ON SUMS INVOLVING COEFFICIENTS OF AUTOMORPHIC L -FUNCTIONS

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ABSTRACT. Let $L(s, \pi)$ be the automorphic L -function associated to an automorphic irreducible cuspidal representation π of GL_m over $\mathbb{Q}$, and let $a_\pi(n)$ be the n th coefficient in its Dirichlet series expansion. In this paper we prove that if at every finite place p , π_p is unramified, then for any $\varepsilon > 0$,

$$A_\pi(x) = \sum_{n \leq x} a_\pi(n) \ll_{\varepsilon, \pi} \begin{cases} x^{\frac{71}{192} + \varepsilon} & \text{if } m = 2, \\ x^{\frac{m^2 - m}{m^2 + 1} + \varepsilon} & \text{if } m \geq 3. \end{cases}$$

1. INTRODUCTION AND MAIN RESULTS

Let $a(n)$ be an arithmetic function. It is an important problem in number theory to establish the asymptotic formula for the summatory function

$$A(x) = \sum_{n \leq x} a(n).$$

The asymptotic behavior of $A(x)$ is often closely linked with the analytic properties of the Dirichlet series

$$\mathcal{A}(s) = \sum_{n=1}^{\infty} a(n)n^{-s}.$$

The Langlands program predicts that the most general L -functions arise from automorphic representations of GL_n over a number field and that such L -functions can be decomposed into products of primitive automorphic L -functions arising from irreducible cuspidal representations of GL_m over $\mathbb{Q}$. Therefore in this paper we focus our attention on primitive automorphic L -functions of GL_m over $\mathbb{Q}$.

To be precise, let us recall some basic facts about primitive automorphic L -functions of GL_m over $\mathbb{Q}$ (see Godement and Jacquet [4], Jacquet and Shalika [8], or Rudnick and Sarnak [11]). Let π be an automorphic irreducible cuspidal

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representation of GL_m over $\mathbb{Q}$ with unitary central character. Then π is a restricted tensor product:

$$\pi = \otimes_p \pi_p.$$

To π one associates an Euler product

$$(1.1) \quad L(s, \pi) = \prod_p L(s, \pi_p)$$

given by a product of local factors. Outside of a finite set of primes, π_p is unramified. To every finite place p where π_p is unramified we associate a semisimple conjugacy class

$$A_\pi(p) = \begin{pmatrix} \alpha_{\pi,p}(1), & & \\ & \ddots & \\ & & \alpha_{\pi,p}(m) \end{pmatrix},$$

and we define the local L -function for the finite place p as

$$(1.2) \quad L(s, \pi_p) = \det(I - p^{-s} A_\pi(p))^{-1} = \prod_{j=1}^m (1 - \alpha_{\pi,p}(j) p^{-s})^{-1}.$$

It is possible to write the local factors at ramified primes p in the form of (1.2) with the convention that some of the $\alpha_{\pi,p}(j)$'s may be zero. In fact, the local factors at the ramified primes can best be described by the Langlands parameters of π_p .

The general Ramanujan conjectures for cuspidal automorphic representations π of GL_m over $\mathbb{Q}$ assert that for p unramified, $|\alpha_{\pi,p}(j)| = 1$. For certain π , this conjecture has been proved. But in general it is still open. In this direction, Serre [12] first observed that the analytic properties of the Rankin-Selberg L -function, in conjunction with Landau's lemma, can lead to

$$(1.3) \quad |\alpha_{\pi,p}(j)| \leq p^{1/2-1/(m^2+1)}.$$

For $m = 2$, this has been refined in [9] to

$$(1.4) \quad |\alpha_{\pi,p}(j)| \leq p^{\frac{7}{64}}.$$

The product (1.1) over primes gives a Dirichlet series representation: for $\mathrm{Res} > 1$,

$$(1.5) \quad L(s, \pi) = \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s}.$$

The aim of this paper is to study the summatory function for the coefficients $a_\pi(n)$ of automorphic L -functions attached to automorphic irreducible cuspidal representations of GL_m over $\mathbb{Q}$, i.e.

$$A_\pi(x) = \sum_{n \leq x} a_\pi(n).$$

Our main result is the following.

Theorem 1.1. *Let $L(s, \pi)$ be the automorphic L -function associated to an automorphic irreducible cuspidal representation π of GL_m over $\mathbb{Q}$, and let $a_\pi(n)$ be its n th coefficient in (1.5). If at every finite place p , π_p is unramified, then we have that for any $\varepsilon > 0$,*

$$A_\pi(x) = \sum_{n \leq x} a_\pi(n) \ll_{\varepsilon, \pi} \begin{cases} x^{\frac{71}{192} + \varepsilon} & \text{if } m = 2, \\ x^{\frac{m^2 - m}{m^2 + 1} + \varepsilon} & \text{if } m \geq 3, \end{cases}$$

where throughout this paper the notation $\ll_{\varepsilon, \pi}$ means that the implied constant depends on ε and π .

Our Theorem 1.1, for which the Ramanujan-Petersson conjecture is not known to hold, can be compared with the results of Iwaniec and Friedlander [3]: if the Ramanujan-Petersson conjecture is assumed, then the coefficients $a(n)$ of a general L -function of degree m with a functional equation and suitable analytic properties satisfy

$$\sum_{n \leq x} a(n) = \text{main term} + O_L(x^{\frac{m-1}{m+1}+\varepsilon}).$$

Our result can also be compared with one result of Miller [10], which states that for any $\varepsilon > 0$ and any real number α ,

$$\sum_{n \leq x} a(m, n) e(n\alpha) \ll_{\varepsilon, m, \Phi} x^{\frac{3}{4}+\varepsilon},$$

where $a(m, n)$ are the Fourier coefficients of a cusp form Φ for $GL(3, \mathbb{Z}) \backslash GL(3, \mathbb{R})$.

As an application of our Theorem 1.1, we shall consider the sum

$$\sum_{n \leq x} t(n^2),$$

where $t(n)$ is the n th normalized Fourier coefficient of a Hecke-Maass cusp form φ corresponding to the eigenvalue $l = \kappa^2 + \frac{1}{4}$ with respect to the full modular group $SL(2, \mathbb{Z})$, which coincides with the eigenvalue of the n th Hecke operator T_n .

Corollary 1.2. *Let $t(n)$ be the n th normalized Fourier coefficient of a Hecke-Maass cusp form φ with respect to the full modular group $SL(2, \mathbb{Z})$. Then for any $\varepsilon > 0$, we have*

$$S(x) = \sum_{n \leq x} t(n^2) \ll_{\varepsilon, \varphi} x^{\frac{3}{5}+\varepsilon},$$

where throughout this paper the notation $\ll_{\varepsilon, \varphi}$ means that the implied constant depends on ε and the Maass cusp form φ .

Our result improves a previous result given by Ivić [6]:

$$S(x) \ll_{\varphi} x \exp\left(-A(\log x)^{\frac{3}{5}}(\log \log x)^{-\frac{1}{5}}\right),$$

where $A > 0$ is a suitable constant.

2. THREE LEMMAS

To prove Theorem 1.1, we need the following three lemmas.

Lemma 2.1. *Let $L(f, s)$ be a Dirichlet series with Euler product of degree $m \geq 1$, which is defined by*

$$L(f, s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s} = \prod_{p < \infty} \prod_{j=1}^m \left(1 - \frac{\alpha_f(p, j)}{p^s}\right)^{-1},$$

where $\alpha_f(p, j)$, $j = 1, \dots, m$, are the local parameters of $L(f, s)$ at prime p . This series and Euler product are absolutely convergent for $\text{Re } s > 1$. Let the gamma

factor be given by

$$L_{\infty}(f, s) = \prod_{j=1}^m \pi^{-\frac{s+\mu_f(j)}{2}} \Gamma\left(\frac{s+\mu_f(j)}{2}\right),$$

where $\mu_f(j)$, $j = 1, \dots, m$, are the local parameters of $L(f, s)$ at ∞ . We also define the completed L -function $\Lambda(f, s)$ by

$$\Lambda(f, s) = q(f)^{\frac{s}{2}} L_{\infty}(f, s) L(f, s),$$

where $q(f)$ is the conductor of $L(f, s)$. We assume that $\Lambda(f, s)$ admits an analytic continuation to the whole complex plane $\mathbb{C}$ and is an entire function. Assume that it also satisfies a functional equation

$$\Lambda(f, s) = \epsilon_f \Lambda(\tilde{f}, 1-s)$$

where ϵ_f is the root number with $|\epsilon_f| = 1$ and $\tilde{f}$ is the dual of f such that $\lambda_{\tilde{f}}(n) = \overline{\lambda_f(n)}$, $\mu_{\tilde{f}}(j) = \overline{\mu_f(j)}$, and $q(\tilde{f}) = q(f)$.

Then for every $\eta \geq 0$ we have

$$\sum_{n \leq x} \lambda_f(n) \ll_f x^{\frac{1}{2} - \frac{1}{2m} + (\frac{m}{2} - \frac{1}{2})\eta} + \sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} |\lambda_f(n)|.$$

Proof. This is a special case of Theorem 4.1 in Chandrasekharan and Narasimhan [2] with

$$\delta = 1, \quad A = \frac{m}{2}, \quad \beta = 1, \quad u = \frac{1}{2} - \frac{1}{2m} \quad \text{and} \quad q = -\infty.$$

We reformulate it in the language used in Chapter 5 of Iwaniec and Kowalski [7].

Lemma 2.2. *With the same notation as in Lemma 2.1, we assume that the Dirichlet series $L(f, s)$ with Euler product of degree $m \geq 1$ has non-negative coefficients, i.e. $\lambda_f(n) \geq 0$, and converges for $\text{Re } s$ sufficiently large. Suppose further that $L(f, s)$ has a meromorphic continuation to $\mathbb{C}$ with, at most, poles of finite order at $s = 0, 1$. Assume also that $L(f, s)$ is of finite order and satisfies a functional equation*

$$\Lambda(f, s) = \epsilon_f \Lambda(f, 1-s).$$

Then we have that for any $\varepsilon > 0$,

$$\sum_{n \leq x} \lambda_f(n) = P(\log x)x + O_{\varepsilon, f}\left(x^{\frac{m-1}{m+1} + \varepsilon}\right),$$

where P is a polynomial depending only on L , whose degree equals the order of the pole of $L(f, s)$ at $s = 1$.

Proof. This is a refined version of Landau's lemma; see Barthel and Ramakrishnan [1].

Lemma 2.3. *Let $b(1), b(2), \dots$ be a sequence of complex numbers. Define the sequence $a(0) = 1, a(1), a(2), \dots$ by means of the formal identity*

$$\exp\left(\sum_{k=1}^{\infty} \frac{b(k)}{k} x^k\right) = \sum_{n=0}^{\infty} a(n) x^n.$$

For $j = 1$ or 2 , define the sequence $A_j(0) = 1$, $A_j(1)$, $A_j(2)$, ... by means of the formal identity

$$\exp\left(\sum_{k=1}^{\infty} \frac{|b(k)|^j}{k} x^k\right) = \sum_{n=0}^{\infty} A_j(n) x^n.$$

Then $A_j(n) \geq |a(n)|^j$.

Proof. See Lemma 3.1 in Soundararajan [13].

3. PROOF OF THEOREM 1.1

Associated with π , an automorphic representation of GL_m over $\mathbb{Q}$, there is also an Archimedean L -factor defined as

$$L(s, \pi_{\infty}) = \prod_{j=1}^m \pi^{-\frac{s+\mu_{\pi}(j)}{2}} \Gamma\left(\frac{s+\mu_{\pi}(j)}{2}\right),$$

where $\mu_{\pi}(j)$, $j = 1, 2, 3, \dots, m$, are local parameters at ∞ . In connection with (1.1), the completed L -function associated to π is defined by

$$\Lambda(s, \pi) = L(s, \pi_{\infty}) L(s, \pi).$$

This completed L -function has analytic continuation, is entire everywhere (note that in our case $m \geq 2$), and satisfies the functional equation

$$(3.1) \quad \Lambda(s, \pi) = \epsilon_{\pi} q_{\pi}^{\frac{1}{2}-s} \Lambda(1-s, \tilde{\pi}),$$

where $\tilde{\pi}$ is the contragredient of π , ϵ_{π} is a complex number of modulus 1, and q_{π} is a positive integer called the arithmetic conductor of π . For any place $p \leq \infty$, $\tilde{\pi}_p$ is equivalent to the complex conjugate $\overline{\pi_p}$, and we have

$$\{\alpha_{\tilde{\pi},p}(j)\} = \{\overline{\alpha_{\pi,p}(j)}\}, \quad \{\mu_{\tilde{\pi}}(j)\} = \{\overline{\mu_{\pi}(j)}\}.$$

Therefore, from Lemma 2.1 and (3.1), we have

$$(3.2) \quad A_{\pi}(x) = \sum_{n \leq x} a_{\pi}(n) \ll_{\pi} x^{\frac{1}{2}-\frac{1}{2m}+(\frac{m}{2}-\frac{1}{2})\eta} + \sum_{x < n \leq x+x^{1-\frac{1}{m}-\eta}} |a_{\pi}(n)|,$$

for every $\eta \geq 0$.

For $m = 2$, from (1.4) we have

$$(3.3) \quad |a_{\pi}(n)| \leq \tau(n) n^{\frac{7}{64}},$$

where $\tau(n)$ is the divisor function. From (3.2) with $m = 2$, we have

$$(3.4) \quad A_{\pi}(x) = \sum_{n \leq x} a_{\pi}(n) \ll_{\pi} x^{\frac{1}{4}+\frac{1}{2}\eta} + \sum_{x < n \leq x+x^{\frac{1}{2}-\eta}} |a_{\pi}(n)|.$$

From (3.3), we obtain

$$(3.5) \quad A_{\pi}(x) \ll_{\pi} x^{\frac{1}{4}+\frac{1}{2}\eta} + x^{\frac{39}{64}-\eta+\varepsilon}.$$

On taking $\eta = \frac{23}{96}$, we have

$$(3.6) \quad A_{\pi}(x) \ll_{\pi} x^{\frac{71}{192}+\varepsilon}.$$

In order to give the result for $m \geq 3$, we recall some basic facts about the Rankin-Selberg L -function $L(s, \pi \times \tilde{\pi})$ associated to π and its contragredient $\tilde{\pi}$. It is defined as a product of local factors:

$$(3.7) \quad L(s, \pi \times \tilde{\pi}) = \prod_p L(s, \pi_p \times \tilde{\pi}_p).$$

For unramified primes p , the local factor is given by

$$(3.8) \quad L(s, \pi_p \times \tilde{\pi}_p) = \prod_{j=1}^m \prod_{k=1}^m (1 - \alpha_{\pi,p}(j) \overline{\alpha_{\pi,p}(k)} p^{-s})^{-1}.$$

It can be defined similarly at primes p where π_p is ramified. By (1.3), the product $\prod_p L(s, \pi_p \times \tilde{\pi}_p)$ converges absolutely on $\text{Res} > 2 - \frac{2}{m^2+1}$ (in fact on $\text{Res} > 1$; see e.g. Jacquet and Shalika [8] or Rudnick and Sarnak [11]). We write this product as a Dirichlet series:

$$(3.9) \quad L(s, \pi \times \tilde{\pi}) = \prod_p \sum_{k=0}^{\infty} \frac{a_{\pi \times \tilde{\pi}}(p^k)}{p^{ks}} = \sum_{n=1}^{\infty} \frac{a_{\pi \times \tilde{\pi}}(n)}{n^s}.$$

The completed Rankin-Selberg L -function is defined by

$$\Lambda(s, \pi \times \tilde{\pi}) = L(s, \pi_{\infty} \times \tilde{\pi}_{\infty}) L(s, \pi \times \tilde{\pi})$$

with

$$L(s, \pi_{\infty} \times \tilde{\pi}_{\infty}) = \prod_{j=1}^{m^2} \pi^{-\frac{s + \mu_{\pi \times \tilde{\pi}}(j)}{2}} \Gamma\left(\frac{s + \mu_{\pi \times \tilde{\pi}}(j)}{2}\right).$$

When π_{∞} is unramified,

$$\{\mu_{\pi \times \tilde{\pi}}(j)\}_{1 \leq j \leq m^2} = \{\mu_{\pi}(j) + \overline{\mu_{\pi}(k)}\}_{1 \leq j \leq m, 1 \leq k \leq m}.$$

It is known that $a_{\pi \times \tilde{\pi}}(n) \geq 0$ and $L(s, \pi \times \tilde{\pi})$ has a simple pole at $s = 1$. The completed Rankin-Selberg L -function $\Lambda(s, \pi \times \tilde{\pi})$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation

$$\Lambda(s, \pi \times \tilde{\pi}) = \epsilon_{\pi \times \tilde{\pi}} q_{\pi \times \tilde{\pi}}^{\frac{1}{2}-s} \Lambda(1-s, \pi \times \tilde{\pi}),$$

where $|\epsilon_{\pi \times \tilde{\pi}}| = 1$ and $q_{\pi \times \tilde{\pi}} > 0$.

Therefore by applying Lemma 2.2 to $L(s, \pi \times \tilde{\pi})$ with degree m^2 , we have

$$(3.10) \quad \sum_{n \leq x} a_{\pi \times \tilde{\pi}}(n) = c_{\pi} x + O_{\varepsilon, \pi}(x^{\frac{m^2-1}{m^2+1} + \varepsilon}),$$

where c_{π} is a positive constant.

From (3.10), we find that for any $\eta \geq 0$,

$$(3.11) \quad \sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} a_{\pi \times \tilde{\pi}}(n) \ll_{\varepsilon, \pi} x^{\frac{m^2-1}{m^2+1} + \varepsilon}.$$

From (3.7), (3.8) and (3.9), we have that for $\text{Res} > 2 - \frac{2}{m^2+1}$,

$$(3.12) \quad \sum_{k=0}^{\infty} \frac{a_{\pi \times \tilde{\pi}}(p^k)}{p^{ks}} = \exp\left(\sum_{v=1}^{\infty} \frac{|\lambda_{\pi}(p^v)|^2}{v} p^{-vs}\right),$$

where

$$\lambda_{\pi}(p^v) = \sum_{j=1}^m \alpha_{\pi,p}(j)^v.$$

From (1.1), (1.2) and (1.5), we have

$$(3.13) \quad \sum_{k=0}^{\infty} \frac{a_{\pi}(p^k)}{p^{ks}} = \exp \left(\sum_{v=1}^{\infty} \frac{\lambda_{\pi}(p^v)}{v} p^{-vs} \right).$$

From (3.12), (3.13) and Lemma 2.3 with $j = 2$, we have that for an unramified prime p ,

$$|a_{\pi}(p^k)|^2 \leq a_{\pi \times \tilde{\pi}}(p^k),$$

and thus in our case

$$|a_{\pi}(n)|^2 \leq a_{\pi \times \tilde{\pi}}(n).$$

Therefore we have

$$(3.14) \quad \sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} |a_{\pi}(n)|^2 \ll \sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} a_{\pi \times \tilde{\pi}}(n).$$

Now we begin to estimate (3.2). By Cauchy's inequality, we find that the short-interval sum in (3.2) satisfies

$$(3.15) \quad \sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} |a_{\pi}(n)| \leq \left(\sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} |a_{\pi}(n)|^2 \right)^{\frac{1}{2}} \left(\sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} 1 \right)^{\frac{1}{2}}.$$

By (3.11) and (3.14), we have

$$(3.16) \quad \sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} |a_{\pi}(n)|^2 \ll_{\varepsilon, \pi} x^{\frac{m^2-1}{m^2+1} + \varepsilon}.$$

From (3.15) and (3.16), we obtain

$$(3.17) \quad \sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} |a_{\pi}(n)| \ll_{\varepsilon, \pi} x^{\frac{1}{2} - \frac{1}{2m} - \frac{\eta}{2} + \frac{m^2-1}{2m^2+2} + \varepsilon}.$$

Inserting (3.17) into (3.2), we have

$$A_{\pi}(x) = \sum_{n \leq x} a_{\pi}(n) \ll_{\varepsilon, \pi} x^{\frac{1}{2} - \frac{1}{2m} + (\frac{m}{2} - \frac{1}{2})\eta} + x^{\frac{1}{2} - \frac{1}{2m} - \frac{\eta}{2} + \frac{m^2-1}{2m^2+2} + \varepsilon}.$$

On taking $\eta = \frac{m^2-1}{m(m^2+1)}$, we get

$$A_{\pi}(x) \ll_{\varepsilon, \pi} x^{\frac{m^2-m}{m^2+1} + \varepsilon}.$$

This completes the proof of Theorem 1.1.

4. PROOF OF COROLLARY 1.2

To prove Corollary 1.2, we recall some basic facts from the books of Iwaniec and Kowalski [7], and of Goldfeld [5]. Associated to each Hecke-Maass cusp form φ for the full modular group $SL(2, \mathbb{Z})$ there is an L -function $L(\varphi, s)$, which is defined, for $\text{Res} > 1$, by

$$\begin{aligned} L(\varphi, s) &= \sum_{n=1}^{\infty} t(n)n^{-s} = \prod_p (1 - t(p)p^{-s} + p^{-2s})^{-1} \\ &= \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\alpha'_p}{p^s}\right)^{-1} \end{aligned}$$

with $\alpha_p + \alpha'_p = t(p)$ and $\alpha_p \alpha'_p = 1$. The symmetric square L -function $L(\text{Sym}^2 \varphi, s)$ is defined, for $\text{Res} > 1$, by

$$\begin{aligned} L(\text{Sym}^2 \varphi, s) &= \zeta(2s) \sum_{n=1}^{\infty} t(n^2)n^{-s} \\ &= \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p'^2}{p^s}\right)^{-1}, \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. Then we have

$$(4.1) \quad \sum_{n=1}^{\infty} t(n^2)n^{-s} = \frac{L(\text{Sym}^2 \varphi, s)}{\zeta(2s)}.$$

This gives

$$(4.2) \quad t(n^2) = \sum_{d^2|n} \mu(d) t^{(2)}\left(\frac{n}{d^2}\right),$$

where $t^{(2)}(n)$ is the n th coefficient of the symmetric square L -function $L(\text{Sym}^2 \varphi, s)$ with $\text{Res} > 1$.

It follows from the Gelbart-Jacquet lift that $L(\text{Sym}^2 \varphi, s)$ is an automorphic L -function of GL_3 . Then from Theorem 1.1 with $m = 3$, we have

$$(4.3) \quad \sum_{n \leq x} t^{(2)}(n) \ll x^{\frac{3}{5} + \varepsilon}.$$

From (4.2) and (4.3), we have

$$S(x) = \sum_{n \leq x} t(n^2) \ll x^{\frac{3}{5} + \varepsilon}.$$

This completes the proof of Corollary 1.2.

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