MINIMAL GRAPHS IN $\mathbb{R}^4$ WITH BOUNDED JACOBIANS

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ABSTRACT. We obtain a Bernstein type result for entire two dimensional minimal graphs in $\mathbb{R}^4$, which extends a previous result due to L. Ni. Moreover, we provide a characterization for complex analytic curves.

1. INTRODUCTION

The famous theorem of Bernstein states that the only entire minimal graphs in the Euclidean space $\mathbb{R}^3$ are planes. More precisely, if $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is an entire (i.e., defined over all of $\mathbb{R}^2$) smooth function whose graph

$$G_f := \{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2\}$$

is a minimal surface, then it is an affine function, and the graph is a plane.

This type of result has been generalized in higher dimension and codimension under various conditions. See [1], [3], [9] and the references therein for the codimension one case and [5], [10], [12] for the higher codimension case.

The aim of this paper is to study the following special case. Let M be a minimal surface in $\mathbb{R}^4$ that can be described as the graph of an entire and smooth vector valued function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $f(x, y) = (f_1(x, y), f_2(x, y))$, that is,

$$M = G_f := \{(x, y, f_1(x, y), f_2(x, y)) \in \mathbb{R}^4 : (x, y) \in \mathbb{R}^2\}.$$

The following question arises in a natural way: *Is it true that the graph G_f of f is a plane in $\mathbb{R}^4$?* In general, the answer is no. An easy counterexample is given by the function $f(x, y) = (x^2 - y^2, 2xy)$. Actually, the graph of any holomorphic function $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ is a minimal surface. So, the problem of finding geometric conditions in order to have a result of Bernstein type is reasonable. In this direction, R. Schoen [8] obtained a Bernstein type result by imposing the assumption that $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism. Moreover, L. Ni [6] by using the result of R. Schoen [8] and results due to J. Wolfson [11] on minimal Lagrangian surfaces, has derived a result of Bernstein type under the assumption that f is an *area-preserving map*; that is, the *Jacobian* $J_f := \det(df)$ satisfies $J_f = 1$, where df denotes the differential of f .

In this paper we prove, first, the following result of Bernstein type, which generalizes the result due to L. Ni.

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Theorem 1.1. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an entire smooth vector valued function such that its graph G_f is a minimal surface in $\mathbb{R}^4$. If the Jacobian J_f of f is bounded, then G_f is a plane.*

As a consequence we derive an easy alternative proof of the following well-known result of Jörgens [4]:

Theorem 1.2 (Jörgens' Theorem). *The only entire solutions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of the Monge-Ampere equation $f_{xx}f_{yy} - f_{xy}^2 = 1$ are the quadratic polynomials.*

There are plenty of entire minimal graphs in $\mathbb{R}^4$, the so called *complex analytic curves*. More precisely, if $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ is any entire holomorphic or anti-holomorphic function, then the graph

$$G_\Phi = \{(z, \Phi(z)) \in \mathbb{C}^2 : z \in \mathbb{C}\}$$

of Φ in $\mathbb{C}^2 = \mathbb{R}^4$ is a minimal surface and is called a complex analytic curve. Such surfaces are locally characterized (see for example L.P. Eisenhart [2]) by the relation $|K| = |K_N|$, where K and K_N stand for the Gauss and normal curvature of the surface, respectively. The following result holds true.

Theorem 1.3. *Let G_f be the graph of an entire smooth vector valued function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with Gaussian curvature K and normal curvature K_N . Assume that G_f is minimal in $\mathbb{R}^4$. Then,*

$$\inf_{K < 0} \frac{|K_N|}{|K|} = 0,$$

unless G_f is a complex analytic curve.

Corollary 1.4. *Let G_f be the graph of an entire smooth vector valued function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with Gaussian curvature K and normal curvature K_N . If $K_N = cK$, where c is a constant, then G_f is a complex analytic curve. More precisely, $K_N = K = 0$ and G_f is a plane, or $|c| = 1$ and G_f is a non-trivial complex analytic curve.*

2. BASIC NOTATION AND DEFINITIONS

A surface M in the Euclidean space $\mathbb{R}^n$ is represented, locally, by a transformation $X : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^n$ of rank 2, given by

$$X(x, y) = (f_1(x, y), f_2(x, y), \dots, f_n(x, y)), \quad (x, y) \in D,$$

where D is an open subset of $\mathbb{R}^2$ and $f_i : D \rightarrow \mathbb{R}$, $i \in \{1, \dots, n\}$, are smooth functions. Following the standard notation of differential geometry, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product on $\mathbb{R}^n$ and by E, F, G the *coefficients of the first fundamental form*, which are given by

$$E = \sum_{i=1}^n \left(\frac{\partial f_i}{\partial x} \right)^2, \quad F = \sum_{i=1}^n \frac{\partial f_i}{\partial x} \frac{\partial f_i}{\partial y}, \quad G = \sum_{i=1}^n \left(\frac{\partial f_i}{\partial y} \right)^2.$$

We recall that the parameters (x, y) are called *isothermal* if and only if $E = G$ and $F = 0$, everywhere on D .

Consider a local orthonormal frame field $\{e_1, e_2; \xi_3, \dots, \xi_n\}$ in $\mathbb{R}^n$ such that, restricted to M , the vectors e_1, e_2 are tangent to M and, consequently, $\xi_3, \dots, \xi_n$ are normal to M . Denote by $\bar{\nabla}$ the usual linear connection on $\mathbb{R}^n$ and let

$$h_{ij}^\alpha = \langle \bar{\nabla}_{e_i} \xi_\alpha, e_j \rangle, \quad i, j \in \{1, 2\}, \quad \alpha \in \{3, \dots, n\},$$

be the *coefficients of the second fundamental form*.

The *mean curvature vector* H and the *Gauss curvature* K of M are given, respectively, by

$$\begin{aligned} H &= \frac{1}{2} \sum_{\alpha=3}^n (h_{11}^\alpha + h_{22}^\alpha) \xi_\alpha, \\ K &= \sum_{\alpha=3}^n \left(h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2 \right). \end{aligned}$$

Moreover, if

$$|h|^2 = \sum_{i,j=1}^2 \sum_{\alpha=3}^n (h_{ij}^\alpha)^2$$

is the square of the length of the second fundamental form h , then the Gauss equation implies

$$2K = 4H^2 - |h|^2.$$

In the case where M is *minimal*, i.e., $H = 0$, the above become

$$(2.1) \quad K = - \sum_{\alpha=3}^n \left\{ (h_{11}^\alpha)^2 + (h_{12}^\alpha)^2 \right\},$$

$$(2.2) \quad 2K = -|h|^2.$$

Another geometric invariant which plays an important role in the theory of surfaces in $\mathbb{R}^4$ is the *normal curvature* K_N of M which is given by

$$K_N = \sum_{i=1}^2 (h_{i1}^3 h_{2i}^4 - h_{2i}^3 h_{1i}^4).$$

In particular, for minimal surfaces we have

$$(2.3) \quad K_N = 2 (h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4).$$

One of the simplest ways to express a surface in $\mathbb{R}^{n+2}$ is in *non-parametric form*, that is to say, as the graph

$$G_f = \{ (x, y, f_1(x, y), \dots, f_n(x, y)) \in \mathbb{R}^{n+2} : (x, y) \in D \}$$

of a vector valued map $f : D \rightarrow \mathbb{R}^n$, $f(x, y) = (f_1(x, y), \dots, f_n(x, y))$, where D is an open subset of $\mathbb{R}^2$. Of course, any surface can be locally described in this manner. By computing the Euler-Lagrange equations for the area integral we see that the surface G_f is minimal if and only if f satisfies the following equation:

$$(2.4) \quad \left(1 + |f_y|^2 \right) f_{xx} - 2 \langle f_x, f_y \rangle f_{xy} + \left(1 + |f_x|^2 \right) f_{yy} = 0.$$

This is the classical *non-parametric minimal surface equation*.

The following result due to R. Osserman [7, Theorem 5.1] is the main tool for the proofs of our results.

Theorem 2.1. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an entire solution of the minimal surface equation. Then there exist real constants a, b , with $b > 0$, and a non-singular linear transformation*

$$x = u, \quad y = au + bv,$$

such that (u, v) are global isothermal parameters for the surface G_f .

Moreover the following identity is useful in the proofs.

Lemma 2.2 (Lagrange's Identity). *For any two vectors $V = (v_1, \dots, v_n)$, $W = (\omega_1, \dots, \omega_n)$ in $\mathbb{R}^n$ we have*

$$\left(\sum_{i=1}^n v_i^2\right) \left(\sum_{i=1}^n \omega_i^2\right) - \left(\sum_{i=1}^n v_i \omega_i\right)^2 = \sum_{i < j} (v_i \omega_j - v_j \omega_i)^2.$$

3. PROOFS OF THE RESULTS

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $f(x, y) = (f_1(x, y), f_2(x, y))$, $(x, y) \in \mathbb{R}^2$, be an entire solution of the minimal surface equation. Then, the graph

$$G_f = \{(x, y, f_1(x, y), f_2(x, y)) \in \mathbb{R}^4 : (x, y) \in \mathbb{R}^2\}$$

of f is a minimal surface in $\mathbb{R}^4$. By virtue of the Theorem 2.1, we can introduce global isothermal parameters (u, v) , via a non-singular transformation

$$x = u, \quad y = au + bv,$$

where a, b are real constants with $b > 0$. Now, the minimal surface G_f is parametrized via the map

$$X(u, v) = (u, au + bv, \varphi(u, v), \psi(u, v)),$$

where $\varphi(u, v) := f_1(u, au + bv)$ and $\psi(u, v) := f_2(u, au + bv)$. Since (u, v) are isothermal parameters, the vectors

$$(3.1) \quad X_u = (1, a, \varphi_u, \psi_u), \quad X_v = (0, b, \varphi_v, \psi_v)$$

are orthogonal and of the same length, that is,

$$(3.2) \quad \begin{aligned} \varphi_u \varphi_v + \psi_u \psi_v &= -ab, \\ E &= 1 + a^2 + \varphi_u^2 + \psi_u^2 = b^2 + \varphi_v^2 + \psi_v^2. \end{aligned}$$

Moreover, the fact that the surface G_f is minimal implies that the functions φ and ψ are harmonic, that is,

$$(3.3) \quad \varphi_{uu} + \varphi_{vv} = 0, \quad \psi_{uu} + \psi_{vv} = 0.$$

Appealing to Lagrange's Identity and taking the relations (3.2) into account, we obtain

$$\begin{aligned} E^2 &= b^2 + \varphi_v^2 + \psi_v^2 + (a\varphi_v - b\varphi_u)^2 \\ &\quad + (a\psi_v - b\psi_u)^2 + (\varphi_u\psi_v - \varphi_v\psi_u)^2 \end{aligned}$$

or, equivalently,

$$(3.4) \quad E^2 = (1 + a^2 + b^2) E - b^2 + (\varphi_u\psi_v - \varphi_v\psi_u)^2.$$

We set $\Phi(u, v) = (\varphi(u, v), \psi(u, v))$. Because of the relation

$$\frac{\partial(\varphi, \psi)}{\partial(u, v)} = \frac{\partial(f_1, f_2)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u, v)}$$

for the Jacobians, we have

$$J_\Phi = bJ_f,$$

where J_f, J_Φ stand for the Jacobians of f and Φ , respectively. So (3.4) becomes

$$(3.5) \quad J_\Phi^2 = E^2 - (1 + a^2 + b^2) E + b^2,$$

a useful identity for us.

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Since the Jacobian J_Φ is bounded, we conclude from (3.5) that E , and thus $\log E$, is bounded from above. On the other hand, the Gaussian curvature K of G_f is given by

$$K = -\frac{\Delta \log E}{2E},$$

where Δ is the usual Laplacian operator on the (u, v) -plane. Appealing to (2.1), we deduce that the Gaussian curvature K is non-positive. Consequently, $\Delta \log E \geq 0$, and thus the function $\log E$ is a subharmonic function defined on the whole plane. Since $\log E$ is also bounded from above, we deduce that E is constant and consequently K is identically zero. Then it follows immediately from (2.2) that the graph G_f of f is totally geodesic and hence a plane. $\square$

Remark 3.1. In a similar way, we can prove the following result: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^n$,

$$f(x, y) = (f_1(x, y), f_2(x, y), \dots, f_n(x, y)),$$

be a vector valued function, defined on the whole $\mathbb{R}^2$, which is a solution of the minimal surface equation. If the quantity

$$\sum_{i < j} \left(\frac{\partial f_i}{\partial x} \frac{\partial f_j}{\partial y} - \frac{\partial f_j}{\partial x} \frac{\partial f_i}{\partial y} \right)^2$$

is bounded, then the graph G_f of f is a plane in $\mathbb{R}^{n+2}$.

Now, we show that one can get Jörgens' well-known result [4] as a consequence of Theorem 1.1.

Proof of Jörgens' Theorem. Obviously $f_{xx} + f_{yy} \neq 0$ everywhere on $\mathbb{R}^2$. By virtue of classical regularity results for elliptic equations, $f(x, y)$ actually is a real analytic function. We consider the function $\Theta : \mathbb{R}^2 \rightarrow \mathbb{R}$, given by

$$\Theta = \frac{f_{xx}f_{yy} - f_{xy}^2 - 1}{f_{xx} + f_{yy}}.$$

The function Θ , thanks to our assumption, is identically zero, so $\Theta_x = \Theta_y = 0$. On the other hand, one can readily verify that the equations $\Theta_x = \Theta_y = 0$ are equivalent to the minimal surface equation (2.4) for the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by $g(x, y) = (f_x(x, y), f_y(x, y))$. Moreover, we have $J_g = 1$. So, according to Theorem 1.1, the graph G_g of g is a plane and the result is immediate. $\square$

For the proof of Theorem 1.3, we need the following auxiliary result.

Lemma 3.2. *Let $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\Phi(u, v) = (\varphi(u, v), \psi(u, v))$, be a map, where φ and ψ are harmonic functions on $\mathbb{R}^2$, i.e., a harmonic map. Then $\inf |J_\Phi| = 0$, unless Φ is an affine map.*

Proof. Suppose to the contrary that Φ is not affine and $\inf |J_\Phi| = c > 0$. Hence $|J_\Phi| \geq c > 0$. Assume at first that $J_\Phi \geq c > 0$. We view Φ as a complex valued function $\Phi : \mathbb{C} \rightarrow \mathbb{C}$, $\Phi = \varphi + i\psi$. Then, for $z = u + iv$, we have

$$\Phi_z = \frac{1}{2}(\varphi_u + \psi_v) + \frac{i}{2}(\psi_u - \varphi_v)$$

and

$$\Phi_{\bar{z}} = \frac{1}{2}(\varphi_u - \psi_v) + \frac{i}{2}(\psi_u + \varphi_v).$$

A simple calculation shows that

$$(3.6) \quad J_{\Phi} = |\Phi_z|^2 - |\Phi_{\bar{z}}|^2.$$

Furthermore, since φ and ψ are harmonic functions it follows that the function Φ_z is holomorphic. From our assumption and (3.6) we get

$$(3.7) \quad |\Phi_z|^2 \geq |\Phi_{\bar{z}}|^2 + c \geq c > 0.$$

Since Φ_z is an entire holomorphic function, Picard's Theorem implies that Φ_z must be constant. Therefore, there are real constants κ, λ such that

$$\varphi_u + \psi_v = 2\kappa \quad \text{and} \quad \psi_u - \varphi_v = 2\lambda.$$

Then from (3.7) we deduce that

$$(\psi_v - \kappa)^2 + (\psi_u - \lambda)^2 \leq \kappa^2 + \lambda^2 - c.$$

By the harmonicity of the functions $\psi_v - \kappa$, $\psi_u - \lambda$ and Liouville's Theorem, we deduce that φ and ψ are affine functions, which contradicts our assumptions.

Assume now that $J_{\Phi} \leq -c < 0$. In this case, we consider the complex valued function $\tilde{\Phi} = \psi + i\varphi$. Since $J_{\tilde{\Phi}} = -J_{\Phi} \geq c > 0$, proceeding as above we deduce that $\tilde{\Phi}$ is affine, and consequently Φ is affine. This is again a contradiction. Thus $\inf |J_{\Phi}| = 0$, and the proof is concluded. $\square$

Proof of Theorem 1.3. Assume that G_f is not a plane and that

$$\inf_{K < 0} \frac{|K_N|}{|K|} > 0.$$

We introduce global isothermal parameters (u, v) such that the minimal surface G_f is parametrized via the map

$$X(u, v) = (u, au + bv, \varphi(u, v), \psi(u, v)),$$

where a, b are real constants with $b > 0$.

We claim that $(a, b) = (0, 1)$. Arguing indirectly, we assume that $(a, b) \neq (0, 1)$. Differentiating (3.2) with respect to u, v and taking (3.3) into account, we find

$$(3.8) \quad \begin{aligned} \varphi_{uu}\varphi_v + \varphi_u\varphi_{uv} &= -\psi_{uu}\psi_v - \psi_u\psi_{uv}, \\ \varphi_{uu}\varphi_u - \varphi_v\varphi_{uv} &= -\psi_{uu}\psi_u + \psi_v\psi_{uv}. \end{aligned}$$

Squaring both of them and summing we obtain

$$(3.9) \quad (\varphi_u^2 + \varphi_v^2)(\varphi_{uu}^2 + \varphi_{uv}^2) = (\psi_u^2 + \psi_v^2)(\psi_{uu}^2 + \psi_{uv}^2).$$

Consider the subset of $\mathbb{R}^2$,

$$M_0 = \{(u, v) \in \mathbb{R}^2 : \omega(u, v) = 0\},$$

where

$$\omega(u, v) := (\varphi_u^2 + \varphi_v^2)(\varphi_{uu}^2 + \varphi_{uv}^2),$$

or, equivalently, in view of (3.9),

$$\omega(u, v) = (\psi_u^2 + \psi_v^2)(\psi_{uu}^2 + \psi_{uv}^2).$$

We claim that the complement $M_1 = \mathbb{R}^2 - M_0$ is dense in $\mathbb{R}^2$. To this purpose it is enough to show that the interior, $\text{int}(M_0)$, of M_0 is empty. Assume to the contrary

that $\text{int}(M_0) \neq \emptyset$ and let U be a connected component of $\text{int}(M_0)$. Then it follows easily that the analytic functions φ and ψ are affine. Thus, by analyticity, G_f is a plane, which is a contradiction.

In the sequel, we work on M_1 . By virtue of (3.8), we get

$$(3.10) \quad \begin{aligned} \varphi_{uv} &= \frac{-(\varphi_u \psi_u + \varphi_v \psi_v) \psi_{uv} - J_\Phi \psi_{uu}}{\varphi_u^2 + \varphi_v^2}, \\ \varphi_{uu} &= \frac{J_\Phi \psi_{uv} - (\varphi_u \psi_u + \varphi_v \psi_v) \psi_{uu}}{\varphi_u^2 + \varphi_v^2}. \end{aligned}$$

Due to (3.1), we readily infer that the vector fields

$$\xi = (-b\varphi_u + a\varphi_v, -\varphi_v, b, 0), \quad \eta = (-b\psi_u + a\psi_v, -\psi_v, 0, b)$$

are normal to G_f and satisfy

$$|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2 = b^2 E^2.$$

We easily check that the vector fields $\{e_1, e_2; \xi_3, \xi_4\}$ given by

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{E}} X_u, \quad e_2 = \frac{1}{\sqrt{E}} X_v, \\ \xi_3 &= \frac{1}{|\xi|} \xi, \quad \xi_4 = \frac{1}{b|\xi|E} \left(|\xi|^2 \eta - \langle \xi, \eta \rangle \xi \right), \end{aligned}$$

constitute an orthonormal frame field along G_f . Moreover, ξ_3 and ξ_4 are normal to G_f . Then a straightforward computation shows that the coefficients of the second fundamental form are given by

$$\begin{aligned} h_{11}^3 &= -\frac{b\varphi_{uu}}{E|\xi|}, \quad h_{11}^4 = \frac{\langle \xi, \eta \rangle \varphi_{uu} - |\xi|^2 \psi_{uu}}{E^2 |\xi|}, \\ h_{12}^3 &= -\frac{b\varphi_{uv}}{E|\xi|}, \quad h_{12}^4 = \frac{\langle \xi, \eta \rangle \varphi_{uv} - |\xi|^2 \psi_{uv}}{E^2 |\xi|}. \end{aligned}$$

So using (2.1) and (2.3) and (3.10), we find

$$(3.11) \quad K = \frac{1}{E^3} \frac{\psi_{uu}^2 + \psi_{uv}^2}{\varphi_u^2 + \varphi_v^2} \left(2b^2 - (1 + a^2 + b^2) E \right)$$

and

$$(3.12) \quad K_N = \frac{2b}{E^3} \frac{\psi_{uu}^2 + \psi_{uv}^2}{\varphi_u^2 + \varphi_v^2} J_\Phi.$$

The second equation of (3.2) yields

$$(3.13) \quad E \geq \frac{1 + a^2 + b^2}{2}.$$

Hence,

$$2b^2 - (1 + a^2 + b^2) E \leq -\frac{1}{2} \left(a^2 + (b-1)^2 \right) \left(a^2 + (b+1)^2 \right) < 0.$$

This shows that

$$M_1 \subset \{(u, v) \in \mathbb{R}^2 : K(u, v) < 0\}.$$

Moreover, due to (3.11), (3.12) and (3.5), we have

$$\frac{K_N^2}{K^2} = 4b^2 W(E),$$

where $W(t)$ is the increasing real valued function

$$W(t) := \frac{t^2 - (1 + a^2 + b^2)t + b^2}{((1 + a^2 + b^2)t - 2b^2)^2}, \quad t \geq \frac{1 + a^2 + b^2}{2}.$$

From our assumption $\inf_{K < 0} \frac{|K_N|}{|K|} > 0$, we get

$$\inf_{M_1} \frac{|K_N|}{|K|} > 0.$$

Bearing (3.13) in mind and since $W(t)$ is increasing, we have

$$\inf_{M_1} \frac{K_N^2}{K^2} = 4b^2 W\left(\inf_{M_1} E\right).$$

Hence $W(\inf_{M_1} E) > 0$ or, equivalently,

$$\left(\inf_{M_1} E\right)^2 - (1 + a^2 + b^2) \inf_{M_1} E + b^2 > 0.$$

Appealing to the identity (3.5), we deduce that $\inf_{M_1} |J_\Phi| > 0$. By continuity, and bearing in mind the fact that M_1 is dense in $\mathbb{R}^2$, we infer that $|J_\Phi|$ is bounded from below away from zero. On the other hand, $\Phi(u, v) = (\varphi(u, v), \psi(u, v))$ is a harmonic map. Therefore, according to Lemma 3.2, G_f is a plane, which contradicts our assumptions. Thus $(a, b) = (0, 1)$ and the equations (3.2) become

$$\begin{aligned} \varphi_u \varphi_v + \psi_u \psi_v &= 0, \\ \varphi_u^2 + \psi_u^2 &= \varphi_v^2 + \psi_v^2. \end{aligned}$$

So, $\varphi_u = \pm \psi_v$, $\varphi_v = \mp \psi_u$ and G_f is a complex analytic curve. $\square$

Proof of Corollary 1.4. In the case where $K \equiv 0$, G_f is a plane. Now consider the case where K is not identically zero. According to Theorem 1.3 we have $c = 0$, unless G_f is a complex analytic curve. We claim that the case $c = 0$ does not occur. Indeed, arguing indirectly suppose that $c = 0$. As in the proof of Theorem 1.3, the set M_1 is dense in $\mathbb{R}^2$. From the assumption $K_N = cK$, we conclude that $K_N = 0$ in M_1 . Furthermore, the relation (3.12) yields that $J_\Phi = 0$. Taking into account the identity (3.5), we get that E is constant, which implies that K is identically zero, a contradiction. Therefore, G_f is a complex analytic curve. $\square$

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