A SPECTRAL THEORETIC APPROACH
TO THE KIRILLOV-DUFLO CORRESPONDENCE

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ABSTRACT. The Kirillov-Duflo orbit correspondance for compact Lie groups is parametrisation of the unitary dual, associating to the irreducible representation of highest weight λ the coadjoint orbit through $\lambda + \delta$, where δ is half the sum of the positive roots and justified by the character formulae of Weyl or Kirillov. In this paper we obtain this correspondence independently of character theory, showing that it arises from a convexity property of the Weyl functional calculus of the infinitesimal generators of the representation.

1. INTRODUCTION

Let G be a semisimple compact connected Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the vector space dual of $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{t}$, denote the Weyl group by W and distinguish a positive Weyl chamber. Let $\mathfrak{t}^{reg}$ denote the set of regular elements of $\mathfrak{t}$, i.e. those H for which iH does not belong to a wall of any Weyl chamber.

Let π be an irreducible representation of G with highest weight λ. Denote by δ the half-sum of the positive roots of $(\mathfrak{g}, \mathfrak{t})$. The Weyl character formula states that the character χ_π is given by the closed-form expression

$$\chi_\pi(\exp H) = \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\delta)(H)}}{\Delta(H)}$$

for all $H \in \mathfrak{t}^{reg}$, where Δ is the Weyl denominator function defined as

$$\Delta(H) := \sum_{w \in W} \operatorname{sgn}(w) e^{w\delta(H)}$$

for all $H \in \mathfrak{t}$. This relation implies a parametrisation of the set of irreducible representations of G by positive integral weights in $\mathfrak{t}^*$, whereby a representation π of highest weight λ is associated to the linear functional $\lambda + \delta$.

A still more precise form of this correspondence is achieved by the Kirillov character formula. On every coadjoint orbit $\mathcal{O}$ of G, the Kirillov-Kostant-Souriau 2-form $\omega_\mathcal{O}$ is defined by

$$\omega_\mathcal{O}(\xi)(ad^*(X)\xi, ad^*(Y)\xi) := \langle \xi, [X, Y] \rangle$$

for all $X, Y \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, and it endows the manifold $\mathcal{O}$ with a symplectic structure. If $2d$ is the dimension of the orbit $\mathcal{O}$, then the expression $\frac{1}{d}\omega_\mathcal{O} \wedge \cdots \wedge \omega_\mathcal{O}$

(d times) is the Liouville measure $\mu_{\mathcal{O}}$ on $\mathcal{O}$. The following relation was first proved by Kirillov [9]:

$$j(H)\chi_\pi(\exp H) = \int_{\mathcal{O}_{\lambda+\delta}} e^{i\beta(H)}\, d\mu_{\mathcal{O}_{\lambda+\delta}}(\beta),$$

where $\mathcal{O}_{\lambda+\delta}$ denotes the coadjoint orbit through $\lambda + \delta$ and

$$j(X) := \det{}^{1/2}\left(\frac{\sinh ad(X/2)}{ad(X/2)}\right)$$

for all $X \in \mathfrak{g}$. Here, a one-to-one correspondence between irreducible representations and coadjoint orbits is explicit. This is known as the "Kirillov-Duflo correspondence" and is applicable to a wide variety of Lie groups (see [13] for a survey).

In the sequel, we derive this correspondence without resorting to character theory.[1] To the unitary irreducible representation π, we associate a vector $d\pi(X)$ of self-adjoint operators given by the values of the derived representation $d\pi$ on a fixed basis of X of $\mathfrak{g}$, orthonormal with respect to the Killing form. It is shown that the set of extremal points of the convex hull of the support of the Weyl functional calculus distribution of $d\pi(X)$ is exactly the orbit $\mathcal{O}_\lambda$ (Theorem 6.12). The bijection $\pi \leftrightarrow \mathcal{O}_{\lambda+\delta}$ is obtained by convolving the Weyl calculus of $d\pi(X)$ with the Fourier transform of the function j defined above (Theorem 6.15).

Key to our analysis is a result of E. Nelson which states that the support of the Weyl calculus of a d-tuple of hermitian elements in a Banach algebra is equal to the joint spectrum of the same d-tuple in a certain commutative Banach algebra of "operants". We present this fact in Section 4 (Theorem 4.9), after some background material is introduced.

The Nelson algebra of operants is a special case of the more general notion of "operating algebras", introduced by E. Albrecht [1], which are discussed in the next section.

2. Operating algebras

Let $\mathfrak{R}$ be a complex unital Banach algebra with unit $1_{\mathfrak{R}}$. Let E be any vector subspace of $\mathfrak{R}$ and let $\mathfrak{D}$ be a complex unital Banach algebra with unit $1_{\mathfrak{D}}$. Let $\phi : E \to \mathfrak{D}$ be a one-to-one linear mapping and denote by $\langle\phi(E)\rangle$ the subalgebra of $\mathfrak{D}$ generated by $\phi(E)$. Denote by $\mathcal{S}_n$ the symmetric group of order n.

Definition 2.1 ([1]). A commutative complex unital Banach algebra $\mathfrak{D}$ is called an *operating algebra with respect to E and $\mathfrak{R}$* if there exists a one-to-one linear mapping $\phi : E \to \mathfrak{D}$ such that $\langle\phi(E)\rangle$ is dense in $\mathfrak{D}$ and a continuous linear mapping $Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)} : \mathfrak{D} \to \mathfrak{R}$ determined uniquely by the conditions $Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(1_{\mathfrak{D}}) = 1_{\mathfrak{R}}$ and

$$(2.1) \qquad Sym_{\mathfrak{R}}^{(\mathfrak{D},\phi)}(y_1 \ldots y_n) := \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} \phi^{-1}y_{\sigma(1)}\cdots\phi^{-1}y_{\sigma(n)}$$

for all $y_1, \ldots, y_n \in \langle\phi(E)\rangle$, $n \in \mathbb{N}$.

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[1]See [7] for an alternative approach using Dirac operators.

Let $S(E)$ be the symmetric algebra of E and let $\hat{} : E \to S(E)$ be the canonical inclusion mapping. It is well-known that any $\alpha \in S(E)$ can be written in the form

$$(2.2) \qquad \alpha = a + \sum_{i=1}^{n} \hat{x}_{j_1}^{(i)} \cdots \hat{x}_{j_i}^{(i)},$$

where $a \in \mathbb{C}$, $n, j_1, \ldots, j_n \in \mathbb{N}$ and $x_k^{(j)} \in E$.

It can be checked [11] that the formula

$$(2.3) \qquad \|\alpha\|_{S(E)} := \inf \; \{|a| + \sum_{i=1}^{n} \|x_{j_1}^{(i)}\|_{\mathfrak{R}} \cdots \|x_{j_i}^{(i)}\|_{\mathfrak{R}}\},$$

where the infimum is taken over all representations of α of the form (2.2), defines an algebra norm on $S(E)$. The completion of $S(E)$ with respect to this norm is a commutative Banach algebra, which we denote $\bar{S}(E)$. For simplicity of notation, we write $Sym_{\mathfrak{R}}^{(E)}$ for $Sym_{\mathfrak{R}}^{(\bar{S}(E), \hat{})}$.

Lemma 2.2. *The complete symmetric algebra $\bar{S}(E)$ is an operating algebra with respect to E and $\mathfrak{R}$.*

Proposition 2.3 ([1], Proposition 3.3). *Every operating algebra $\mathfrak{D}$ with respect to E and $\mathfrak{R}$ is isometrically isomorphic to the completion of a quotient of $S(E)$ endowed with some algebra norm.*

3. NUMERICAL RANGES AND HERMITIAN ELEMENTS

Let $\mathfrak{R}^*$ be the dual space of continuous linear functionals on $\mathfrak{R}$. By the Hahn-Banach theorem the following set is not void:

$$D(\mathfrak{R}, 1) := \{f \in \mathfrak{R}^* \; : \; f(1_{\mathfrak{R}}) = 1 = \|f\|\}.$$

This enables us to make the next

Definition 3.1. Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. The *numerical range* of x with respect to $\mathfrak{R}$ is the set $V_{\mathfrak{R}}(x) := \{(f(x_1), \ldots, f(x_d)) \, | \, f \in D(\mathfrak{R}, 1)\}$.

We note that if $\mathfrak{R}$ is a C^*-algebra, then $D(\mathfrak{R}, 1)$ is exactly the set of states of $\mathfrak{R}$.

Let $\mathcal{H}$ be a Hilbert space and denote by $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. It is well-known that the defining representation is the only irreducible representation of $L(\mathcal{H})$; hence, by the GNS construction, the pure states of $L(\mathcal{H})$ are exactly the set of functionals $\phi : L(\mathcal{H}) \to \mathbb{C}$ of the form $\phi(x) := \langle xu, u \rangle$ $(x \in L(\mathcal{H}))$ for some fixed unit vector $u \in \mathcal{H}$.

Let $A \subseteq \mathbb{C}^n$. We write $\operatorname{co} A$ for the convex hull of A. By the preceding discussion and the Kreĭn-Milman theorem we immediately have the following.

Theorem 3.2. *Let $\mathfrak{R} = L(\mathcal{H})$ and let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. Then*

$$(3.1) \qquad V_{L(\mathcal{H})}(x) := \operatorname{co} \{(\langle x_1 u, u \rangle, \ldots, \langle x_d u, u \rangle) \; : \; u \in \mathcal{H}, \|u\| = 1\}.$$

Lemma 3.3. *Let $T : \mathbb{C}^d \to \mathbb{C}^d$ be a linear mapping. Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. Then $V_{\mathfrak{R}}(Tx) = TV_{\mathfrak{R}}(x)$.*

Theorem 3.4 ([5], Theorem 1.2.3). *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. The numerical range $V_{\mathfrak{R}}(x)$ is a compact, convex subset of $\mathbb{C}^d$.*

Proof. See [5], Theorem 3, Chapter 1, Section 2. $\qquad\qquad\square$

An element $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ satisfying the following equivalence will be called **hermitian**.

Theorem 3.5 ([5], Lemma 2.2.5). *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. Then $V_{\mathfrak{R}}(x) \subseteq \mathbb{R}^d$ if and only if $\|e^{i\xi \cdot x}\| = 1$ for all $\xi \in \mathbb{R}^d$.*

Definition 3.6. Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. The two sets

$$\sigma_{\mathfrak{R}}^L(x) := \{\lambda \in \mathbb{C}^d : \sum_i \mathfrak{R} \cdot (\lambda_i 1_{\mathfrak{R}} - x_i) \neq \mathfrak{R}\}$$

and

$$\sigma_{\mathfrak{R}}^R(x) := \{\lambda \in \mathbb{C}^d : \sum_i (\lambda_i 1_{\mathfrak{R}} - x_i) \cdot \mathfrak{R} \neq \mathfrak{R}\}$$

are called respectively the *left spectrum* and the *right spectrum* of x with respect to $\mathfrak{R}$. The set $\sigma_{\mathfrak{R}}(x) := \sigma_{\mathfrak{R}}^L(x) \cup \sigma_{\mathfrak{R}}^R(x)$ is called the *spectrum* of x with respect to $\mathfrak{R}$.

Theorem 3.7 ([5], Theorem 1.2.12). *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. Then $\operatorname{co} \sigma_{\mathfrak{R}}(x) \subseteq V_{\mathfrak{R}}(x)$.*

Following [5], we denote by N the set of all algebra norms equivalent to the given norm. We write $V_{\mathfrak{R}}^{(p)}(x)$ for the numerical range of x with respect to the norm p.

Theorem 3.8 ([5], Theorem 1.2.13). *Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ be a d-tuple of commuting elements. Then*

$$\operatorname{co} \sigma_{\mathfrak{R}}(x) = \bigcap_{p \in N} V_{\mathfrak{R}}^{(p)}(x).$$

For the remainder of this section we fix $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$. Suppose that E is the linear span of $1_{\mathfrak{R}}$ and $x_1, \ldots, x_d$, the linear mapping $\phi : E \to \mathfrak{D}$ satisfies $\|\phi\| = 1$, and $\phi(1_{\mathfrak{R}}) = 1_{\mathfrak{D}}$. Let $\phi(x)$ denote the d-tuple $(\phi(x_1), \ldots, \phi(x_d))$.

Proposition 3.9 ([1], Lemma 5.3). *The inclusion*

$$V_{\mathfrak{D}}(\phi(x)) \subseteq V_{\mathfrak{R}}(x)$$

holds with equality if ϕ is an isometry.

Corollary 3.10. *Suppose that $\mathfrak{R}$ is a C^*-algebra and that $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ is a d-tuple of pairwise commuting elements. Then*

$$(3.2) \qquad\qquad \operatorname{co} \sigma_{\mathfrak{D}}(\phi(x)) \subseteq V_{\mathfrak{R}}(x).$$

Proof. By Theorem 3.8,

$$\operatorname{co} \sigma_{\mathfrak{D}}(\phi(x)) = \bigcap_{p \in N} V_{\mathfrak{D}}^{(p)}(\phi(x))$$

and by Lemma 3.9, $V_{\mathfrak{D}}^{(p)}(\phi(x)) \subseteq V_{\mathfrak{R}}^{(p)}(x)$ with equality if ϕ is an isometry. Since $\bigcap_{p \in N} V_{\mathfrak{R}}^{(p)}(x) \subseteq V_{\mathfrak{R}}(x)$, the result follows. $\qquad\square$

Theorem 3.11 ([1], Theorem 5.7). *Suppose that $\mathfrak{R}$ is a C^*-algebra, that $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ is a d-tuple of pairwise commuting elements and that $\mathfrak{D}$ is an operating algebra. Then*

$$\operatorname{co} \sigma_{\mathfrak{D}}(\phi(x)) = V_{\mathfrak{D}}(\phi(x)) = V_{\mathfrak{R}}(x) = \sigma_{\mathfrak{B}}(x) = \operatorname{co} \sigma_{\mathfrak{R}}(x),$$

where $\mathfrak{B}$ is an arbitrary C^*-subalgebra of $\mathfrak{R}$ containing $1_{\mathfrak{R}}, x_1, \ldots, x_d$.

4. The Weyl functional calculus

Let $x = (x_1, \ldots, x_d) \in \mathfrak{R}^d$ be a hermitian element and let E be the vector subspace of $\mathfrak{R}$ spanned by $1_{\mathfrak{R}}, x_1, \ldots, x_d$. Let $\phi : E \to \mathfrak{D}$ be a linear mapping with $\|\phi\| = 1$ and $\phi(1_{\mathfrak{R}}) = 1_{\mathfrak{D}}$. By Lemma 3.9, $\phi(x)$ is a hermitian element of $\mathfrak{D}$. Hence, for any function $f : \mathbb{R}^d \to \mathbb{C}$ with integrable Fourier transform $\hat{f}$, the following $\mathfrak{D}$-valued Bochner integral converges:

$$f_{\mathfrak{R}}^{(\mathfrak{D}, \phi)}(x) := \int_{\mathbb{R}^d} \hat{f}(\lambda) e^{i\lambda \cdot \phi(x)} \, d\lambda.$$

Here the Fourier transform $\hat{f}$ and the normalisation of Lebesgue measure dx are chosen exactly so that when $\mathfrak{R} = \mathfrak{D} = \mathbb{C}$ and ϕ is the identity mapping id, then $f(x) := f_{\mathbb{C}}^{(\mathbb{C}, id)}(x)$ has the usual meaning.

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space [8] of rapidly decreasing functions on $\mathbb{R}^d$. The mapping $W_{\mathfrak{R}}^{(\mathfrak{D}, \phi)}(x)$ given by $\langle W_{\mathfrak{R}}^{(\mathfrak{D}, \phi)}(x), f \rangle := f_{\mathfrak{R}}^{(\mathfrak{D}, \phi)}(x)$ for all $f \in \mathcal{S}(\mathbb{R}^d)$ is continuous and linear on $\mathcal{S}(\mathbb{R}^d)$ and hence defines a tempered distribution taking values in the Banach algebra $\mathfrak{D}$.

Definition 4.1. The $\mathfrak{D}$-valued tempered distribution $W_{\mathfrak{R}}^{(\mathfrak{D}, \phi)}(x)$ is called the *Weyl calculus* of x with respect to $\mathfrak{D}$ and ϕ.

Let id be the identity mapping on $\mathfrak{R}$. We write $W_{\mathfrak{R}}(x)$ for $W_{\mathfrak{R}}^{(\mathfrak{R}, id)}(x)$.

The following lemma is proved in [12] using the Trotter product formula for bounded self-adjoint operators on a Hilbert space. As every C^*-algebra is isometrically isomorphic to a closed subalgebra of bounded operators on some Hilbert space, we are able to state this lemma in slightly more generality.

Lemma 4.2 ([12], p.92). *Suppose that $\mathfrak{D}$ is a unital C^*-algebra and let x_1, x_2 be hermitian elements of $\mathfrak{D}$. Then*

$$\| \exp(x_1 + ix_2) \| \leq \| \exp x_1 \|.$$

For $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, let $\Im z := (\Im z_1, \ldots, \Im z_d)$ and $\Re z := (\Re z_1, \ldots, \Re z_d)$. The Hahn-Banach theorem ensures that there are sufficiently many linear functionals to obtain the following vector-valued extension of the usual Paley-Wiener theorem [8].

Theorem 4.3 (Paley-Wiener). *An $\mathfrak{R}$-valued tempered distribution u on $\mathbb{R}^d$ has compact support if and only if u is the Fourier transform of an analytic function $e : \mathbb{C}^d \to \mathfrak{R}$ for which there exist constants $C \geq 0$, $s \geq 0$ such that $\|e(\zeta)\|_{\mathfrak{R}} \leq C(1 + |\zeta|)^s e^{r|\Im \zeta|}$ for all $\zeta \in \mathbb{C}^d$.*

Theorem 4.4. *Suppose that $\mathfrak{D}$ is a unital C^*-algebra. Then the distribution $W_{\mathfrak{R}}^{(\mathfrak{D}, \phi)}(x)$ has compact support.*

Proof. We verify that the entire function $e(z) := \exp(i\langle z, \phi(x) \rangle)$, for $z \in \mathbb{C}^d$, satisfies the estimate required by Theorem 4.3. We have

$$\| \exp i\langle z, \phi(x) \rangle \| = \| \exp \left(\langle \Im z, \phi(x) \rangle - i \langle \Re z, \phi(x) \rangle \right) \|$$
$$\leq \| \exp \langle \Im z, \phi(x) \rangle \|$$
$$\leq \exp \| \langle \Im z, \phi(x) \rangle \|$$
$$\leq \exp |\Im z| \| \phi(x) \|$$
$$\leq \exp |\Im z| \|x\|. \qquad \square$$

Let $C^\infty(\mathbb{R}^d)$ be the space of infinitely differentiable $\mathbb{C}$-valued functions on $\mathbb{R}^d$.

Corollary 4.5. *Suppose that $\mathfrak{D}$ is a unital C^*-algebra. Then $W_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(x)$ extends uniquely to a distribution on $C^\infty(\mathbb{R}^d)$.*

The following theorem gives a characterisation of the Weyl calculus distribution by its values on polynomials.

Theorem 4.6 ([2], Theorem 2.8). *Suppose that $\mathfrak{R}$ is a unital C^*-algebra and let the monomial $p:\mathbb{R}^d \to \mathbb{R}$ be given by $p(\lambda_1,\ldots,\lambda_d) := \lambda_1^{k_1} \cdots \lambda_d^{k_d}$ for $k_1,\ldots,k_d \in \mathbb{N}$. Then $W_{\mathfrak{R}}(x)$ is the unique $\mathfrak{R}$-valued distribution such that*

$$\langle W_{\mathfrak{R}}(x),\, p \rangle = \frac{k_1! \cdots k_d!}{k!} \sum_{\pi} x_{\pi(1)} \cdots x_{\pi(k)},$$

where $k = k_1 + \cdots + k_d$ and π runs over the set of all maps from $\{1,\ldots,k\}$ into $\{1,\ldots,d\}$ that assume the value j exactly k_j times for $j = 1,\ldots,d$.

Corollary 4.7. *Suppose that $f \in C^\infty(\mathbb{R}^d)$ and for some $g \in C^\infty(\mathbb{R}^{d'})$, $d' < d$, $f(\xi_1,\ldots,\xi_d) = g(\xi_1,\ldots,\xi_{d'})$ for all $\xi_1,\ldots,\xi_d \in \mathbb{R}$. Then*

$$\langle W_{\mathfrak{R}}(x),\, f \rangle = \langle W_{\mathfrak{R}}(x'),\, g \rangle,$$

where $x' = (x_1,\ldots,x_{d'})$.

If $\mathfrak{D}$ is an operating algebra, let

$$I := I(\mathfrak{D},\,\phi) = \{ y_1 \in \mathfrak{D} \mid Sym_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}(y_1 y_2) = 0 \text{ for all } y_2 \in \mathfrak{D} \}.$$

Then I is a closed ideal of $\mathfrak{D}$, which allows us to define the quotient Banach algebra $\mathfrak{D}' := \mathfrak{D}/I$. Denote by ϕ' the map induced by $\phi:E \to \mathfrak{D}$.

Lemma 4.8. *Suppose that $\mathfrak{D}$ is an operating algebra with respect to E and $\mathfrak{R}$. Then $\mathfrak{D}'$ is also an operating algebra with respect to E and $\mathfrak{R}$.*

Proof. Clearly $\phi'(1_{\mathfrak{R}}) = 1_{\mathfrak{D}'}$ and $\langle \phi'(E) \rangle$ is dense in $\mathfrak{D}'$. Since $Sym_{\mathfrak{R}}^{(\mathfrak{D},\,\phi)}$ vanishes on I, the map $Sym_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi)}$ is well-defined and has all the properties required by Definition 2.1. $\qquad\square$

Theorem 4.9 ([1], Theorem 5.10; [11], Theorem 8). *Suppose that $\mathfrak{D}$ is a unital commutative C^*-algebra and an operating algebra with respect to E and $\mathfrak{R}$. Then*

(4.1) $$\operatorname{supp} W_{\mathfrak{R}}^{(\mathfrak{D}',\,\phi')}(x) = \operatorname{supp} W_{\mathfrak{R}}(x) = \sigma_{\mathfrak{D}'}(\phi'(x)).$$

5. A CANONICAL QUOTIENT OPERATING ALGEBRA

Suppose that V is a normed vector subspace of $\mathfrak{R}$ containing $1_{\mathfrak{R}}$. As in Section 2, denote by $S(V)$ the symmetric algebra of V and by $\bar{S}(V)$ the completion of $S(V)$ with respect to norm (2.3). Let $\hat{\ }:V \to \bar{S}(V)$ be the canonical inclusion mapping.

Definition 5.1. Let $\mathcal{I} = \mathcal{I}(V) := \{ \alpha \in \bar{S}(V) \mid Sym_{\mathfrak{R}}^{(V)}(\alpha\beta) = 0 \text{ for all } \beta \in \bar{S}(V) \}$. The quotient $\mathcal{A} = \mathcal{A}(V) := \bar{S}(V)/\mathcal{I}$ is called the Nelson algebra of *operants* over V.

Let $\tilde{\ }:V \to \mathcal{A}$ be the canonical projection.

Lemma 5.2. $\tilde{1}_{\mathfrak{R}} = 1_{\mathcal{A}}$.

Proof. The unit in $\mathcal{A}$ is clearly the image of $1 \in \mathbb{C}$. However, for any $\alpha \in S(V)$,

$$
\begin{aligned}
Sym_{\mathfrak{R}}^{(V)}(1 - \hat{1}_{\mathfrak{R}})\alpha &= Sym_{\mathfrak{R}}^{(V)}(\alpha - \hat{1}_{\mathfrak{R}}\alpha) \\
&= Sym_{\mathfrak{R}}^{(V)}\alpha - Sym_{\mathfrak{R}}^{(V)}\hat{1}_{\mathfrak{R}}\alpha \\
&= 0.
\end{aligned}
$$

By continuity, $Sym_{\mathfrak{R}}^{(V)}(1 - \hat{1}_{\mathfrak{R}})\alpha = 0$ for all $\alpha \in \bar{S}(V)$. The result follows. $\square$

Lemma 5.3. *The maps* $\hat{} : E \to S(E)$ *and* $\tilde{} : E \to \mathcal{A}(E)$ *are isometries. Furthermore,* $\|\hat{}\| = \|\tilde{}\| = 1$.

Theorem 5.4 ([11], Theorem 8). *Suppose that* $\mathfrak{R}$ *is a unital* C^**-algebra. Then*

$$
\text{(5.1)} \qquad \operatorname{supp} W_{\mathfrak{R}}^{(\mathcal{A}(E), \hat{})}(x) = \operatorname{supp} W_{\mathfrak{R}}(x) = \sigma_{\mathcal{A}}(\tilde{x}).
$$

Proof. By Lemma 2.2, $\bar{S}(E)$ is an operating algebra with respect to E and $\mathfrak{R}$; hence by Lemma 4.8, $\mathcal{A}$ is an operating algebra with respect to E and $\mathfrak{R}$. Since $\|\tilde{}\| = 1$, the distribution $W_{\mathfrak{R}}^{(\mathcal{A}(E), \hat{})}(x)$ is defined and equality (5.1) follows from Theorem 4.9. $\square$

By Proposition 2.3 all operating algebras with respect to E and $\mathfrak{R}$ are completions of quotients of $S(E)$ for some norm; hence $\mathcal{A}$ is the freest $(E, \mathfrak{R})$-operating algebra for which (4.1) holds.

Example 5.5. Let $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$. Let I be the identity of dimension 2 and set $E := Span_{\mathbb{C}}\{I, A\}$. Then $S(E) \cong \mathbb{C}[A]$. Since $A^2 - I = 0$ and $Sym := Sym_{L(\mathbb{C}^2)}^{(E)}$ is a homomorphism we have $(\hat{A}^2 - \hat{I})\bar{S}(E) \subseteq \mathcal{I}(E)$. On the other hand, if $c_0\hat{I} + c_1\hat{A} + \cdots + c_n\hat{A}^n \in \mathcal{I}(E)$, then

$$
\begin{aligned}
c_0 I + c_1 A + \cdots + c_n A^n &= Sym(c_0\hat{I} + c_1\hat{A} + \cdots + c_n\hat{A}^n) \\
&= 0
\end{aligned}
$$

and it follows that $\mathcal{I}(E) = (1 - \hat{I})\bar{S}(E) + (\hat{A}^2 - \hat{I})\bar{S}(E)$ by continuity.

Suppose that $\phi \in \sigma(\bar{S}(E))$. Then in particular $\phi(\mathcal{I}(E)) = \{0\}$, or equivalently $\phi(1 - \hat{I}) = \phi(\hat{A}^2 - I) = 0$; hence $\phi(\hat{I}) = 1$ and $\phi(A) = \{-1, 1\}$, and it follows that $\sigma(\hat{A}) = \{-1, 1\}$.

However, for $f \in \mathcal{S}(\mathbb{R})$ we have

$$
\begin{aligned}
\langle W_{L(\mathbb{C}^2)}(A), f \rangle &= \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi A} \, d\xi \\
&= \begin{pmatrix} f(1) & 0 \\ 0 & f(-1) \end{pmatrix}
\end{aligned}
$$

and it follows that $\operatorname{supp} W_{L(\mathbb{C}^2)}(A) = \{-1, 1\}$. Hence we have shown that $\sigma(A) = \operatorname{supp} W_{L(\mathbb{C}^2)}(A)$.

6. Spectra and matrix coefficients

Let G be a real, semisimple, compact and connected Lie group with Lie algebra $\mathfrak{g}$. Let $X := (X_1, \ldots, X_d)$ be any basis of $\mathfrak{g}$ which is orthogonal with respect to the Killing form and such that for some $0 < l \leq d$, $X' := (X_1, \ldots, X_l)$ is a basis of a Cartan subalgebra of $\mathfrak{g}$ which we fix and denote by $\mathfrak{t}$. Let W be the Weyl group. Let $(\pi, \mathcal{H})$ be a unitary irreducible representation of G of highest weight λ. We

write $d\pi$ for the Lie derivative of π. In this section, we study the support of the Weyl calculus of $\frac{1}{i}d\pi(X) := (\frac{1}{i}d\pi(X_1), \ldots, \frac{1}{i}d\pi(X_d))$.

Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let $\lambda_1, \ldots, \lambda_n \in \mathfrak{t}^*$ be the weights of π. Denote by $p : \mathfrak{g}^* \to \mathfrak{t}^*$ the canonical projection of $\mathfrak{g}^*$ onto $\mathfrak{t}^*$; i.e., $p(\xi) := \xi|_{\mathfrak{t}^*}$ for all $\xi \in \mathfrak{g}^*$.

We write $W_\pi(X)$ for $W_{L(\mathcal{H})}(\frac{1}{i}d\pi(X))$.

Definition 6.1. The **moment set** of π is the set
$$I_\pi := \{\frac{1}{i}\langle d\pi(\,\cdot\,)u,\, u\rangle \mid u \in \mathcal{H},\ \|u\| = 1\}.$$

Theorem 6.2 ([3, 14]). *Let π be an irreducible representation of G of highest weight λ. If the set $\{\lambda - w\lambda : w \in W\}$ does not contain a root, then $I_\pi = \mathrm{co}\,\mathcal{O}_\lambda$; otherwise, I_π is not convex and $I_\pi \subsetneq \mathrm{co}\,\mathcal{O}_\lambda$.*

Lemma 6.3. $p(I_\pi) = \mathrm{co}\,\{\lambda_1, \ldots, \lambda_n\}$.

Proposition 6.4. $\mathrm{co}\,\mathrm{supp}\,W_{\pi|_T}(X') = \mathrm{co}\,(W \cdot \lambda)$.

Proof. Clearly $I_{\pi|_T} = p(I_\pi)$. In view of Lemma 6.3, the result now follows from Theorem 3.11 and Theorem 3.2, and the fact that $\lambda_1, \ldots, \lambda_n \in \mathrm{co}\,(W \cdot \lambda)$. $\square$

Lemma 6.5. $\mathcal{O}_\lambda \subseteq I_\pi$.

Proof. We define the inclusion $\mathfrak{t}^* \subseteq \mathfrak{g}^*$ by the Killing form. Since I_π is invariant under Ad^* and $\mathcal{O}_\lambda \cap \mathfrak{t}^* = W \cdot \lambda$, the statement follows from the containment $W \cdot \lambda \subseteq I_\pi$. $\square$

Proposition 6.6 ([3], Proposition 16). $G \cdot \mathrm{co}\,\{\lambda_1, \ldots, \lambda_n\} = \mathrm{co}\,\mathcal{O}_\lambda$.

By Lemma 3.3, the numerical range $V_{L(\mathcal{H})}(\frac{1}{i}d\pi(X))$ does not depend on the choice of orthogonal basis X. We write V_π for $V_{L(\mathcal{H})}(\frac{1}{i}d\pi(X))$.

Theorem 6.7. $V_\pi = \mathrm{co}\,\mathcal{O}_\lambda$.

Proof. We have $I_\pi = G \cdot (I_\pi \cap \mathfrak{t}^*) \subseteq G \cdot p(I_\pi) = \mathrm{co}\,\mathcal{O}_\lambda$, where the last equality follows from Lemma 6.3 and Proposition 6.6. By Lemma 6.5 we have $\mathrm{co}\,\mathcal{O}_\lambda \subseteq \mathrm{co}\,I_\pi$. Hence $\mathrm{co}\,I_\pi = \mathrm{co}\,\mathcal{O}_\lambda$, and the result follows from Theorem 3.2. $\square$

Lemma 6.8. $\mathrm{supp}\,W_\pi(X) = G \cdot \mathrm{supp}\,W_\pi(X)$.

Proof. Suppose that $\xi \in \mathfrak{g}^* \setminus \mathrm{supp}\,W_\pi(X)$. Then there exists $\phi \in \mathcal{S}(\mathfrak{g}^*)$ such that ϕ is nonvanishing in a neighbourhood of ξ and $\langle W_\pi(X),\, \phi\rangle = 0$. Let $g \in G$. Set $\tilde{\phi} := \phi \circ Ad^*(g^{-1})$. Then $\tilde{\phi} \in \mathcal{S}(\mathfrak{g}^*)$ and $\tilde{\phi}$ is nonzero near $g \cdot \xi$. Now,

$$\begin{aligned}
\langle W_\pi(X),\, \tilde{\phi}\rangle &= \int_{\mathfrak{g}} \hat{\phi}(g \cdot X) e^{d\pi(X)}\, dX \\
&= \int_{\mathfrak{g}} \hat{\phi}(X) e^{d\pi(g^{-1} \cdot X)}\, dX \\
&= \pi(g^{-1}) \int_{\mathfrak{g}} \hat{\phi}(X) e^{d\pi(X)}\, dX\, \pi(g) \\
&= \pi(g^{-1})\langle W_\pi(X),\, \phi\rangle \pi(g) \\
&= 0.
\end{aligned}$$

Hence $g \cdot \xi \in \mathfrak{g}^* \setminus \mathrm{supp}\,W_\pi(X)$, and it follows that $G \cdot \mathrm{supp}\,W_\pi(X) \subseteq \mathrm{supp}\,W_\pi(X)$. The reverse inclusion is clear. This completes the proof. $\square$

Theorem 6.9. $G \cdot \operatorname{supp} W_{\pi|_T}(X') \subseteq \operatorname{supp} W_\pi(X)$.

Proof. By Corollary 4.7, $\operatorname{supp} W_{\pi|_T}(X') \subseteq \operatorname{supp} W_\pi(X)$, and the result follows by Lemma 6.8. □

Let A be a subset of $\mathbb{R}^d$. The set of extremal points of A, denoted $\operatorname{Ext} A$, is the collection of points in A that are not contained in any open interval, the endpoints of which lie in A.

Lemma 6.10 ([3], Lemme 15)**.** $\operatorname{Ext} \operatorname{co} \{\lambda_1 \ldots, \lambda_n\} = W \cdot \lambda$.

Corollary 6.11. $\operatorname{Ext} \operatorname{co} \mathcal{O}_\lambda = \mathcal{O}_\lambda$.

Theorem 6.12. $\operatorname{co} \operatorname{supp} W_\pi(X) = \operatorname{co} \mathcal{O}_\lambda$.

Proof. By Corollary 6.9,

$$
\begin{aligned}
\operatorname{co} \operatorname{supp} W_\pi(X) \quad &\supseteq \quad \operatorname{co} G \cdot \operatorname{supp} W_{\pi|_T}(X') \\
&\supseteq \quad G \cdot \operatorname{co} \operatorname{supp} W_{\pi|_T}(X') \\
&= \quad G \cdot V_{\pi|_T},
\end{aligned}
$$

where the last equality follows from Theorem 3.11. By Theorem 6.7, $V_{\pi|_T} = \operatorname{co} \{\lambda_1, \ldots, \lambda_n\}$, and we have $\operatorname{co} \operatorname{supp} W_\pi(X) \supseteq \operatorname{co} \mathcal{O}_\lambda$ by Lemma 6.6. By Theorem 3.7, $\operatorname{co} \operatorname{supp} W_\pi(X) \subseteq V_\pi$ and the reverse inclusion follows. This completes the proof. □

Theorem 6.13 ([8], Theorem 4.3.3)**.** *Let u_1, u_2 be compactly supported distributions on $\mathbb{R}^d$. Then*

(6.1) $$\operatorname{co} \operatorname{supp} u_1 * u_2 = \operatorname{co} \operatorname{supp} u_1 + \operatorname{co} \operatorname{supp} u_2.$$

Lemma 6.14. $\operatorname{co} \mathcal{O}_{\lambda+\delta} = \operatorname{co} \mathcal{O}_\lambda + \operatorname{co} \mathcal{O}_\delta$.

Proof. We have

$$
\begin{aligned}
\operatorname{Ext} (\operatorname{co} \mathcal{O}_\lambda + \operatorname{co} O_\delta) \quad &\subseteq \quad \operatorname{Ext} (\operatorname{Ext} \operatorname{co} \mathcal{O}_\lambda + \operatorname{Ext} \operatorname{co} \mathcal{O}_\delta) \\
&= \quad \operatorname{Ext} (\mathcal{O}_\lambda + \mathcal{O}_\delta) \\
&= \quad \mathcal{O}_{\lambda+\delta}.
\end{aligned}
$$

Therefore, by the Kreĭn-Milman theorem, $\operatorname{co} \mathcal{O}_{\lambda+\delta}$ is the closure of $\operatorname{co} \mathcal{O}_\lambda + \operatorname{co} \mathcal{O}_\delta$. Since the coadjoint orbits of G are compact and the convex hull of a closed set is closed, the result follows. □

Identifying the Lie algebra $\mathfrak{g}$ with $\mathbb{R}^{\dim G}$, let $\hat{j}$ denote the distributional Fourier transform of the j-function. It is well known that $\hat{j} = \mu_{\mathcal{O}_\delta}$ (see [4], Corollary 7.25).

Theorem 6.15. $\operatorname{co} \operatorname{supp} \hat{j} * W_\pi(X) = \operatorname{co} \mathcal{O}_{\lambda+\delta}$.

Proof. This is immediate from Theorem 6.13 and Theorem 6.12. □

Hence by Corollary 6.11, we reobtain the Kirillov-Duflo correspondence.

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