

HYPERCYCLICITY OF WEIGHTED CONVOLUTION OPERATORS ON HOMOGENEOUS SPACES

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ABSTRACT. Let $1 \leq p < \infty$. We show that a weighted translation operator on the L^p space of a homogeneous space is hypercyclic under some condition on the weight. This condition is also necessary in the discrete case and is equivalent to hereditary hypercyclicity of the operator. The condition can be strengthened to characterise topologically mixing weighted translation operators on discrete spaces.

1. INTRODUCTION

Given a continuous linear self-map T on a Fréchet space X , we denote its iterates by

$$T^1 = T, \quad \dots, \quad T^{n+1} = T^n \circ T, \quad \dots \quad (n = 1, 2, \dots).$$

The operator T is said to be *hypercyclic* if there is a vector $x \in X$ such that the orbit $\{x, Tx, \dots, T^n x, \dots\}$ is dense in X , in which case x is called a *hypercyclic vector* for T . A Fréchet space admits a hypercyclic operator if, and only if, it is separable and infinite-dimensional [1, 6, 11]. Hypercyclic operators have been studied by many authors since the work of Birkhoff [9] and MacLane [24]; we refer to [21, 22] for recent surveys and [2, 3, 7, 12, 14, 25, 26] for some recent works.

Let \mathbb{N} denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Rolewicz [27] began the study of hypercyclic operators in classical Banach spaces and showed that a scalar multiple λB of the backward shift $B(x_0, x_1, \dots) = (x_1, x_2, \dots)$ on $\ell^2(\mathbb{N}_0)$ is hypercyclic for any complex number λ with $|\lambda| > 1$ while B itself is not hypercyclic. Hypercyclicity of generalized backward shifts on Banach spaces has been considered in [20] and hypercyclic bilateral weighted shifts on $\ell^2(\mathbb{Z})$ have been characterised by Salas [28] in terms of their weights.

Following a recent study of convolution operators on groups and homogeneous spaces in [13], it is natural to consider the question of hypercyclicity for these operators. Hypercyclicity of convolution operators on spaces of ultradifferentiable functions has been studied in [10]. Although Birkhoff's seminal result [9] shows the hypercyclicity of the translation operator on the space of entire functions, in contrast, a translation operator, or a convolution operator by a measure of unit mass, on L^p spaces of locally compact groups is never hypercyclic. In this paper,

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we give a sufficient condition for a weighted translation operator on the L^p space of a homogeneous space to be hypercyclic. This condition is also necessary in the discrete case which subsumes the above result of Salas, and further, it is equivalent to hereditary hypercyclicity of the weighted translation operator. By strengthening the condition and analogous arguments, we also characterise topologically mixing weighted translation operators on discrete homogeneous spaces. Hypercyclicity in weighted L^p spaces has also been studied in [16, 17].

We will make use of the following form of the hypercyclic criterion in [8], derived from the original one obtained by Kitai [23], and by Gethner and Shapiro [19] independently.

Lemma 1.1. *Let T be a continuous linear self-map on a Fréchet space X . Then T is hypercyclic if it satisfies the following criteria:*

- (i) (T^n) admits a subsequence (T^{n_k}) converging to zero pointwise on a dense subset of X ;
- (ii) there is a dense subset Y of X and a sequence of maps $S_{n_k} : Y \rightarrow X$ such that (S_{n_k}) tends to zero pointwise on Y and $(T^{n_k} S_{n_k})$ tends to the identity pointwise on Y .

We note that the above criteria are not necessary for hypercyclicity [4, 15], although they have been shown in [8] to be equivalent to $T \oplus T$ being hypercyclic and also equivalent to T being *hereditarily hypercyclic*; that is, there is an increasing sequence (n_k) in \mathbb{N} such that every subsequence (T^{m_k}) of (T^{n_k}) admits a vector $z \in X$ for which the orbit $\{T^{m_k} z\}_{k=1}^\infty$ is dense in X . If, moreover, the sequence (n_k) in the hypercyclic criterion satisfies the *syndetic condition* $\sup_k \{n_{k+1} - n_k\} < \infty$, then it has been shown in [14] that T is *topologically mixing*; that is, given nonempty open sets $U, V \subset X$, there exists $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n > N$. In fact, an operator satisfies the hypercyclic criterion for a syndetic sequence if and only if it does so for the full sequence (n) [5, Remark 3.3].

2. WEIGHTED CONVOLUTION OPERATORS

In what follows, let G be a locally compact second countable group with identity e and a right-invariant Haar measure λ which is the counting measure if G is discrete. Then G is a union of a nested sequence

$$G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$$

of compact sets with G_n contained in the interior of G_{n+1} . Let G act transitively on a locally compact Hausdorff space Ω by a continuous right action

$$(v, g) \in \Omega \times G \mapsto vg \in \Omega$$

so that Ω is homeomorphic to, and hence identifies with, a *right coset space* G/H of G by an isotropy subgroup H . We assume throughout that H is *compact*, in which case G/H admits a G -invariant measure ν satisfying $\nu = \lambda \circ q^{-1}$ where $q : G \rightarrow G/H$ will always denote the quotient map [18, p.58]. We call Ω a *homogeneous space* of G and consider convolution operators on Ω .

Let $C_c(G/H)$ be the space of continuous functions on G/H with compact support. Let $1 \leq p \leq \infty$ and let $L^p(\Omega) = L^p(G/H)$ be the complex Lebesgue space of G/H with respect to ν , and write $L^p(G)$ for $H = \{e\}$. Given a complex Borel

measure σ on G with (finite) total variation $\|\sigma\| = |\sigma|(G)$, we define a convolution operator $T_\sigma : L^p(G/H) \longrightarrow L^p(G/H)$ by

$$T_\sigma(f) = f * \sigma \quad (f \in L^p(G/H)),$$

where the convolution

$$f * \sigma(Hx) = \int_G f(Hxy^{-1}) d\sigma(y)$$

exists ν -almost everywhere and is defined to be 0 otherwise. The operator T_σ is bounded with norm $\|T_\sigma\|_p \leq \|\sigma\|$. Hence T_σ is not hypercyclic if $\|\sigma\| \leq 1$.

A continuous function $w : G \rightarrow (0, \infty)$ is called a *weight for G/H* if it satisfies

$$(1) \quad w(hx) = w(x) \quad (x \in G, h \in H)$$

so that $w'(Hx) := w(x)$ is a well-defined function on G/H . If such a weight w is in $L^\infty(G)$, we can define a *weighted convolution operator*

$$T_{\sigma,w} : f \in L^p(G/H) \mapsto T_{\sigma,w}f \in L^p(G/H),$$

where

$$T_{\sigma,w}f(Hx) = w(x)(f * \sigma)(Hx) \quad (f \in L^p(G/H)).$$

The operator $T_{\sigma,w}$ is not hypercyclic if $\|\sigma\|\|w\|_\infty \leq 1$. One can also consider the weighted convolution operator $\tilde{T}_{\sigma,w} : L^p(G) \longrightarrow L^p(G)$ with $\tilde{T}_{\sigma,w}f = w(f * \sigma)$. It is a ‘lift’ of $T_{\sigma,w}$ in the following commutative diagram:

$$\begin{array}{ccc} L^p(G) & \xrightarrow{\tilde{T}_{\sigma,w}} & L^p(G) \\ J \uparrow & & \downarrow Q \\ L^p(G/H) & \xrightarrow{T_{\sigma,w}} & L^p(G/H) \end{array}$$

where the continuous linear maps $J : L^p(G/H) \longrightarrow L^p(G)$ and $Q : L^p(G) \longrightarrow L^p(G/H)$ are injective and surjective respectively, with $\|Q\| \leq 1$, and are given by

$$Jf = f \circ q, \quad Qg(Hx) = \int_H g(\xi x) d\xi \quad (f \in L^p(G/H), g \in L^p(G))$$

with $d\xi$ being the normalized Haar measure on the compact group H (cf. [13]). We have $T_{\sigma,w}Q = Q\tilde{T}_{\sigma,w}$ [13, p. 77], which gives the following result.

Lemma 2.1. *Let $1 \leq p < \infty$. If the weighted convolution operator $\tilde{T}_{\sigma,w}$ possesses a hypercyclic vector $g \in L^p(G)$, then Qg is a hypercyclic vector for $T_{\sigma,w}$ on $L^p(G/H)$.*

For each $a \in G$, we denote by δ_a the unit mass at a and by δ_a^n the n -fold convolution $\delta_a * \cdots * \delta_a$. We define $\delta_a^0 = \delta_e$ and write T_a for the convolution operator T_{δ_a} .

Given a weight $w \in L^\infty(G)$ for G/H , the weighted convolution operator $T_{\delta_a,w}$ is written simply $T_{a,w}$, which is a weighted translation operator. If we also have $w^{-1} \in L^\infty(G)$, then the weighted convolution operator $T_{a^{-1},w^{-1}*\delta_{a^{-1}}}$ is the inverse of $T_{a,w}$. We write $S_{a,w}$ for $T_{a^{-1},w^{-1}*\delta_{a^{-1}}}$ to simplify notation. Thus, for each $f \in L^p(G/H)$, we have

$$\begin{aligned} T_{a,w}f(Hx) &= w(x)f(Hxa^{-1}), \\ S_{a,w}f(Hx) &= \frac{1}{w(xa)}f(Hxa). \end{aligned}$$

Without the assumption of $w^{-1} \in L^\infty(G)$, we can still define the operator $S_{a,w}$ from the subspace $C_c(G/H)$ to $L^p(G/H)$, and we will use the same notation for this map since no confusion is likely. The same remark applies to $T_{a,w}$ if $w \notin L^\infty(G)$. Evidently, the translation operator T_a is not hypercyclic. However if one considers the weighted convolution operator $T_{a,w}$, then hypercyclicity can occur for certain weights. Indeed, we are going to describe these weights for arbitrary homogeneous spaces $\Omega = G/H$ and show, for a discrete group G , that these are the only weights making $T_{a,w}$ hereditarily hypercyclic.

Let $p \in [1, \infty]$ with conjugate exponent q and let $\langle \cdot, \cdot \rangle : L^p(\Omega) \times L^q(\Omega) \rightarrow \mathbb{C}$ be the duality. A simple computation gives

$$\langle T_{\sigma,w} f, g \rangle = \langle f, T_{\tilde{\sigma}}(wg) \rangle \quad (f \in L^p(\Omega), g \in L^q(\Omega)),$$

where $\tilde{\sigma}$ is the measure $d\tilde{\sigma}(x) = d\sigma(x^{-1})$. Therefore the dual map $T_{\sigma,w}^* : L^q(\Omega) \rightarrow L^q(\Omega)$ is given by $T_{\sigma,w}^*(g) = T_{\tilde{\sigma}}(wg)$ for $g \in L^q(\Omega)$. In particular, if $\sigma = \delta_a$, we have

$$T_{a,w}^*(g) = T_{\delta_{a^{-1}}}(wg) = T_{a^{-1}, w * \delta_{a^{-1}}}(g) \quad (g \in L^q(\Omega))$$

and $T_{a,w}^*$ is a weighted convolution operator on $L^q(\Omega)$.

By a *null set* in G , we mean a Borel set $K \subset G$ with $\lambda(K) = 0$.

Proposition 2.2. *Let G be a locally compact second countable group with $a \in G$. Let $\Omega = G/H$ be a homogeneous space of G . Let $w : G \rightarrow (0, \infty)$ be a weight for G/H satisfying $w \in L^\infty(G)$. Let $1 \leq p < \infty$ and $T_{a,w}$ be the weighted convolution operator on $L^p(\Omega)$ defined above. Then condition (ii) below implies (i).*

- (i) $T_{a,w}$ is hereditarily hypercyclic.
- (ii) Both sequences (depending on a)

$$w_n := \prod_{s=1}^n w * \delta_{a^{-1}}^s \quad \text{and} \quad \overline{w}_n := \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit respectively subsequences (w_{n_k}) and (\overline{w}_{n_k}) which converge pointwise to 0 λ -a.e. and are uniformly bounded on each non-null compact subset K of G .

Proof. Let (w_{n_k}) and (\overline{w}_{n_k}) be subsequences of (w_n) and (\overline{w}_n) respectively satisfying (ii). We show that $T_{a,w}$ satisfies the hypercyclic criterion.

We make use of the sequence of maps $S_{a,w}^{n_k} : C_c(G/H) \rightarrow L^p(G/H)$. Let $f \in C_c(G/H) \setminus \{0\}$ with compact support $\text{supp } f$. Then we have $T_{a,w}^{n_k}(S_{a,w}^{n_k} f) = f$. We show that $\|T_{a,w}^{n_k} f\|_p \rightarrow 0$ as $n_k \rightarrow \infty$. There exists a compact set $K \subset G$ with $q(K) = \text{supp } f$ (see, for instance, [18, 2.46]). It follows that $q^{-1}(\text{supp } f) = q^{-1}(q(K)) = HK$, which is compact and non-null. Let (w_{n_k}) be bounded on HK by M say. Let $\varepsilon > 0$ and choose, by Egoroff's theorem, a Borel set $E \subset HK$ such that $\lambda(HK \setminus E) < \frac{\varepsilon}{M^p \|f\|_\infty^p}$ and $(w_{n_k}^p)$ converges to 0 uniformly on E . There exists

$N \in \mathbb{N}$ such that $w_{n_k}^p < \frac{\varepsilon}{\|f\|_p^p}$ on E for $n_k > N$. We have, by change of variables,

$$\begin{aligned}
\|T_{a,w}^{n_k} f\|_p^p &= \int_{G/H} |T_{a,w}^{n_k} f(Hx)|^p d\nu(Hx) \\
&= \int_{(\text{supp } f)a^{n_k}} |w(x)w(xa^{-1}) \cdots w(xa^{-(n_k-1)})|^p |f(Hxa^{-n_k})|^p d\nu(Hx) \\
&= \int_{HKa^{n_k}} |w(x)w(xa^{-1}) \cdots w(xa^{-(n_k-1)})|^p |f(Hxa^{-n_k})|^p d\lambda(x) \\
&= \int_{HK} |w(xa^{n_k})w(xa^{n_k-1}) \cdots w(xa)|^p |f(Hx)|^p d\lambda(x) \\
&= \int_E w_{n_k}^p(x) |f(Hx)|^p d\lambda(x) + \int_{HK \setminus E} w_{n_k}^p(x) |f(Hx)|^p d\lambda(x) \\
&\leq \frac{\varepsilon}{\|f\|_p^p} \|f\|_p^p + M^p \|f\|_\infty^p \lambda(HK \setminus E) < 2\varepsilon
\end{aligned}$$

for $n_k > N$. Similar arguments using the sequence (\bar{w}_{n_k}) yield

$$\|S_{a,w}^{n_k} f\|_p^p = \int_{HKa^{-n_k}} \frac{1}{|w(xa)w(xa^2) \cdots w(xa^{n_k})|^p} |f(Hxa^{n_k})|^p d\lambda(x) \longrightarrow 0.$$

Hence $T_{a,w}^{n_k}$ satisfies the hypercyclic criterion in Lemma 1.1 since $C_c(G/H)$ is dense in $L^p(G/H)$. Therefore $T_{a,w}$ is hereditarily hypercyclic by [8, Theorem 2.3]. \square

Remark 2.3. In condition (ii) above, if the sequence (n_k) is syndetic, then $T_{a,w}$ is topologically mixing by a remark after Lemma 1.1. If we have $w^{-1} \in L^\infty(G)$ instead of $w \in L^\infty(G)$, then condition (ii) implies that $S_{a,w}$ is hereditarily hypercyclic on $L^p(G/H)$ by changing the roles of $T_{a,w}$ and $S_{a,w}$ in the above proof.

We note that, if $a = e$, then condition (ii) in Proposition 2.2 fails and, in fact, we have $\bar{w}_n = w_n^{-1}$ in this case. Also, a pointwise convergence sequence of continuous functions need not be uniformly bounded on a compact set. For example, the sequence $w_n(x) = 2n^2 x e^{-n^2 x^2}$ is not uniformly bounded on $[0, 1]$.

There are examples of hypercyclic operators with hypercyclic dual [26, 28]. The following result shows that $T_{a,w}$ and its dual $T_{a,w}^*$ can both be hypercyclic for certain weights w .

Corollary 2.4. *The dual $T_{a,w}^*$ of a weighted translation operator $T_{a,w} : L^p(G/H) \longrightarrow L^p(G/H)$ is hypercyclic if the weight $w * \delta_{a^{-1}}$ satisfies condition (ii) for a^{-1} in Proposition 2.2.*

Example 2.5. Fix $t \in (0, 1)$. We define a weight $w : \mathbb{R} \rightarrow (0, \infty)$ for \mathbb{R} by

$$w(x) = \begin{cases} t & \text{if } 1 \leq x, \\ t^x & \text{if } -1 \leq x \leq 1, \\ \frac{1}{t} & \text{if } x \leq -1. \end{cases}$$

Then w and w^{-1} are bounded and continuous on \mathbb{R} , with w satisfying condition (ii) for $a > 0$ in Proposition 2.2. Indeed, let $K = [b, c]$ say. Pick $n_0 \in \mathbb{N}$ such that $b + n_0 a > 1$. Since w is decreasing, we have

$$\begin{aligned}
0 &< w_n(x) = w(x+a)w(x+2a) \cdots w(x+na) \\
&\leq w(b+a)w(b+2a) \cdots w(b+na) \\
&\leq w(b+a)w(b+2a) \cdots w(b+n_0 a) \quad (x \in K, n \geq n_0).
\end{aligned}$$

It follows that (w_n) is uniformly bounded on K by some constant M . For each $x \in [b, c]$, we have $w(x+s) = t$ for all $s \geq n_0 a$. This implies, for $n > n_0$,

$$\begin{aligned} w_n(x) &= w(x+a)w(x+2a) \cdots w(x+n_0 a)w(x+(n_0+1)a) \cdots w(x+na) \\ &\leq Mt^{n-n_0} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence (w_n) converges to 0 uniformly on $[b, c]$. For the sequence (\bar{w}_n) , we have

$$\begin{aligned} \bar{w}_n(x) &= \frac{1}{w(x)w(x-a) \cdots w(x-(n-1)a)} \\ &\leq \frac{1}{w(c)w(c-a) \cdots w(c-(n-1)a)} \\ &\leq \frac{1}{w(c)w(c-a) \cdots w(c-(n_1-1)a)} \quad (x \in K, n \geq n_1), \end{aligned}$$

where n_1 is chosen so that $c - (n_1 - 1)a < -1$. Likewise, $(\bar{w}_n) \rightarrow 0$ uniformly on $[b, c]$.

In fact, the above example is a special case of the following lemma.

Lemma 2.6. *Let w be a weight for a locally compact second countable group G . The following conditions are equivalent.*

- (i) *Given $\varepsilon > 0$, a compact set $D \subset G$ and $N \in \mathbb{N}$, there exists $m > N$ satisfying $w_m(x) < \varepsilon$ and $\bar{w}_m(x) < \varepsilon$ for all $x \in D$.*
- (ii) *Both sequences (w_n) and (\bar{w}_n) admit subsequences (w_{n_k}) and (\bar{w}_{n_k}) which converge uniformly to 0 on each compact subset K of G .*

If G is discrete, then D can be replaced by a singleton.

Proof. We show (i) \Rightarrow (ii). Since G is a union $\bigcup_{k=1}^{\infty} G_k$ of nested compact sets G_k with G_k contained in the interior of G_{k+1} , it suffices to prove convergence on G_t for each $t \in \mathbb{N}$.

Let $\varepsilon = \frac{1}{2}$ and $D = G_1$. Then there exists n_1 such that $w_{n_1}(x) < \frac{1}{2}$ and $\bar{w}_{n_1}(x) < \frac{1}{2}$ for all $x \in G_1$. Inductively, for each $k > 1$, there exists $n_k > n_{k-1}$ such that $w_{n_k}(x) < \frac{1}{2^k}$ and $\bar{w}_{n_k}(x) < \frac{1}{2^k}$ for all $x \in G_k$.

Now let $\varepsilon > 0$ and choose $k_0 \in \mathbb{N}$ with $k_0 > t$ and $\frac{1}{2^{k_0}} < \varepsilon$. Then, for all $k > k_0$, we have

$$w_{n_k}(x) < \frac{1}{2^k} < \frac{1}{2^{k_0}} < \varepsilon \quad \text{and} \quad \bar{w}_{n_k}(x) < \varepsilon$$

on $G_k \supset G_t$. Hence (w_{n_k}) and (\bar{w}_{n_k}) converge uniformly to 0 on G_t . \square

We now consider discrete groups and derive necessary and sufficient conditions for a weighted translation operator to be hypercyclic. A *torsion element* of a group G is an element of finite order.

Lemma 2.7. *Let G be a discrete group and $a \in G$. Then a is not a torsion element if and only if for any finite subset $D \subset G$, there exists $N \in \mathbb{N}$ such that $D \cap Da^{\pm n} = \emptyset$ for $n > N$.*

Proof. Given that a is not a torsion element, we observe that, for every $d \in D$, there exists N_d such that $da^n \notin D$ for $n > N_d$. Otherwise, there is some $d \in D$ such that $da^{n_j} \in D$ for a strictly increasing sequence (n_j) in \mathbb{N} . Since D is finite, we must have $da^{n_j} = da^{n_t}$ for some $n_j \neq n_t$, which contradicts the fact that a is not a torsion element. Let $N = \max\{N_d : d \in D\}$. Then $D \cap Da^n = \emptyset$ for $n > N$. The condition $D \cap Da^{-n} = \emptyset$ can be proved similarly.

On the other hand, if $a \in G$ is a torsion element with order m , then for any finite subset $D \subset G$, there exist infinitely many n 's such that $D \cap Da^n \neq \emptyset$. Indeed, $D \cap Da^n = D \neq \emptyset$ for $n \in m\mathbb{Z}$. \square

Theorem 2.8. *Let G be a discrete group and H a finite subgroup. Choose $a \in G$ that is not a torsion element. Let $w : G \rightarrow (0, \infty)$ be a weight for G/H such that $w \in \ell^\infty(G)$. Let $1 \leq p < \infty$ and $T_{a,w}$ be the weighted convolution operator on $\ell^p(G/H)$ defined by a and w . The following conditions are equivalent.*

- (i) $T_{a,w}$ is hypercyclic.
- (ii) $T_{a,w}$ is hereditarily hypercyclic.
- (iii) Both sequences (depending on a)

$$w_n = \prod_{s=1}^n w * \delta_{a^{-1}}^s \quad \text{and} \quad \overline{w}_n = \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit subsequences (w_{n_k}) and (\overline{w}_{n_k}) which converge to 0 pointwise in G . In particular, if G is torsion free, then the above conditions are equivalent for all $a \in G \setminus \{e\}$.

Proof. By Proposition 2.2, (iii) implies (i) and (ii) since a compact subset of a discrete group is finite. We only need to show (i) implies (iii).

Let $T_{a,w}$ be hypercyclic. Let $\varepsilon > 0$ and $z \in G$. Fix $N \in \mathbb{N}$. Let $\chi_z \in \ell^p(G/H)$ be the characteristic function

$$\chi_z(Hx) = \begin{cases} 1 & \text{if } x \in Hz, \\ 0 & \text{otherwise.} \end{cases}$$

Choose $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$. Since the set of hypercyclic vectors for $T_{a,w}$ is dense, there exist a hypercyclic vector $f \in \ell^p(G/H)$ for $T_{a,w}$ and some $m > N$ such that

$$\|f - \chi_z\|_p < \delta \quad \text{and} \quad \|T_{a,w}^m f - \chi_z\|_p < \delta.$$

By Lemma 2.7, we may choose m sufficiently large so that $Hz \cap Hza^{\pm m} = \emptyset$. Since

$$\|f - \chi_z\|_p^p = \sum_{Hx \in G/H} |f(Hx) - \chi_z(Hx)|^p \nu(Hx) = \sum_{x \in G} |f(Hx) - \chi_z(Hx)|^p < \delta^p,$$

where $\nu(Hx) = \lambda(H)$, we have

$$|f(Hx) - \chi_z(Hx)| < \delta \quad (x \in G).$$

This gives

$$\begin{aligned} |f(Hx)| &> 1 - \delta && \text{for } x \in Hz, \\ |f(Hx)| &< \delta && \text{for } x \notin Hz. \end{aligned}$$

From $\|T_{a,w}^m f - \chi_z\|_p < \delta$, we also deduce that

$$(2) \quad \left| w(x)w(xa^{-1}) \cdots w(xa^{-(m-1)})f(Hxa^{-m}) - \chi_z(Hx) \right| < \delta \quad (x \in G).$$

In particular,

$$\overline{w}_m(z)^{-1} |f(Hza^{-m})| > 1 - \delta.$$

Since $Hz \cap Hza^{-m} = \emptyset$, we have

$$\overline{w}_m(z) < \frac{|f(Hza^{-m})|}{1 - \delta} < \frac{\delta}{1 - \delta} < \varepsilon.$$

From (2), we have

$$|w(xa^m)w(xa^{m-1}) \cdots w(xa)f(Hx) - \chi_z(Hxa^m)| < \delta \quad (x \in G)$$

and hence, as $Hx \cap Hxa^m = \emptyset$, one obtains

$$w_m(z)|f(Hz)| < \delta.$$

It follows that

$$w_m(z) < \frac{\delta}{|f(Hz)|} < \frac{\delta}{1-\delta} < \varepsilon.$$

This proves that (w_n) and (\overline{w}_n) satisfy condition (i) in Lemma 2.6 for each point $z \in G$, and hence they admit subsequences (w_{n_k}) and (\overline{w}_{n_k}) which converge pointwise to 0 on G . \square

Remark 2.9. The above result implies that if $T_{a,w} : \ell^p(G/H) \rightarrow \ell^p(G/H)$ is hypercyclic for some $p \in [1, \infty)$, then it is so for all $p \in [1, \infty)$. As in Remark 2.3, if $w^{-1} \in \ell^\infty(G)$, then Theorem 2.8 applies to $S_{a,w}$. We also note that torsion freeness is not required in Theorem 2.8 for the case of $\ell^p(G)$ since Hx reduces to $\{x\}$ in the above proof.

Corollary 2.10. *Let $a \in G$ and $w \in \ell^\infty(G)$ be as in Theorem 2.8 for the homogeneous space G/H . Then $T_{a,w} : \ell^p(G/H) \rightarrow \ell^p(G/H)$ is hypercyclic if and only if the lift $\tilde{T}_{a,w} : \ell^p(G) \rightarrow \ell^p(G)$ is hypercyclic.*

Proof. This follows from Lemma 2.1 and Theorem 2.8. \square

Example 2.11. The weighted shift with weight sequence (a_n) studied in [28] is the weighted convolution operator $S_{a,w}$ on $\ell^2(\mathbb{Z})$ with $a = -1 \in \mathbb{Z}$ and the weight $w(n) = a_n^{-1}$. By Remark 2.9 and Lemma 2.6, $S_{a,w}$ is hypercyclic if and only if given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists an arbitrarily large n such that for all $|j| \leq q$, we have

$$\prod_{s=1}^n w(j-s) = w_n(j) < \varepsilon \quad \text{and} \quad \prod_{s=0}^{n-1} w(j+s) = \overline{w}_n(j)^{-1} > \frac{1}{\varepsilon},$$

which is the condition in [28, Theorem 2.1].

Using similar arguments as in the proof of Theorem 2.8, one can also characterise topologically mixing weighted translation operators on $\ell^p(G/H)$, which extends a result in [14, Theorem 1.2] for $\ell^2(\mathbb{Z})$.

Corollary 2.12. *Let $T_{a,w}$ be the operator defined in Theorem 2.8. The following conditions are equivalent.*

- (i) $T_{a,w}$ is topologically mixing.
- (ii) Both sequences (depending on a)

$$w_n = \prod_{s=1}^n w * \delta_{a^{-1}}^s \quad \text{and} \quad \overline{w}_n = \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

converge to 0 pointwise in G .

If G is torsion free, then the above conditions are equivalent for all $a \in G \setminus \{e\}$.

Proof. We see from Remark 2.3 that condition (ii) implies (i). For the converse, let $\varepsilon > 0$ and fix $z \in G$ with the characteristic function $\chi_z \in \ell^p(G/H)$ as defined in the proof of Theorem 2.8. Choose $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$ and let $U = \{f \in \ell^p(G/H) : \|f - \chi_z\| < \delta\}$. By the topologically mixing assumption, there exists $N \in \mathbb{N}$ such that

$$T_{a,w}^n(U) \cap U \neq \emptyset \quad (n > N).$$

We can therefore pick, for each $n > N$, a function $f_n \in U$ with $T_{a,w}^n f_n \in U$ which gives

$$\|f_n - \chi_z\|_p < \delta \quad \text{and} \quad \|T_{a,w}^n f_n - \chi_z\|_p < \delta.$$

Using this for each f_n and repeating the arguments in the proof of Theorem 2.8, we arrive at

$$\overline{w}_n(z) < \varepsilon \quad \text{and} \quad w_n(z) < \varepsilon$$

for all $n > N$, proving that (w_n) and (\overline{w}_n) converge to 0 pointwise in G . \square

We conclude with a remark that the weighted translation operator $T_{a,w}$ above and its dual $T_{a,w}^*$ can never be simultaneously topologically mixing since $T_{a,w}^* = T_{a^{-1}, w * \delta_{a^{-1}}}$, and for $a^{-1} \in G$ the two sequences for the weight $w * \delta_{a^{-1}}$ in condition (ii) above are given by

$$(w * \delta_{a^{-1}})_n = \prod_{s=1}^n (w * \delta_{a^{-1}}) * \delta_a^s = \overline{w}_n^{-1}$$

and $(\overline{w * \delta_{a^{-1}}})_n = w_n^{-1}$.

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