is not so 'symmetric' as the Geiss-Leclerc-Schröer formula. In order to overcome this difficulty, we should consider flags of composition series instead of Grassmannians of submodules as in [7]. In this section, we prove a multiplication formula as an analog of the Geiss-Leclerc-Schröer formula in [7].

Let $A=\mathbb{C}Q/\mathcal{I}$ be an algebra associated to a finite and connected quiver Q and let $\mathcal{S}=\{S_1,\cdots,S_n\}$ be a finite set of finite-dimensional simple A-modules. Let $\mathcal{C}(\mathcal{S})$ be a full subcategory of Ext-symmetry of $\mathrm{mod}(A)$ associated to $\mathcal{S}$.

Let $A_{\underline{d}}$ be the constructible subset of $\mathbb{E}_{\underline{d}}(A)$ consisting of A-modules in $\mathcal{C}(\mathcal{S})$ with dimension vector $\underline{d}$. Let $\mathcal{X}$ be the set of pairs $(\mathbf{j},\mathbf{c})$ where $\mathbf{c}=(c_1,\cdots,c_m)\in\{0,1\}^m$ and $\mathbf{j}=(j_1,\cdots,j_m)$ is a sequence of integers such that $S_{j_k}\in\mathcal{S}$ for $1\le k\le m$. Given $x\in A_{\underline{d}}$ and $(\mathbf{j},\mathbf{c})\in\mathcal{X}$, we define an x-stable flag of type $(\mathbf{j},\mathbf{c})$ as a composition series of x,

$$\mathfrak{f}_x = \left(V=(\mathbb{C}^{\underline{d}},x)\supseteq V^1\supseteq\cdots\supseteq V^m=0\right),$$

of A-submodules of V such that $|V^{k-1}/V^k| = c_kS_{j_k}$, where S_{j_k} is the simple module in $\mathcal{S}$. Let $\Phi_{\mathbf{j},\mathbf{c},x}$ be the variety of x-stable flags of type $(\mathbf{j},\mathbf{c})$. We simply write $\Phi_{\mathbf{j},x}$ when $\mathbf{c}=(1,1,\cdots,1)$. Define

$$\Phi_{\mathbf{j}}(A_{\underline{d}}) = \{(x,\mathfrak{f}) \mid x\in A_{\underline{d}}, \mathfrak{f}\in\Phi_{\mathbf{j},x}\}.$$

As in Proposition 1.3, we consider a projection: $p:\Phi_{\mathbf{j}}(A_{\underline{d}})\to A_{\underline{d}}$. The function $p_*(1_{\Phi_{\mathbf{j}}(A_{\underline{d}})})$ is constructible by Theorem 1.2.

Proposition 2.1. *For any type $\mathbf{j}$, the function $A_{\underline{d}}\to\mathbb{C}$ mapping x to $\chi(\Phi_{\mathbf{j},x})$ is constructible.*

Let $d_{\mathbf{j},\mathbf{c}}:\mathbb{E}_{\underline{d}}(A)\to\mathbb{C}$ be the function defined by $d_{\mathbf{j},\mathbf{c}}(x)=\chi(\Phi_{\mathbf{j},\mathbf{c},x})$ for $x\in\mathbb{E}_{\underline{d}}(A)$. It is a constructible function as in Proposition 2.1. We simply write $d_{\mathbf{j}}$ if $\mathbf{c}=(1,\cdots,1)$. Define $\mathcal{M}(\underline{d})$ to be the vector space spanned by $d_{\mathbf{j}}$. For fixed $A_{\underline{d}}$, there are finitely many types $\mathbf{j}$ such that $\Phi_{\mathbf{j}}(A_{\underline{d}})$ is not empty. Hence, there exists a finite subset $S(\underline{d})$ of $A_{\underline{d}}$ such that

$$A_{\underline{d}} = \bigcup_{M\in S(\underline{d})}\langle M\rangle,$$

where $\langle M\rangle=\{M'\in A_{\underline{d}}\mid \chi(\Phi_{\mathbf{j},M'})=\chi(\Phi_{\mathbf{j},M})$ for any type $\mathbf{j}\}$.

For any $M\in A_{\underline{d}}$, we define the evaluation form $\delta_M:\mathcal{M}(\underline{d})\to\mathbb{C}$ mapping a constructible function $f\in\mathcal{M}(\underline{d})$ to $f(M)$. We have

$$\langle M\rangle = \{M'\in A_{\underline{d}}\mid \delta_{M'}=\delta_M\}.$$

Lemma 2.2. *For $M,N\in\mathcal{C}(\mathcal{S})$, we have $\delta_{M\oplus N}=\delta_M\cdot\delta_N$.*

The lemma is equivalent to showing that

$$\chi(\Phi_{\mathbf{j},M\oplus N}) = \sum_{\mathbf{c}'+\mathbf{c}''\sim 1} \chi(\Phi_{\mathbf{j},\mathbf{c}',M}) \cdot \chi(\Phi_{\mathbf{j},\mathbf{c}'',N}).$$

Here, $\mathbf{c}'+\mathbf{c}'' \sim 1$ means that $c'_k + c''_k = 1$ for $k = 1, \cdots, m$. The proof of the lemma depends on the fact that under the action of $\mathbb{C}^*$, $\Phi_{\mathbf{j},M\oplus N}$ and its stable subset have the same Euler characteristic. We refer to [6] for details.

The following formula is just the multiplication formula in [7, Theorem 1] when A is a preprojective algebra and $\mathcal{S}$ is the set of all simple A-modules.

Theorem 2.3. *With the above notation, for $M, N \in \mathcal{C}(\mathcal{S})$, we have*

$$\chi(\mathbb{P}\mathrm{Ext}^1_A(M,N))\delta_{M\oplus N} = \sum_{L\in S(\underline{e})} (\chi(\mathbb{P}\mathrm{Ext}^1_A(M,N)_{\langle L\rangle}) + \chi(\mathbb{P}\mathrm{Ext}^1_A(N,M)_{\langle L\rangle}))\delta_L,$$

where $\underline{e} = \underline{\dim}M + \underline{\dim}N$.

In the proof of Theorem 1.5, a key point is to consider the linear maps β_{M_1,N_1} and β'_{M_1,N_1} dual to each other by the property of Ext-symmetry. Now we extend this idea to the present situation as in [7]. Let

$$\mathfrak{f}_M = (M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_m = 0)$$

be a flag of type $(\mathbf{j}, \mathbf{c}')$ and let

$$\mathfrak{f}_N = (N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_m = 0)$$

be a flag of type $(\mathbf{j}, \mathbf{c}'')$ such that $c'_k + c''_k = 1$ for $k = 1, \cdots, m$. We write $\mathbf{c}'+\mathbf{c}'' \sim 1$. For $k = 1, \cdots, m$, let $\iota_{M,k}$ and $\iota_{N,k}$ be the inclusion maps $M_k \to M_{k-1}$ and $N_k \to N_{k-1}$, respectively. Define [7, Section 2]

$$\beta_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N} : \bigoplus_{k=0}^{m-2} \mathrm{Ext}\,^1_A(N_k, M_{k+1}) \to \bigoplus_{k=0}^{m-2} \mathrm{Ext}\,^1_A(N_k, M_k)$$

by the following map:

$$\begin{tikzcd} N_k \arrow[r, "\varepsilon_k"] \arrow[d, "\iota_{N,k}"'] \arrow[dr] & M_{k+1}[1] \arrow[d, "\iota_{M,k+1}"] \\ N_{k-1} \arrow[r, "\varepsilon_{k-1}"] & M_k[1] \end{tikzcd}$$

satisfying

$$\beta_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N}(\varepsilon_0, \cdots, \varepsilon_{m-2}) = \iota_{M,1} \circ \varepsilon_0 + \sum_{k=1}^{m-2} (\iota_{M,k+1} \circ \varepsilon_k - \varepsilon_{k-1} \circ \iota_{N,k}).$$

Depending on the fact that $\mathcal{C}(\mathcal{S})$ is of Ext-symmetry, we can write its dual

$$\beta'_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N} : \bigoplus_{k=0}^{m-2} \mathrm{Ext}\,^1_A(M_k, N_k) \to \bigoplus_{k=0}^{m-2} \mathrm{Ext}\,^1_A(M_{k+1}, N_k)$$

by the following map:

$$\begin{tikzcd} M_{k+1} \arrow[r, "\eta_{k+1}"] \arrow[d, "\iota_{M,k+1}"'] \arrow[dr] & N_{k+1}[1] \arrow[d, "\iota_{N,k+1}"] \\ M_k \arrow[r, "\eta_k"] & N_k[1] \end{tikzcd}$$

satisfying

$$\beta'_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N}(\eta_0,\cdots,\eta_{m-2})=\sum_{k=0}^{m-3}(\eta_k\circ\iota_{M,k+1}-\iota_{N,k+1}\circ\eta_{k+1})+\eta_{m-2}\circ\iota_{M,m-1}.$$

Now, we prove Theorem 2.3.

Proof. Define

$$EF_{\mathbf{j}}(M,N)=\{(\varepsilon,\mathfrak{f})\mid\varepsilon\in\mathrm{Ext}\,^1_A(M,N)_L,L\in A_{\underline{e}},\mathfrak{f}\in\Phi_{\mathbf{j},L}\}.$$

The action of $\mathbb{C}^*$ on $\mathrm{Ext}^1_A(M,N)$ induces an action on $EF_{\mathbf{j}}(M,N)$. The orbit space under the action of $\mathbb{C}^*$ is denoted by $\mathbb{P}EF_{\mathbf{j}}(M,N)$, and the orbit of $(\varepsilon,\mathfrak{f})$ is denoted by $\mathbb{P}(\varepsilon,\mathfrak{f})$. We have the natural projection

$$p:\mathbb{P}EF_{\mathbf{j}}(M,N)\to\mathbb{P}\mathrm{Ext}\,^1_A(M,N).$$

The fibre for any $\mathbb{P}\varepsilon\in\mathbb{P}\mathrm{Ext}\,^1_A(M,N)_L$ is isomorphic to $\Phi_{\mathbf{j},L}$. By Theorem 1.1, we have

$$\chi(\mathbb{P}EF_{\mathbf{j}}(M,N))=\sum_{L\in S(\underline{e})}\chi(\mathbb{P}\mathrm{Ext}\,^1_A(M,N)_{\langle L\rangle})\chi(\Phi_{\mathbf{j},L}).$$

We also have the natural morphism

$$\phi:\mathbb{P}EF_{\mathbf{j}}(M,N)\to\bigcup_{\mathbf{c}'+\mathbf{c}''\sim 1}\Phi_{\mathbf{j},\mathbf{c}',M}\times\Phi_{\mathbf{j},\mathbf{c}'',N}$$

mapping $\mathbb{P}(\varepsilon,\mathfrak{f})$ to $(\mathfrak{f}_M,\mathfrak{f}_N)$, where $(\mathfrak{f}_M,\mathfrak{f}_N)$ is naturally induced by ε and $\mathfrak{f}$ and $t.(\varepsilon,\mathfrak{f})$ induces the same $(\mathfrak{f}_M,\mathfrak{f}_N)$ for any $t\in\mathbb{C}^*$. By [7, Lemma 2.4.2], we know that

$$p(\phi^{-1}(\mathfrak{f}_M,\mathfrak{f}_N))=\mathbb{P}(p_0(\mathrm{Ker}(\beta'_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N}))),$$

where $p_0:\bigoplus_{k=0}^{m-2}\mathrm{Ext}\,^1_A(M_k,N_k)\to\mathrm{Ext}\,^1_A(M,N)$ is a projection. On the other hand, by [8, Lemma 7], the morphism

$$p\mid_{\phi^{-1}(\mathfrak{f}_M,\mathfrak{f}_N)}:\phi^{-1}(\mathfrak{f}_M,\mathfrak{f}_N)\to\mathbb{P}(p_0(\mathrm{Ker}(\beta'_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N})))$$

has fibres isomorphic to an affine space. Hence, by Theorem 1.1, we have

$$\chi(\phi^{-1}(\mathfrak{f}_M,\mathfrak{f}_N))=\chi(\mathbb{P}(p_0(\mathrm{Ker}(\beta'_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N})))).$$

Dually, we define

$$EF_{\mathbf{j}}(N,M)=\{(\varepsilon,\mathfrak{f})\mid\varepsilon\in\mathrm{Ext}\,^1_A(N,M)_L,L\in A_{\underline{e}},\mathfrak{f}\in\Phi_{\mathbf{j},L}\}.$$

The orbit space under $\mathbb{C}^*$-action is denoted by $\mathbb{P}EF_{\mathbf{j}}(N,M)$. We have the natural projection

$$q:\mathbb{P}EF_{\mathbf{j}}(N,M)\to\mathbb{P}\mathrm{Ext}\,^1_A(N,M).$$

The fibre for any $\mathbb{P}\varepsilon\in\mathbb{P}\mathrm{Ext}\,^1_A(N,M)_L$ is isomorphic to $\Phi_{\mathbf{j},L}$. By Theorem 1.1, we have

$$\chi(\mathbb{P}EF_{\mathbf{j}}(N,M))=\sum_{L\in S(\underline{e})}\chi(\mathbb{P}\mathrm{Ext}\,^1_A(N,M)_{\langle L\rangle})\chi(\Phi_{\mathbf{j},L}).$$

As in the proof of Theorem 1.5, there is a natural morphism

$$\varphi_0:EF_{\mathbf{j}}(N,M)\to\bigcup_{\mathbf{c}'+\mathbf{c}''\sim 1}\Phi_{\mathbf{j},\mathbf{c}',M}\times\Phi_{\mathbf{j},\mathbf{c}'',N}$$

such that

$$\varphi_0((\varepsilon,\mathfrak{f}))=\varphi_0(t.(\varepsilon,\mathfrak{f}))$$

for any $(\varepsilon, \mathfrak{f}) \in EF_{\mathbf{j}}(N, M)$ and $t \in \mathbb{C}^*$. Hence, we have the morphism

$$\varphi : \mathbb{P}EF_{\mathbf{j}}(N, M) \to \bigcup_{\mathbf{c}'+\mathbf{c}''\sim 1} \Phi_{\mathbf{j},\mathbf{c}',M} \times \Phi_{\mathbf{j},\mathbf{c}'',N}.$$

By [7, Lemma 2.4.3], we know that

$$q(\varphi^{-1}(\mathfrak{f}_M, \mathfrak{f}_N)) = \mathbb{P}\mathrm{Ext}\,^1_A(N, M) \cap \mathrm{Im}(\beta_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N}).$$

Similar to the above dual situation, by [8, Lemma 7], the morphism

$$q\,|_{\varphi^{-1}(\mathfrak{f}_M,\mathfrak{f}_N)} : \varphi^{-1}(\mathfrak{f}_M, \mathfrak{f}_N) \to \mathbb{P}\mathrm{Ext}\,^1_A(N, M) \cap \mathrm{Im}(\beta_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N})$$

has fibres isomorphic to an affine space. Hence, by Proposition 1.1, we have

$$\chi(\varphi^{-1}(\mathfrak{f}_M, \mathfrak{f}_N)) = \chi(\mathbb{P}\mathrm{Ext}\,^1_A(N, M) \cap \mathrm{Im}(\beta_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N})).$$

However, since $\beta_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N}$ and $\beta'_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N}$ are dual to each other, we have

$$(p_0(\mathrm{Ker}(\beta'_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N})))^{\perp} = \mathrm{Ext}\,^1_A(N, M) \cap\ \mathrm{Im}(\beta_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N}).$$

Thus we have

$$\begin{aligned} & \chi(\mathbb{P}(p_0(\mathrm{Ker}(\beta'_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N})))) + \chi(\mathbb{P}\mathrm{Ext}\,^1_A(N, M) \cap \mathrm{Im}(\beta_{\mathbf{j},\mathbf{c}',\mathbf{c}'',\mathfrak{f}_M,\mathfrak{f}_N})) \\ = \quad & \dim_{\mathbb{C}}\mathrm{Ext}\,^1_A(M, N). \end{aligned}$$

Therefore, using Proposition 1.1, we obtain

$$\mathbb{P}EF_{\mathbf{j}}(M, N) + \mathbb{P}EF_{\mathbf{j}}(N, M) = \dim_{\mathbb{C}}\mathrm{Ext}\,^1(M, N) \cdot \sum_{\mathbf{c}'+\mathbf{c}''\sim 1} \chi(\Phi_{\mathbf{j},\mathbf{c}',M}) \cdot \chi(\Phi_{\mathbf{j},\mathbf{c}'',N}).$$

Now, we have obtained the identity

$$\begin{aligned} & \dim_{\mathbb{C}}\mathrm{Ext}\,^1(M, N) \cdot \sum_{\mathbf{c}'+\mathbf{c}''\sim 1} \chi(\Phi_{\mathbf{j},\mathbf{c}',M}) \cdot \chi(\Phi_{\mathbf{j},\mathbf{c}'',N}) \\ = & \sum_{L\in S(\underline{e})} \chi(\mathbb{P}\mathrm{Ext}\,^1_A(M, N)_{\langle L\rangle})\chi(\Phi_{\mathbf{j},L}) + \sum_{L\in S(\underline{e})} \chi(\mathbb{P}\mathrm{Ext}\,^1_A(N, M)_{\langle L\rangle})\chi(\Phi_{\mathbf{j},L}) \end{aligned}$$

for any type $\mathbf{j}$. Using Lemma 2.2 and Proposition 1.1, we finish the proof of Theorem 2.3. $\square$

3. Examples

In this section, we give some examples of module subcategories of Ext-symmetry.

(I) Let A be a preprojective algebra associated to a connected quiver Q without loops. Let $\mathcal{S}$ be the set of all simple A-modules. Then $\mathcal{C}(\mathcal{S})$ is of Ext-symmetry [7, Theorem 3].

(II) Let $A = \mathbb{C}Q/\langle \alpha\alpha^* - \alpha^*\alpha \rangle$ be an associative algebra associated to the following quiver:

$$Q := \ \alpha \circlearrowleft \bullet \circlearrowright \alpha^* \ .$$

Let $M = (\mathbb{C}^m, X_\alpha, X_{\alpha^*})$ and $N = (\mathbb{C}^n, Y_\alpha, Y_{\alpha^*})$ be two finite-dimensional A-modules. Following the characterization of $\mathrm{Ext}^1_A(M, N)$ in Section 1.3, we consider the following isomorphism between complexes (see [3, Lemma 1] or [7, Section 8.2]):

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathbb{C}}(M_\bullet, N_\bullet) & \xrightarrow{d^0_{M,N}} & \mathrm{Hom}_{\mathbb{C}}(M_\bullet, N_\bullet) \bigoplus \mathrm{Hom}_{\mathbb{C}}(M_\bullet, N_\bullet) & \xrightarrow{d^1_{M,N}} & \mathrm{Hom}_{\mathbb{C}}(M_\bullet, N_\bullet)\ , \\
\big\downarrow \mathbf{1} & & \big\downarrow (\mathbf{1}, -\mathbf{1}) & & \big\downarrow -\mathbf{1} \\
\mathrm{Hom}_{\mathbb{C}}(M_\bullet, N_\bullet) & \xrightarrow{d^{1,*}_{N,M}} & \mathrm{Hom}_{\mathbb{C}}(M_\bullet, N_\bullet) \bigoplus \mathrm{Hom}_{\mathbb{C}}(M_\bullet, N_\bullet) & \xrightarrow{d^{0,*}_{N,M}} & \mathrm{Hom}_{\mathbb{C}}(M_\bullet, N_\bullet)
\end{array}$$

where $M_\bullet = \mathbb{C}^m, N_\bullet = \mathbb{C}^n$. Here, we define

$$\begin{aligned} d^0_{M,N}(A) &= (Y_\alpha A - AX_\alpha, Y_{\alpha^*}A - AX_{\alpha^*}), d^1_{M,N}(B,B^*) \\ &= Y_{\alpha^*}B + B^*X_\alpha - Y_\alpha B^* - BX_{\alpha^*}, \\ d^{0,*}_{N,M}(B,B^*) &= BX_{\alpha^*} + B^*X_\alpha - Y_{\alpha^*}B - Y_\alpha B^*, d^{1,*}_{N,M}(A) \\ &= (Y_\alpha A - AX_\alpha, -Y_{\alpha^*}A + AX_{\alpha^*}) \end{aligned}$$

for any $n \times m$ matrices A, B and B^*. The second complex is dual to the complex

$$\mathrm{Hom}_{\mathbb{C}}(N_\bullet, M_\bullet) \xrightarrow{d^0_{N,M}} \mathrm{Hom}_{\mathbb{C}}(N_\bullet, M_\bullet) \bigoplus \mathrm{Hom}_{\mathbb{C}}(N_\bullet, M_\bullet) \xrightarrow{d^1_{N,M}} \mathrm{Hom}_{\mathbb{C}}(N_\bullet, M_\bullet)$$

with respect to the nondegenerate bilinear form

$$\Phi : \mathrm{Hom}_{\mathbb{C}}(N_\bullet, M_\bullet) \times \mathrm{Hom}_{\mathbb{C}}(N_\bullet, M_\bullet) \to \mathbb{C}$$

mapping (X, Y) to $tr(XY)$. As in Section 1.3, we have functorially

$\mathrm{Ext}^1_A(M,N) = \mathrm{Ker}(d^1_{M,N})/\mathrm{Im}(d^0_{M,N})$ and $\mathrm{DExt}^1_A(N,M) = \mathrm{Ker}(d^{0,*}_{N,M})/\mathrm{Im}(d^{1,*}_{N,M})$.

Hence, we have a bifunctorial isomorphism:

$$\mathrm{Ext}^1_A(M,N) \cong \mathrm{DExt}^1_A(N,M).$$

(III) Deformed preprojective algebras were introduced by Crawley-Boevey and Holland in [4]. Fix $\lambda = (\lambda_i)_{i\in Q_0}$ where $\lambda_i \in \mathbb{C}$. The deformed preprojective algebra of weight λ is an associative algebra

$$A(\lambda) = \mathbb{C}\overline{Q}/\langle \sum_{\alpha\in Q_1} (\alpha\alpha^* - \alpha^*\alpha) - \sum_{i\in Q_0} \lambda_i e_i \rangle,$$

where $\overline{Q} = Q \cup Q^*$ is the double of a quiver Q without loops. Let M, N be finite-dimensional A-modules. As in Section 1.3, we know $D(M,N)$ is just the kernel of the following linear map:

$$\bigoplus_{\alpha\in\overline{Q}_1} \mathrm{Hom}_{\mathbb{C}}(M_{s(\alpha)}, N_{t(\alpha)}) \xrightarrow{d^1_{M,N}} \bigoplus_{i\in Q_0} \mathrm{Hom}_{\mathbb{C}}(M_i, N_i),$$

where $d^1_{M,N}$ maps $(f_\alpha)_{\alpha\in\overline{Q}_1}$ to $(g_i)_{i\in Q_0}$ such that

$$g_i = \sum_{\alpha\in Q_1, s(\alpha)=i} (N_{\alpha^*}f_\alpha + f_{\alpha^*}M_\alpha) - \sum_{\alpha\in Q_1, t(\alpha)=i} (N_\alpha f_{\alpha^*} + f_\alpha M_{\alpha^*}).$$

In the same way as in [7, Section 8.2], we obtain a bifunctorial isomorphism

$$\mathrm{Ext}^1_A(M,N) \cong \mathrm{DExt}^1_A(N,M).$$

(IV) It is easy to construct examples of module subcategories of Ext-symmetry over an algebra which is not of Ext-symmetry. Let $A = \mathbb{C}Q/\langle \beta\beta^* - \beta^*\beta \rangle$ be a quotient algebra associated to the quiver

$$Q := \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, bend left, "\beta"] & 3 \arrow[l, bend left, "\beta^*"] \end{tikzcd}.$$

Let S_1, S_2 and S_3 be finite-dimensional simple A-modules associated to three vertices, respectively. Since $\dim_{\mathbb{C}}\mathrm{Ext}^1(S_1, S_2) = 1$ and $\mathrm{Ext}^1(S_2, S_1) = 0$, A is not an algebra of Ext-symmetry. However, for $\mathcal{S} = \{S_1, S_3\}$ or $\{S_2, S_3\}$, $\mathcal{C}(\mathcal{S})$ is of Ext-symmetry.

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