PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 10, October 2009, Pages 3543–3545
S 0002-9939(09)09895-5
Article electronically published on May 6, 2009

HAVING CUT-POINTS IS NOT A WHITNEY REVERSIBLE PROPERTY

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(Communicated by Alexander N. Dranishnikov)

Abstract. We show that the property of having cut-points is not a Whitney reversible property. This answers in the negative a question posed by Illanes and Nadler.

1. Introduction

In this note, all spaces are separable metrizable spaces and maps are continuous. We denote the interval [0, 1] by I. A compact metric space is called a *compactum*, and *continuum* means a connected compactum. If X is a continuum, $C(X)$ denotes the space of all subcontinua of X with the topology generated by the Hausdorff metric.

A topological property P is called a *Whitney property* provided that if a continuum X has property P, so does $\mu^{-1}(t)$ for each Whitney map (see p. 105 of [2]) μ for $C(X)$ and for each $t \in [0, \mu(X))$. A topological property P' is called a *Whitney reversible property* provided that whenever X is a continuum such that $\mu^{-1}(t)$ has property P' for all Whitney maps μ for $C(X)$ and all $t \in (0, \mu(X))$, then X has property P'. These properties have been studied by many authors (see [2]).

A point p in a continuum X is called a *cut-point of* X provided that $X \setminus \{p\}$ is disconnected. In this paper we prove that the property of having cut-points is not a Whitney reversible property (it is known that the property of having cut-points is not a Whitney property; see Exercise 43.4 of [2]). This answers in the negative question 43.3 of [2] posed by Illanes and Nadler.

2. Main theorem

A map $f : X \to Y$ between continua is called an *atomic map* if $f^{-1}(f(A)) = A$ for each $A \in C(X)$ such that $f(A)$ is nondegenerate.

A subcontinuum T of a continuum X is *terminal* if every subcontinuum of X which intersects both T and its complement must contain T. It is known that a map f of a continuum X onto a continuum Y is atomic if and only if every fiber of f is a terminal subcontinuum of X.

The main aim of this paper is to prove Theorem 2.2. To prove this theorem, we need the next result, proved by Anderson [1].

Received by the editors December 23, 2008, and, in revised form, January 6, 2009.
2000 *Mathematics Subject Classification.* Primary 54B20; Secondary 54F15.
Key words and phrases. Whitney reversible property, cut-point, terminal continuum, atomic map.

Lemma 2.1 (see Theorem of [1]). *For each continuum Y, there exist a continuum X and a monotone open map $f : X \to Y$ such that $f^{-1}(y)$ is a nondegenerate terminal subcontinuum of X for each $y \in Y$.*

Theorem 2.2. *There exists a continuum Z such that:*

(A) Z does not have a cut-point, and

(B) $\mu^{-1}(s)$ has a cut-point for each Whitney map $\mu : C(Z) \to [0, \mu(Z)]$ and for each $s \in (0, \mu(Z))$.

Proof. By Lemma 2.1 there exist a continuum X and a monotone open map $f : X \to I$ such that $f^{-1}(y)$ is a nondegenerate terminal subcontinuum of X for each $y \in I$. Let Z be the quotient space obtained from X by shrinking $f^{-1}(1)$ to the point. Let $p : X \to Z$ be the natural projection and $q = f \circ p^{-1} : Z \to I$. Note that q is a monotone open map such that $q^{-1}(1)$ is a one-point set and $q^{-1}(y)$ is a nondegenerate terminal subcontinuum of Z for each $y \in [0, 1)$ (hence q is atomic).

We show that Z has the required properties. At first we prove that Z does not have a cut-point. Let $z \in Z$. If $\{z\} = q^{-1}(1)$, then $Z \setminus \{z\} = q^{-1}([0, 1))$. Since q is monotone, $q^{-1}([0, 1))$ is connected. Hence in this case z is not a cut-point of Z. Assume that $z \in q^{-1}(t)$ for some $t \in [0, 1)$ and that z is a cut-point of Z. Then there exist nonempty open subsets $O, H \subset Z$ such that $Z \setminus \{z\} = O \cup H$ and $O \cap H = \emptyset$. Then $O \cup \{z\}$ and $H \cup \{z\}$ are nondegenerate continua (see Proposition 6.3 of [3]). We may assume that $q^{-1}(t) \cap H \neq \emptyset$. Since $O \cup \{z\}$ is a nondegenerate continuum, there exists $\{z_i\}_{i=1}^{\infty} \subset O$ such that $\lim_{i\to\infty} z_i = z$. For each $i = 1, 2, ...$, let $t_i = q(z_i)$. Note that $q^{-1}(t_i) \subset O$ for each $i = 1, 2, ...$ Since q is an open map, $\lim_{i\to\infty} q^{-1}(t_i) = q^{-1}(t)$. Then $q^{-1}(t) \subset O \cup \{z\}$. This is a contradiction because $q^{-1}(t) \cap H \neq \emptyset$. So Z does not have a cut-point.

Next we prove that $\mu^{-1}(s)$ has a cut-point for each Whitney map $\mu : C(Z) \to [0, \mu(Z)]$ and for each $s \in (0, \mu(Z))$. Take $a, b \in (0, 1)$ such that $\mu(q^{-1}([a, b])) = s$ (this is possible because q is a monotone open map and $q^{-1}(1)$ is a one-point set). Now we show that

$$\mu^{-1}(s) = \{C \in \mu^{-1}(s) \mid C \subset q^{-1}([0, b])\} \cup \{C \in \mu^{-1}(s) \mid C \subset q^{-1}([a, 1])\}. \tag{1}$$

If not, there exists $D \in \mu^{-1}(s)$ such that $D \cap q^{-1}([0, a)) \neq \emptyset \neq D \cap q^{-1}((b, 1])$. Then $q(D)$ contains $[a, b]$ as a proper subcontinuum of $q(D)$. Since q is atomic, $D = q^{-1}(q(D))$. So D contains $q^{-1}([a, b])$ as a proper subcontinuum of D. This is a contradiction because $\{D, q^{-1}([a, b])\} \subset \mu^{-1}(s)$. Hence (1) holds.

It is easy to see that

$$\{C \in \mu^{-1}(s) \mid C \subset q^{-1}([0, b])\} \cap \{C \in \mu^{-1}(s) \mid C \subset q^{-1}([a, 1])\} = \{q^{-1}([a, b])\} \tag{2}$$

and

$$\{C \in \mu^{-1}(s) \mid C \subset q^{-1}([0, b])\} \quad \text{and} \quad \{C \in \mu^{-1}(s) \mid C \subset q^{-1}([a, 1])\} \tag{3}$$

are nondegenerate subcontinua of $\mu^{-1}(s)$.

By (1), (2) and (3) we see that $q^{-1}([a, b])$ is a cut-point of $\mu^{-1}(s)$. □

By this result, we see that the property of having cut-points is not a Whitney reversible property.

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