

A NOTE ON CLASSIFICATION OF SUBMODULES IN $H^2(D^2)$

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ABSTRACT. The Hardy spaces $H^2(D^2)$ can be viewed as a module over the polynomial ring $C[z_1, z_2]$. Based on a study of *core operator*, a new equivalence relation for submodules, namely *congruence*, was introduced. This paper gives a classification of congruent submodules by the rank of core operators.

0. INTRODUCTION

In this paper D denotes the unit disk of the complex plane \mathbf{C} and T denotes the unit circle. The polynomial ring $C[z_1, z_2]$ acts on the Hardy space over the bidisk $H^2(D^2)$ by multiplication of functions, which turns $H^2(D^2)$ into a module over $C[z_1, z_2]$. It is clear that a closed subspace $M \subset H^2(D^2)$ is a submodule if and only if it is invariant under multiplications by both z_1 and z_2 . For example, if I is an ideal in $C[z_1, z_2]$, then its closure in $H^2(D^2)$ (which we denote by $[I]$) is a submodule. There are also many submodules that are unrelated to ideals in $C[z_1, z_2]$. For instance, W. Rudin displayed two submodules in [Ru]: one is of infinite rank, and the other contains no nontrivial bounded functions. In an attempt to understand the structure of submodules, two canonical equivalence relations were considered. Two submodules M and N are said to be *unitarily equivalent (or similar)* if there is a unitary (or, respectively, invertible) module map between them. Much is known about the two equivalence relations (cf. Chen and Guo [CG]). A most notable fact is the rigidity phenomenon discovered by Douglas, Paulsen, Sah and Yan in [DPSY]. To be precise, let I_1 and I_2 be two ideals in $C[z_1, z_2]$ such that each has at most countably many zeros in D^2 . If there are bounded module maps $A : [I_1] \rightarrow [I_2]$ and $B : [I_2] \rightarrow [I_1]$ both with dense range, then $[I_1] = [I_2]$. Hence $[I_1]$ and $[I_2]$ are unitarily equivalent or similar only if they are identical. The following example provides a simple illustration of this fact.

Example 1. Let $\lambda = (\lambda_1, \lambda_2)$ be any point in D^2 and

$$H_\lambda = \{f \in H^2(D^2) : f(\lambda) = 0\}.$$

Then H_λ is a submodule. The rigidity theorem above implies that as long as $\alpha \neq \beta$, H_α and H_β are not unitarily equivalent.

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However, H_α and H_β are intuitively the same type of submodules. The rigidity phenomenon indicates that, for the purpose of classifying submodules, one needs a more flexible equivalence relation. *Congruence* of submodules was defined in [Ya2]. While it is still far from a complete classification of all submodules, the congruence relation is able to make good progress in this quest, as we will see in the next section.

1. CORE OPERATOR AND CONGRUENCE

In this paper $K(\lambda, z) = (1 - \overline{\lambda_1}z_1)^{-1}(1 - \overline{\lambda_2}z_2)^{-1}$ is the reproducing kernel for $H^2(D^2)$. The reproducing kernel for a submodule M is denoted by $K^M(\lambda, z)$. The *core function* $G^M(\lambda, z)$ for M is

$$G^M(\lambda, z) := \frac{K^M(\lambda, z)}{K(\lambda, z)} = (1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)K^M(\lambda, z),$$

and the *core operator* C^M (or simply C) on $H^2(D^2)$ is given by

$$C^M(f)(z) := \int_{T^2} G^M(\lambda, z)f(\lambda)dm(\lambda), \quad z \in D^2,$$

where $dm(\lambda)$ is the normalized Lebesgue measure on T^2 . The core operator is introduced in [GY]. More studies can be found in [Ya1] and [Ya2]. A basic fact is that on every submodule M , C is a bounded self-adjoint operator with $\|C\| = 1$. Moreover, it is not hard to check that $C = 0$ on M^\perp , so C will be restricted to M in our study.

For a submodule M we let (R_1, R_2) be the pair of multiplications by z_1 and z_2 on M . Clearly, (R_1, R_2) is a pair of commuting isometries on M . One relation between the core operator and the pair (R_1, R_2) is the identity

$$(1-1) \quad C = 1 - R_1R_1^* - R_2R_2^* + R_1R_2R_1^*R_2^*.$$

A submodule M is said to be *c-compact* (*c-finite*) if its core operator C is compact (or, respectively, of finite rank). There are many *c-finite* submodules, and as indicated in [Ya2], almost all known examples of submodules are *c-compact* (in fact Hilbert-Schmidt). Two submodules M and N are said to be *congruent* if C^M and C^N are congruent, e.g., there is a bounded invertible linear operator J from N to M such that $C^M = JC^NJ^*$.

Example 2. Now consider the action L of $\text{Aut}(D^2)$ on $H^2(D^2)$ defined by

$$(L_x f)(z) = f(x(z)), \quad x \in \text{Aut}(D^2),$$

where $\text{Aut}(D^2)$ is the group of bi-holomorphic self-maps on D^2 . One sees that L_x is bounded invertible and $L_x(M)$ is a submodule. Moreover, by [Ya2],

$$C^{L_x(M)} = L_x C^M L_x^*.$$

Hence M and $L_x(M)$ are congruent. In particular, H_α and H_β in Example 1 are congruent.

An invertible symmetric matrix A is said to have signature (p, q) if there is a nondegenerate matrix T such that TAT^* is a diagonal matrix with p 1s and $q - 1$ s. Since signature is a complete invariant of congruence relation for invertible self-adjoint matrices, it follows easily that two *c-finite* submodules M and N are congruent if and only if C^M and C^N , when restricted to the orthogonal complement of their kernels, have the same signature (cf. [Ya2]). The main purpose of this paper

is to improve on this fact and show that the rank itself is a complete invariant for congruent c -finite submodules.

The following lemma from [Ya2] is needed.

Lemma 1.1. C^2 is unitarily equivalent to the diagonal block matrix

$$\begin{pmatrix} [R_1^*, R_1][R_2^*, R_2][R_1^*, R_1] & 0 \\ 0 & [R_2^*, R_1]^*[R_2^*, R_1] \end{pmatrix}.$$

For an operator A with an eigenvalue λ , we let $E_\lambda(A)$ denote the corresponding eigenspace. It is shown in [GY] that

$$E_1(C) = M \ominus (z_1M + z_2M), \quad E_{-1}(C) = (z_1M \cap z_2M) \ominus z_1z_2M.$$

The next lemma is concerned with eigenvalues in the open interval $(-1, 1)$.

Lemma 1.2. Let M be a submodule, and let λ be a nonzero eigenvalue of C in $(-1, 1)$. Then $-\lambda$ is also an eigenvalue, and moreover $\dim E_\lambda(C) = \dim E_{-\lambda}(C)$.

Proof. Assume λ is a nonzero eigenvalue of C in $(-1, 1)$. For any nontrivial $f \in E_\lambda(C)$, we have

$$R_2^*Cf = \lambda R_2^*f.$$

It follows from (1-1) that

$$\begin{aligned} \lambda R_2^*f &= R_2^*(I - R_1R_1^* - R_2R_2^* + R_1R_2R_1^*R_2^*)f \\ &= R_2^* - R_2^*R_1R_1^* - R_2^* + R_1R_1^*R_2^*)f \\ (1-2) \quad &= -(R_2^*R_1 - R_1R_2^*)R_1^*f. \end{aligned}$$

Parallely, we have

$$(1-3) \quad \lambda R_1^*f = -(R_1^*R_2 - R_2R_1^*)R_2^*f.$$

We first observe that $R_2^*f \neq 0$. Since if $R_2^*f = 0$, by (1-3), R_1^*f is also equal to 0. This means that $f \in M \ominus (z_1M + z_2M)$, which contradicts the fact that $\lambda \neq 1$.

Putting (1-3) into (1-2), we have

$$(1-4) \quad [R_2^*, R_1][R_1^*, R_2]R_2^*f = \lambda^2 R_2^*f.$$

In conclusion, $R_2^* : E_\lambda(C) \longrightarrow E_{\lambda^2}([R_2^*, R_1][R_1^*, R_2])$ is a well-defined injective map. In particular,

$$(1-5) \quad \dim E_\lambda(C) \leq \dim E_{\lambda^2}([R_2^*, R_1][R_1^*, R_2]).$$

On the other hand, if we multiply the equation $Cf = \lambda f$ by $[R_1^*, R_1]$ and using (1-1), we have

$$\begin{aligned} \lambda[R_1^*, R_1]f &= [R_1^*, R_1](I - R_1R_1^* - R_2R_2^* + R_1R_2R_1^*R_2^*)f \\ &= [R_1^*, R_1](I - R_2R_2^*)f + [R_1^*, R_1](-R_1R_1^* + R_1R_2R_1^*R_2^*)f \\ (1-6) \quad &= [R_1^*, R_1][R_2^*, R_2]f. \end{aligned}$$

Parallely, multiplying the equation $Cf = \lambda f$ by $[R_2^*, R_2]$ and using (1-1), we have

$$(1-7) \quad \lambda[R_2^*, R_2]f = [R_2^*, R_2][R_1^*, R_1]f.$$

First we observe that $[R_1^*, R_1]f \neq 0$. Since if $[R_1^*, R_1]f = 0$, then by (1-7), $[R_2^*, R_2]f$ is also 0. These imply that $f \in z_1M \cap z_2M$. Since it is easy to see that $z_1z_2M \subset \ker C$, $f \in z_1M \cap z_2M \ominus z_1z_2M = E_{-1}(C)$, and this contradicts the fact that $\lambda \neq -1$.

Now combining (1-6) and (1-7), we have

$$(1-8) \quad [R_1^*, R_1][R_2^*, R_2][R_1^*, R_1]f = \lambda^2[R_1^*, R_1]f.$$

Since $[R_1^*, R_1] = [R_1^*, R_1]^2$, these observations show that

$$[R_1^*, R_1] : E_\lambda(C) \longrightarrow E_{\lambda^2}([R_1^*, R_1][R_2^*, R_2][R_1^*, R_1])$$

is a well-defined injective map. In particular,

$$(1-9) \quad \dim E_\lambda(C) \leq \dim E_{\lambda^2}([R_1^*, R_1][R_2^*, R_2]).$$

It now follows from Lemma 1.1 that

$$\dim E_{\lambda^2}((C)^2) \geq 2 \dim E_\lambda(C),$$

which implies that

$$\dim E_{-\lambda}(C) \geq \dim E_\lambda(C).$$

The same line of arguments starting with $-\lambda$ will prove the inequality in the other direction, and the proof is complete. \square

If C is compact, then $\overline{\text{ran}(C)}$ can be decomposed as

$$\overline{\text{ran}(C)} = E_1 \oplus \left(\bigoplus_{0 < \lambda_j < 1} E_{\lambda_j} \right) \oplus E_{-1} \oplus \left(\bigoplus_{-1 < \lambda_j < 0} E_{\lambda_j} \right).$$

For simplicity, we let $d_1 = \dim E_1$, $d_{-1} = \dim E_{-1}$, and

$$D = \bigoplus_{0 < \lambda_j < 1} \lambda_j P_j,$$

where P_j is the orthogonal projection from M onto E_{λ_j} . Then Lemma 1.2 indicates that C is unitarily equivalent to the diagonal block matrix

$$(1-10) \quad \begin{pmatrix} I_{d_1} & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & -I_{d_{-1}} & 0 \\ 0 & 0 & 0 & -D \end{pmatrix}.$$

Theorem 1.3. *Two c -finite submodules M and N are congruent if and only if C^M and C^N have the same rank.*

Proof. If M and N are congruent c -finite submodules, then C^M and C^N have the same signature by [Ya2], and hence C^M and C^N have the same rank.

For the sufficiency, it is shown in [GY] that if C is trace class, then $\text{tr} C = 1$. In view of (1-10), this fact implies $d_1 = d_{-1} + 1$. So if C^M and C^N have the same rank, then by (1-10) they have the same signature. Hence M and N are congruent. \square

Example 3. It is known that $\text{rank}(C) = 1$ if and only if $M = \phi H^2(D^2)$ for some inner function ϕ (cf. [GY]). So by Theorem 1.3, M is congruent to $H^2(D^2)$ if and only if M is of the form $\phi H^2(D^2)$.

It follows from (1-10) and the fact that $d_1 = d_{-1} + 1$ that for a c -finite submodule, the rank of C is always an odd number. So next in line is the case $\text{rank} C = 3$.

Example 4. If $q_1(z_1)$, $q_2(z_2)$ are two nontrivial one-variable inner functions over the unit disk D , then

$$M = q_1(z_1)H^2(D^2) + q_2(z_2)H^2(D^2)$$

is a submodule with interesting properties (cf. Izuchi, Nakazi and Seto [INS]). It is not difficult to compute that $\text{rank} C = 3$.

Another type of submodule M with $\text{rank} C = 3$ is of the form

$$M = \phi H^2(D^2) \oplus \frac{\phi H(z)}{w - G(z)} H^2(z),$$

where ϕ is an inner function, $G(z)$ and $H(z)$ are in the unit ball of $H^\infty(D)$ that satisfy some conditions, and $H^2(z)$ is $H^2(D)$ in the variable z (cf. K. J. Izuchi and K. H. Izuchi [II]).

Question. Is it possible to characterize all submodules M with $\text{rank} C = 3$?

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