UNITARY EQUIVALENCES FOR ESSENTIAL EXTENSIONS OF C^* -ALGEBRAS

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ABSTRACT. Let A be a unital separable C^* -algebra and $B = C \otimes \mathcal{K}$, where C is a unital C^* -algebra. Let $\tau : A \rightarrow M(B)/B$ be a unital full essential extension of A by B . We show that there is a bijection between elements in a quotient group of $K_0(B)$ onto the strong unitary equivalence classes of unital full essential extensions σ for which $[\sigma] = [\tau]$ in $KK^1(A, B)$. Consequently, when this group is zero, unitarily equivalent full essential extensions are strongly unitarily equivalent. When B is a non-unital but σ -unital simple C^* -algebra with continuous scale, we also study the problem when two approximately unitarily equivalent essential extensions are strongly approximately unitarily equivalent. A group is used to compute the strongly approximate unitary equivalence classes in the same approximate unitary equivalent class of essential extensions.

1. INTRODUCTION

Let $\mathcal{K}$ be the C^* -algebra of all compact operators on the Hilbert space l^2 and let X be a compact metric space. Suppose that $\tau_1, \tau_2 : A \rightarrow B(l^2)/\mathcal{K}$ are two essential extensions, where $A = C(X)$. The Brown-Douglas-Fillmore theorem states that τ_1 and τ_2 are unitarily equivalent if and only if $[\tau_1] = [\tau_2]$ in $KK^1(A, \mathcal{K})$, i.e., there is a partial isometry $v \in B(l^2)/\mathcal{K}$ such that $v^*v = \tau_2(1)$, $vv^* = \tau_1(1)$ and $v^*\tau_1(a)v = \tau_2(a)$ for all $a \in A$. If both τ_1 and τ_2 are unital, i.e., $\tau_1(1) = \tau_2(1) = 1_{B(l^2)/\mathcal{K}}$, then there exists a unitary $U \in B(l^2)$ such that $\pi(U)^*\tau_1(a)\pi(U) = \tau_2(a)$ for all $a \in A$, if $[\tau_1] = [\tau_2]$, where $\pi : B(l^2) \rightarrow B(l^2)/\mathcal{K}$ is the quotient map. In other words, unitary equivalence is the same as strong unitary equivalence in this case. This fact is particularly important in the classification of essentially normal operators. However, this is no longer true when A is not a commutative C^* -algebra in general. Suppose that $A = \mathcal{O}_n$ for $n \geq 3$. Then it is known that there are $\tau_1, \tau_2 : \mathcal{O}_n \rightarrow B(l^2)/\mathcal{K}$ for which there is a unitary $u \in B(l^2)/\mathcal{K}$ such that $\text{Ad } u \circ \tau_1 = \tau_2$, but they are not strongly unitarily equivalent.

The usual way to get around this is to stabilize the situation. By identifying $M_2(B(l^2)/\mathcal{K})$ with $B(l^2)/\mathcal{K}$, one obtains $v' \in M_2(B(l^2)/\mathcal{K})$ such that $v + v'$ is a unitary in $U_0(M_2(B(l^2)/\mathcal{K}))$ so that there is a unitary $U \in M_2(B(l^2))$ (which is again identified with $B(l^2)$) with $\pi(U) = v + v'$. Hence we have $\pi(U)^*\tau_1(a)\pi(U) = \tau_2(a)$ for all $a \in A$. Since $KK^1(A, B)$ cannot detect the difference between (weak)

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unitary equivalence and strong unitary equivalence, one may choose to ignore this difference by stabilizing the situation. But this does not change the fact that, for example in the above-mentioned case that $A = \mathcal{O}_n$, there is no unitary $U \in B(l^2)/\mathcal{K}$ such that $\pi(U)^*\tau_1(a)\pi(U) = \tau_2(a)$ for all $a \in A$. As one sees in the classification of essentially normal operators, it is vital sometimes to have the strong unitary equivalence instead of the (weak) unitary equivalence.

Let us review the case that $A = C(X)$ and B is a non-unital but σ -unital C^* -algebra. Assume that $M(B)/B$ is a purely infinite simple C^* -algebra (such as the case that $B = \mathcal{K}$). Suppose that $\tau : C(X) \rightarrow M(B)/B$ is a unital essential extension. Let $x \in K_1(M(B)/B)$ and let $u \in M(B)/B$ be a unitary such that $[u] = x$. Consider $\tau_1 = \text{Ad } u \circ \tau$. Let $\xi \in X$ be a point. Consider $D = \{\tau(f) : f \in C(X) \text{ and } f(\xi) = 0\}$. By a result of Pedersen (see [16]), $D^\perp = \{a \in M(B)/B : ba = ab = 0 \text{ for all } a \in D\} \neq \{0\}$. It is a hereditary C^* -subalgebra of $M(B)/B$. A result of S. Zhang says that $M(B)/B$ has real rank zero. Consequently there is a non-zero projection $p \in D^\perp$. It is clear that $p\tau(a) = \tau(a)p$ for all $a \in C(X)$. Since $M(B)/B$ is purely infinite simple, a result of Cuntz provides a unitary $w_1 \in p(M(B)/B)p$ such that $[w_1] = [u^*]$. Now put $v = w_1 \oplus (1 - p)$. Then $[v] = [u^*]$. But $v\tau(a) = \tau(a)v$ for all $a \in A$. So $\text{Ad } v \circ \tau = \tau$. Put $z = vu$. Then $\text{Ad } z \circ \tau = \tau_1$. But $z \in U_0(M(B)/B)$, and there is a unitary $U \in M(B)$ so that $\pi(U) = z$. In other words we have just sketched the proof of the fact that, in this case, unitary equivalence is the same as strong unitary equivalence (see Theorem 1.9 of [9]). Unfortunately this argument cannot be applied to more general C^* -algebras A .

In this short paper, we show that there is a K -theoretical obstruction to prevent, in general, unitary equivalence from being the same as strong unitary equivalence. We will describe it and show that when this K -theoretical obstruction disappears, then unitary equivalence is indeed the same as strong unitary equivalence. We find that, for a fix $z \in KK^1(A, B)$, there is a bijection between elements of a quotient of $K_0(B)$ and strong unitary equivalence classes of full essential extensions which are represented by z . When $A = C(X)$, this quotient of $K_0(B)$ is zero, which explains K -theoretically the reason why these two unitary equivalence relations are the same. This is a direct application of the Universal Coefficient Theorem and some more recent results about classification of full essential extensions.

2. PRELIMINARIES

Definition 2.1. Let A be a unital C^* -algebra and C be a non-unital but σ -unital C^* -algebra. Let $\tau_1, \tau_2 : A \rightarrow M(B)/B$ be two essential extensions. Extensions τ_1 and τ_2 are said to be *unitarily equivalent* if there exists a partial isometry $v \in M(B)/B$ such that $v^*v = \tau_2(1_A)$, $vv^* = \tau_1(1_A)$ and

$$v^*\tau_1(a)v = \tau_2(a) \text{ for all } a \in A.$$

Extension τ_1 is said to be *unital* if $\tau_1(1_A) = 1_{M(B)/B}$. It should be noted that if τ_2 is also unital and τ_1 and τ_2 are unitarily equivalent, then, in the above, v can be chosen to be a unitary in $M(B)/B$.

The two essential extensions are said to be *strongly unitarily equivalent* if there exists a unitary $U \in M(B)$ such that

$$\text{Ad } \pi(U) \circ \tau_1 = \tau_2.$$

It is obvious that if two essential extensions are strongly unitarily equivalent, then they are unitarily equivalent.

Definition 2.2. Let A be a C^* -algebra and D be a unital C^* -algebra. A homomorphism $h : A \rightarrow D$ is said to be *full* if the (closed) ideal generated by $h(a)$ is D for every non-zero $a \in A$.

Let B be a non-unital but σ -unital C^* -algebra. An essential extension $\tau : A \rightarrow M(B)/B$ is *full* if τ is a full homomorphism.

If $M(B)/B$ is simple, then all essential extensions are full. If A is unital and simple, then all unital essential extensions are full.

Definition 2.3 (cf. [15]). Let B be a non-unital but σ -unital C^* -algebra. We say $M(B)/B$ has property (P) if for any full element $b \in M(B)/B$ there exist $x, y \in M(B)/B$ such that $xb y = 1$.

If $M(B)/B$ is simple, then $M(B)/B$ has property (P). This is always the case if B is purely infinite (see also Remark 2.6 below). It is proved in [15] that if $B = C \otimes \mathcal{K}$, where $C \cong C(X)$ for some finite CW complex, then $M(B)/B$ has property (P). Other C^* -algebras which have property (P) are discussed in [15].

Let A be a separable C^* -algebra. A is said to satisfy the Universal Coefficient Theorem if for any σ -unital C^* -algebra C one has the following short exact sequence:

$$0 \rightarrow \text{ext}_{\mathbb{Z}}(K_*(A), K_{*-1}(B)) \xrightarrow{\delta} KK^*(A, B) \xrightarrow{\Gamma} \text{Hom}(K_*(A), K_*(B)) \rightarrow 0.$$

C^* -algebras in the so-called “bootstrap” class $\mathcal{N}$ of amenable C^* -algebras satisfy the Universal Coefficient Theorem (UCT) (by [19]). When A is amenable, one has $KK^1(A, B) = \text{Ext}(A, B)$.

The following classification of full extensions was established in [15].

Theorem 2.4. *Let A be a unital separable amenable C^* -algebra which satisfies the Universal Coefficient Theorem. Let $B = C \otimes \mathcal{K}$, where C is a unital C^* -algebra so that $M(B)/B$ has property (P). Then two full essential extensions $\tau_1, \tau_2 : A \rightarrow M(B)/B$ are unitarily equivalent if and only if*

$$[\tau_1] = [\tau_2] \text{ in } KK^1(A, B).$$

Moreover, for any $z \in KK^1(A, B)$ there exists a full essential extension $\tau : A \rightarrow M(B)/B$ such that $[\tau] = z$.

The basic question that we consider in this short paper is the following: Suppose that $[\tau_1] = [\tau_2]$ in $KK^1(A, B)$ so that τ_1 and τ_2 are unitarily equivalent. Are they strongly unitarily equivalent? If the answer is negative in general, when are they strongly unitarily equivalent?

For the non-stable case, the following was proved in [14].

Theorem 2.5. *Let A a unital separable amenable C^* -algebra which satisfies the UCT and let B be a non-unital but σ -unital simple C^* -algebra for which $M(B)/B$ is simple. Let $\tau_1, \tau_2 : A \rightarrow M(B)/B$ be two essential extensions. Then τ_1 and τ_2 are approximately unitarily equivalent if and only if $[\tau_1] = [\tau_2]$ in $KL(A, M(B)/B)$.*

Remark 2.6. Let B be a non-unital but σ -unital simple C^* -algebra. It is shown in [13] that $M(B)/B$ is simple if and only if B has continuous scale (see [13]). It was also shown that when $M(B)/B$ is simple it is purely infinite (see [13]).

Definition 2.7. Let $\{A_n\}$ be a sequence of C^* -algebras. Denote by $l^\infty(\{A_n\})$ the C^* -product of $\{A_n\}$ and by $c_0(\{A_n\})$ the C^* -direct sum of $\{A_n\}$. We will also use $q_\infty(\{A_n\})$ for the quotient $l^\infty(\{A_n\})/c_0(\{A_n\})$. When $A_n = A$ for all n , we

will use $l^\infty(A)$, $c_0(A)$ and $q_\infty(A)$ instead of $l^\infty(\{A_n\})$, $c_0(\{A_n\})$ and $q_\infty(\{A_n\})$, respectively.

3. STRONG UNITARY EQUIVALENCE

Definition 3.1. Let A be a C^* -algebra and let B be a unital C^* -algebra. Let $G \subset U(B)$ be a normal subgroup. Suppose that $\phi_1, \phi_2 : A \rightarrow B$ are homomorphisms. We say ϕ_1 and ϕ_2 are G -strongly unitarily equivalent if there exists a unitary $u \in G$ such that

$$\text{Ad } u \circ \phi_1(a) = \phi_2(a) \text{ for all } a \in A.$$

In the case where $G = U_0(B)$, we simply say that ϕ_1 and ϕ_2 are *strongly unitarily equivalent*.

We say ϕ_1 and ϕ_2 are G -strongly approximately unitarily equivalent if there exists a sequence of unitaries $u_n \in G$ such that

$$\lim_{n \rightarrow \infty} \text{Ad } u_n \circ \phi_2(a) = \phi_1(a) \text{ for all } a \in A.$$

If $G = U_0(B)$, we simply say that ϕ_1 and ϕ_2 are *strongly approximately unitarily equivalent*.

Lemma 3.2. Let A be a separable C^* -algebra, let B be a unital C^* -algebra and let $h_1, h_2 : A \rightarrow B$ be two homomorphisms. Suppose that G is a normal subgroup of $U(B)$.

(1) Suppose that $h_2 = \text{Ad } u \circ h_1$ for some $u \in U(B)$. Then h_1 and h_2 are G -strongly unitarily equivalent if and only if there is a unitary $v \in U(B)$ such that $[u] = [v]$ in $U(B)/G$ and $vh_1(a) = h_1(a)v$ for all $a \in A$.

(2) Suppose that there exists a sequence of unitaries $u_n \in B$ such that

$$h_2(a) = \lim_{n \rightarrow \infty} \text{Ad } u_n \circ h_1(a) \text{ for all } a \in A.$$

Then h_2 and h_1 are G -strongly approximately unitarily equivalent if and only if there exists a sequence of unitaries $v_k \in B$ such that $[v_k] = [u_{n(k)}]$ in $U(B)/G$ for a subsequence $\{n(k)\}$ and

$$\lim_{k \rightarrow \infty} \|v_k h_2(a) - h_2(a) v_k\| = 0 \text{ for all } a \in A.$$

Proof. (1) Let $w \in G$ such that $w^* \text{Ad } u \circ h_1(a) w = h_1(a)$ for all $a \in A$. Put $v = wu$. Then $[v] = [u] \in U(B)/G$ and $vh_1(a) = h_1(a)v$ for all $a \in A$.

Conversely, suppose that there is $v \in U(B)$ with $[u] = [v]$ in $U(B)/G$ such that $vh_1(a) = h_1(a)v$ for all $a \in A$. Put $w = u^*v$. Then $w \in G$. However, we have

$$w^* u^* h_1(a) u w = v^* h_1(a) v = h_1(a) \text{ for all } a \in A.$$

(2) Suppose that there exists a sequence of unitaries $w_n \in G$ such that

$$\lim_{n \rightarrow \infty} \text{Ad } w_n \circ h_2(a) = h_1(a).$$

Then one has two subsequences $\{m(k)\}$ and $\{n(k)\}$ such that

$$\lim_{n \rightarrow \infty} \text{Ad } w_{m(k)} u_{n(k)} \circ h_1(a) = h_1(a) \text{ for all } a \in A.$$

Choose $v_k = w_{m(k)} u_{n(k)}$. Then $[v_k] = [u_{m(k)}]$ in $U(B)/G$. Moreover

$$\lim_{n \rightarrow \infty} \|v_k h_1(a) - h_1(a) v_k\| = 0 \text{ for all } a \in A.$$

Conversely, suppose that there exists $v_k \in U(B)$ such that $[v_k] = [u_{n(k)}]$ in $U(B)/G$ and

$$\lim_{n \rightarrow \infty} \|v_k h_2(a) - h_2(a) v_k\| = 0 \text{ for all } a \in A.$$

Therefore there exists a subsequence $\{k_m\}$ such that

$$\lim_{m \rightarrow \infty} \|v_{k_m}^* u_{n(k_m)}^* h_1(a) u_{n(k_m)} v_{k_m} - h_2(a)\| = 0 \text{ for all } a \in A.$$

Put $w_m = u_{n(k_m)} v_{k_m}^*$. Then $w_m \in G$ and

$$\lim_{m \rightarrow \infty} \|\text{Ad } w_m \circ h_1(a) - h_2(a)\| = 0 \text{ for every } a \in A.$$

□

Definition 3.3. Let G and F be two groups and $u \in G$ be a distinguished element. Define

$$H_u(G, F) = \{\phi(u) : \phi \in \text{Hom}(G, F)\}.$$

So $x \in H_u(G, F)$ if and only if there is $\phi \in \text{Hom}(G, F)$ such that $\phi(u) = x$. When G and F are abelian, $H_u(G, F)$ is a subgroup of F . Let A and B be two C^* -algebras. Suppose that A is unital. We write $H_{[1_A]}(K_0(A), K_i(B)) = H_1(K_0(A), K_i(B))$. If $K_0(A) = \mathbb{Z}/n\mathbb{Z}$ for some $n > 1$ and $K_i(B) = \mathbb{Z}$, then $H_1(K_0(A), K_i(B)) = \{0\}$. If $K_0(A) = \mathbb{Z}$ and $[1_A] = 1$ in $\mathbb{Z}$, then $H_1(K_0(A), K_i(B)) = K_i(B)$. Suppose that $K_0(A) = \mathbb{Z}/p^2\mathbb{Z}$ with $[1_A] = \bar{1}$ and $K_i(B) = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$, where $(p, q) = 1$. Then $H_1(K_0(A), K_i(B)) = \mathbb{Z}/p\mathbb{Z}$. If $K_i(B)$ is divisible, then $H_1(K_0(A), K_i(B)) = K_i(B)$.

Proposition 3.4. Let A be a separable C^* -algebra and let C be a unital C^* -algebra. Let $G \subset U(C)$ be a normal subgroup and let $h_1, h_2 : A \rightarrow C$ be two monomorphisms. Suppose that $h_2 = \text{Ad } u \circ h_1$ for some $u \in U(A)$. If h_2 and h_1 are G -strongly unitarily equivalent, then $[u] \in H_1(K_0(A), K_1(C)) + \bar{G}$, where $\bar{G}$ is the image of G in $K_1(C)$.

Proof. Suppose that h_1 and h_2 are G -strongly unitarily equivalent. Then by Lemma 3.2, there is $v \in U(C)$ such that $vh_1(a) = h_1(a)v$ and $[v] = [u]$ in $U(C)/G$. Thus we obtain a homomorphism $\Phi : A \otimes C(S^1) \rightarrow C$ defined by $\phi(a \otimes f) = h_1(a)f(v)$ for $a \in A$ and $f \in C(S^1)$. Consequently, there is a homomorphism $\phi_1 : K_1(A \otimes C(S^1)) \rightarrow K_1(C)$ such that $[\psi(1_A \otimes \iota)] = [v]$, where $\iota : S^1 \rightarrow S^1$ is the identity map. There is embedding from $K_0(A)$ into $K_1(A \otimes C(S^1))$ which maps $[1_A]$ to $[1_A \otimes \iota]$. Thus we obtain a homomorphism $\phi : K_0(A) \rightarrow K_1(C)$ such that $\phi([1_A]) = [v]$. This implies $[u] \in H_1(K_0(A), K_1(C)) + \bar{G}$. □

Suppose that A satisfies the UCT and B is a σ -unital C^* -algebra. In what follows $\Gamma : KK^1(A, B) \rightarrow \text{Hom}(K_*(A), K_{*-1}(B))$ is the surjective map given by the UCT.

Lemma 3.5. Let A be a separable unital C^* -algebra in $\mathcal{N}$ and let B be a σ -unital C^* -algebra. Let $\tau \in KK^1(A, B)$ and $\zeta \in H_1(K_0(A), K_0(B))$. Then there is an essential extension $\tau_1 : A \otimes C(S^1) \rightarrow Q(B \otimes \mathcal{K})$ such that

$$[\tau_1|_A] = [\tau] \text{ and } \Gamma([\tau_1])([1_A \otimes \iota]) = \zeta \text{ (in } K_1(Q(B \otimes \mathcal{K})) = K_0(B)),$$

where $\iota(z) = z$ for $z \in S^1$ and $Q(B \otimes \mathcal{K}) = M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$.

Proof. Let $j : A \rightarrow A \otimes C(S^1)$ be defined by $j(a) = a \otimes 1$. If $A \otimes C(S^1)$ is identified with $C(S^1, A)$, then $j(a)(t) = a$ for all $t \in S^1$. By the Künneth formula for

tensor product, we obtain $K_0(A \otimes C(S^1)) = K_0(A) \oplus K_1(A)$ and $K_1(A \otimes C(S^1)) = K_1(A) \oplus K_0(A)$. Moreover, $j_{*0} : K_0(A) \rightarrow K_0(A) \oplus K_1(A)$ may be written as $j_{*0}(x) = x \oplus 0$, and $j_{*1} : K_1(A) \rightarrow K_1(A) \oplus K_0(A)$ may be written as $j_{*1}(y) = y \oplus 0$.

Let $\Gamma(\tau)_i : K_i(A) \rightarrow K_{i+1}(B)$ be the map given by the UCT, $i = 0, 1$. Define $\gamma_0 : K_0(A \otimes C(S^1)) = K_0(A) \oplus K_1(A) \rightarrow K_1(B)$ by $\gamma_0((x, y)) = \Gamma(\tau)_0(x)$.

Since $\zeta \in H_1(K_0(A), K_0(B))$, there exists $\phi : K_0(A) \rightarrow K_0(B)$ such that $\phi([1_A]) = \zeta$. Define $\gamma_1 : K_1(A \otimes C(S^1)) = K_1(A) \oplus K_0(A) \rightarrow K_0(B)$ by $\gamma_1((x, y)) = \Gamma(\tau)_1(x) + \phi(y)$. By the UCT, there is $\tau' : A \otimes C(S^1) \rightarrow Q(B \otimes \mathcal{K})$ such that $\Gamma(\tau') = (\gamma_0, \gamma_1)$. Define $\psi = \tau'|_A : A \rightarrow Q(B \otimes \mathcal{K})$. It follows that $\Gamma(\psi) = \Gamma(\tau' \circ j) = \Gamma(\tau)$. Let $\tau_0 : A \rightarrow Q(B \otimes \mathcal{K})$ be an essential extension so that $[\tau_0] = [\tau] - [\psi]$. Then $\Gamma([\tau_0]) = 0$. By the UCT, $[\tau_0] \in \text{ext}_{\mathbb{Z}}(K_i(A), K_i(B))$. Suppose that $[\tau_0]$ is represented by the following two short exact sequences:

$$0 \rightarrow K_0(B) \rightarrow K_0(E) \rightarrow K_0(A) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K_1(B) \rightarrow K_1(E) \rightarrow K_1(A) \rightarrow 0.$$

Let $G_0 = K_0(E) \oplus K_1(A)$ and $G_1 = K_1(E) \oplus K_0(A)$. Then (G_0, G_1) gives short exact sequences

$$0 \rightarrow K_i(B) \rightarrow G_i \rightarrow K_0(A) \oplus K_1(A) \rightarrow 0.$$

It also gives an element $\delta \in \text{ext}_{\mathbb{Z}}(K_i(A \otimes C(S^1)), K_i(B))$. By the UCT again, there is $\sigma : A \otimes C(S^1) \rightarrow Q(B \otimes \mathcal{K})$ such that $[\sigma] = \delta$. Let $\tau'_0 = \sigma|_A$. Then τ'_0 gives the following short exact sequences:

$$0 \rightarrow K_0(B) \rightarrow K_0(E) \rightarrow K_0(A) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K_1(B) \rightarrow K_1(E) \rightarrow K_1(A) \rightarrow 0$$

and $[\tau'_0] = [\tau_0]$. There is $\sigma' : A \otimes C(S^1) \rightarrow Q(B \otimes \mathcal{K})$ such that $[\sigma'] = -[\sigma]$. There is $\tau_1 : A \otimes C(S^1) \rightarrow Q(B \otimes \mathcal{K})$ so that $[\tau_1] = [\tau'] - [\sigma]$. Let $\tau'' : A \otimes C(S^1) \rightarrow Q(B \otimes \mathcal{K})$ so that $[\tau''] = -[\tau']$. Then $\Gamma(\tau_1) = \Gamma(\tau')$. Moreover,

$$[\tau] - [\tau_1|_A] = [\tau] + [\tau''|_A] + [\sigma'|_A] = [\tau] - [\psi] + [\sigma'|_A] = [\tau_0] + [\sigma'|_A] = 0.$$

Note that $\Gamma([\tau_1])([1_A \otimes \iota]) = \phi([1_A]) = \zeta$. □

Lemma 3.6. *Let $B = C \otimes \mathcal{K}$, where C is a unital C^* -algebra and $M(B)/B$ has property (P). Let $u \in U(M(B)/B)$ be a unitary so that $[u] = 0$ in $K_1(M(B)/B)$. Suppose that $\phi : C(S^1) \rightarrow M(B)/B$ defined by $\phi(f) = f(u)$ for $f \in C(S^1)$ is full. Then $u \in U_0(M(B)/B)$.*

Moreover, if $z \in K_1(M(B)/B)$, then there exists $u \in U(M(B)/B)$ such that $[u] = z$.

Proof. Let $v \in U_0(M(B)/B)$ be a unitary with $sp(v) = S^1$. Define a monomorphism $\phi_0 : C(S^1) \rightarrow M_2(M(B)/B)$ by $\phi_0(f) = f(v) \oplus f(1) \cdot e$, where $e = 1_{M(B)/B}$. Since B is stable, there is $z \in M_2(M(B)/B)$ such that $zz^* = e \oplus e$, $z^*z = e \oplus 0$. Put $\phi_1 = \text{Ad } z \circ \phi_0$. Put $v_0 = \phi_1(\iota)$, where ι is the identity function on the unit circle. Since ϕ is full, by 2.17 of [15], it is absorbing. In particular, there exists $W \in M_2(M(B)/B)$ with $W^*W = e \oplus 0$ and $WW^* = e \oplus e$ such that $\text{Ad } W \circ \phi_1 = \phi$. Therefore $u = W^*(u \oplus v_0)W$. Thus, there is $W_1 \in M_2(M(B)/B)$ with $W_1^*W_1 = e$, $W_1W_1^* = e \oplus e$ such that

$$u = W_1^*(u \oplus e)W_1.$$

Since B is stable and $[u] = 0$ in $K_1(M(B)/B)$, we may assume that $\text{diag}(u, e) \in U_0(M_2(M(B)/B))$. We may write

$$\text{diag}(u, e) = \prod_{k=1}^n \exp(ia_k),$$

where $a_k \in M_2(M(B)/B)$ are self-adjoint elements. It follows that $u = W_1^*(\prod_{k=1}^n \exp(ia_k))W_1$ is connected to $W_1^*W_1 = e$ by a continuous path of unitaries in $U(M(B)/B)$. So $u \in U_0(M(B)/B)$.

To see the last part of the statement, by [15], there is a full essential extension $\tau : C(S^1) \rightarrow M(B)/B$ such that $[\tau(\iota)] = z$, where $\iota : S^1 \rightarrow S^1$ is the canonical unitary in $C(S^1)$. $\square$

Let $z \in KK^1(A, B)$. In the following theorem, we denote by $\mathcal{T}_s^1(z)$ the set of strong unitary equivalence classes of unital full extensions represented by z .

Theorem 3.7. *Let A be a unital separable C^* -algebra in $\mathcal{N}$ and let $B = C \otimes \mathcal{K}$, where C is a unital C^* -algebra so that $M(B)/B$ has property (P). Let $\tau : A \rightarrow M(B)/B$ be a full essential extension. Suppose that $u \in U(M(B)/B)$. Then $\text{Ad } u \circ \tau$ is strongly unitarily equivalent to τ if and only if $u \in H_1(K_0(A), K_0(B))$.*

Moreover, suppose that τ is unital and there is a bijection

$$\kappa : K_0(B)/H_1(K_0(A), K_0(B)) \rightarrow \mathcal{T}_s^1([\tau]).$$

Proof. For the first part of the theorem, by Proposition 3.4, one needs to prove the “if” part of the statement.

Assume that $[u] \in H_1(K_0(A), K_0(B))$. Then, by Lemma 3.5, there is an essential extension $\sigma : A \otimes C(S^1) \rightarrow Q(C \otimes \mathcal{K})$ such that $[\sigma|_A] = [\tau]$ and $[\sigma(1 \otimes \iota)] = [u]$. It follows from 2.17 of [15] that we may assume that σ is full. By applying 2.17 of [15], there is $w \in U(M(B)/B)$ such that

$$\text{Ad } w \circ \sigma|_A = \tau.$$

Put $v = w^*(\sigma(1 \otimes \iota) + (1 - \sigma(1_A)))w$. Then $[v] = [u]$ and $v\tau(a) = \tau(a)v$ for all $a \in A$. It follows from Lemma 3.2 that $\text{Ad } u \circ \tau$ is strongly unitarily equivalent to τ .

Now we assume that τ is unital; by replacing σ by $\text{Ad } w_1 \circ \sigma$ for some isometry, we may assume that σ is also unital. Fix $z \in KK^1(A, B)$ for which $z = [\tau]$. We define a map $\kappa : K_0(B)/H_1(K_0(A), K_0(B)) \rightarrow \mathcal{T}_s^1(z)$ as follows. For each $x \in K_0(B)/H_1(K_0(A), K_0(B))$, choose $u \in U(M(B)/B)$ such that the image of $[u]$ in $K_0(B)/H_1(K_0(A), K_0(B))$ is x . Define $\kappa(x)$ to be the strong unitary equivalence class represented by $\text{Ad } u \circ \tau$. Let $u_1, u_2 \in M(B)/B$. We have shown that $\text{Ad } u_1 \circ \tau$ and $\text{Ad } u_2 \circ \tau$ are strongly unitarily equivalent if and only if

$$[u_1^*u_2] \in H_1(K_0(A), K_0(B)).$$

This implies that κ is well defined and is injective. For any $\sigma \in \mathcal{T}_s^1(z)$, there is $u \in M(B)/B$ such that $\sigma = \text{Ad } u \circ \tau$. This shows that κ is also surjective. $\square$

Corollary 3.8. *Let A be a separable C^* -algebra in $\mathcal{N}$ and $B = C \otimes \mathcal{K}$, where C is unital, such that $M(B)/B$ has property (P). Suppose that*

$$K_0(B) = H_1(K_0(A), K_0(B)).$$

Suppose that $\tau_1, \tau_2 : A \rightarrow M(B)/B$ are two unital full essential extensions. Then they are strongly unitarily equivalent if and only if they are unitarily equivalent.

Corollary 3.9. *In Corollary 3.8, if $K_0(A) = G \oplus \mathbb{Z}$ with $[1_A] = (0, 1)$, then two unital full essential extensions are strongly unitarily equivalent if and only if they are unitarily equivalent.*

Proof. There is a homomorphism $h : K_0(A) \rightarrow \mathbb{Z}$ which maps $[1_A]$ to 1 in $\mathbb{Z}$. For any $\xi \in K_0(B)$, define $\kappa : \mathbb{Z} \rightarrow K_0(B)$ by $\kappa(n) = n\xi$. Put $\phi = \kappa \circ h$. Thus $\xi \in H_1(K_0(A), K_0(B))$. Therefore $K_0(B) = H_1(K_0(A), K_0(B))$. Then Corollary 3.8 applies. $\square$

Corollary 3.10. *In Corollary 3.8, if A is a unital separable commutative C^* -algebra, then two unital full essential extensions are strongly unitarily equivalent if and only if they are unitarily equivalent.*

Proof. Since $A = C(X)$, $K_0(A) = G \oplus \mathbb{Z}$ with $[1_A] = (0, 1)$. $\square$

Let A be a unital C^* -algebra. If $\tau : A \rightarrow M(B)/B$ is not unital, then there are fewer strong unitary equivalence classes of full essential extensions $\sigma : A \rightarrow M(B)/B$ with $[\tau] = [\sigma]$ which are not unital.

Definition 3.11. Let $B = C \otimes \mathcal{K}$ and A be unital. Suppose that $\tau : A \rightarrow M(B)/B$ is not unital and $p = \tau(1_A)$. Let

$$G_p = \{p + v \in U(M(B)/B) : v \in U((1-p)(M(B)/B)(1-p))\}.$$

Then G_p is a subgroup. Denote by $\overline{G_p}$ the image of G_p in $K_0(M(B)/B)$. Note that if $1-p$ is full, then there exists a projection $e \leq 1-p$ and $w \in M(B)/B$ such that $w^*w = e$ and $ww^* = 1_{M(B)/B}$. Then $\overline{G_p} = K_1(M(B)/B)$.

Let $\sigma : A \rightarrow M(B)/B$ be another extension. Suppose that $\tau(1_A) = p$ and $\sigma(1_A) = q$. Note that if p is not unitarily equivalent to q (in $M(B)/B$), then τ and σ cannot possibly be strongly unitarily equivalent.

From the following result, one also knows that if p and q are unitarily equivalent and $1-p$ is full, then σ and τ are strongly unitarily equivalent if $[\sigma] = [\tau]$ in $KK^1(A, B)$.

Theorem 3.12. *Let A be a unital separable C^* -algebra in $\mathcal{N}$ and $B = C \otimes \mathcal{K}$, where C is unital for which $M(B)/B$ has property (P). Let $\tau : A \rightarrow M(B)/B$ be a full essential extension.*

If $\tau(1_A) = p$ and $p \neq 1$, there is a bijection from $K_0(B)/(H_1(K_0(A), K_0(B)) + \overline{G_p})$ onto the set of strong unitary equivalence classes of full essential extensions $\sigma : A \rightarrow M(B)/B$ for which $[\sigma] = [\tau]$ and $\sigma(1_A)$ is unitarily equivalent to p .

Proof. Suppose that $\sigma(1_A)$ is unitarily equivalent to p and $[\sigma] = [\tau]$. Then there exists a unitary $w \in M(B)/B$ such that $\sigma_1 = \text{Ad } w \circ \tau$. Suppose that $[w] \in H_1(K_0(A), K_0(B)) + \overline{G_p}$. There is $z \in U(M(B)/B)$ such that $z = p + v$ with $v^*v = vv^* = 1-p$ and $[zw] \in H_1(K_0(A), K_0(B))$. Note that $\text{Ad } zw \circ \tau = \sigma_1$. Thus we may assume that $[w] \in H_1(K_0(A), K_0(B))$. It follows from Theorem 3.7 that σ and τ are strongly unitarily equivalent.

On the other hand, note that since p is full, $K_1(p(M(B)/B)p) = K_1(M(B)/B) \cong K_0(B)$. Suppose that $z_1, z_2 \in K_1(p(M(B)/B)p)$ such that

$$\overline{z_1} \neq \overline{z_2} \text{ in } K_0(B)/(H_1(K_0(A), K_0(B)) + \overline{G_p}).$$

It follows from Lemma 3.6 that there are unitaries $v_1, v_2 \in p(M(B)/B)p$ such that $[v_1] = z_1$ and $[v_2] = z_2$. Consider extensions $\text{Ad } v_i \circ \tau : A \rightarrow p(M(B)/B)p$ for $i = 1, 2$. Let G'_p be the pre-image of $\overline{G_p}$ in $U(M(B)/B)$. It follows from Proposition 3.4 that they are not G'_p -strongly unitarily equivalent. It follows that they are not strongly unitarily equivalent as homomorphisms to $M(B)/B$. $\square$

Proposition 3.13. *Let A be a unital separable C^* -algebra in $\mathcal{N}$ and $B = C \otimes \mathcal{K}$, where C is unital such that $M(B)/B$ has property (P). Suppose that $K_1(M(B)/B) = H_1(K_0(A), K_1(M(B)/B))$. Suppose also that $\tau_1, \tau_2 : A \rightarrow M(B)/B$ are two unital full essential extensions. Then τ_1 and τ_2 are homotopic if and only if $[\tau_1] = [\tau_2]$ in $KK^1(A, B)$.*

Proof. If τ_1 and τ_2 are homotopic, then $[\tau_1] = [\tau_2]$ in $KK^1(A, B)$.

Conversely, if $[\tau_1] = [\tau_2]$, by Theorem 3.7, then they are strongly unitarily equivalent. There exists a unitary $u \in U_0(M(B)/B)$ such that $\text{Ad } u \circ \tau_1 = \tau_2$. Let $\{u_t : t \in [0, 1]\}$ be a continuous path of unitaries in $M(B)/B$ such that $u_0 = u$ and $u_1 = 1_{M(B)/B}$. Define $\Sigma : A \rightarrow C([0, 1], M(B)/B)$ by $\Sigma(a)(t) = \text{Ad } u_t \circ \tau_1$ for $a \in A$. Then $\Sigma(a)(0) = \tau_2$ and $\Sigma(a)(1) = \tau_1$. $\square$

4. APPROXIMATE UNITARY EQUIVALENCES

Definition 4.1. Let A be a separable C^* -algebra and let B be a unital C^* -algebra. Let $\{x_n\}$ be a sequence of elements in $K_i(B)$. Denote by $H_1^{ap}(K_0(A), K_i(B))$ the subset of those sequences $\{x_n\}$ of $K_i(B)$ such that there exists an increasing sequence of finitely generated subgroups $G_n \subset K_0(A)$ with $[1_A] \in G_n$ and $\bigcup_{n=1}^\infty G_n = K_0(A)$, and group homomorphisms $h_n : G_n \rightarrow K_0(B)$ such that $h_n([1_A]) = x_n$. It forms a subgroup of $\prod K_i(B)$.

Suppose that $\Pi_i : \prod_{n \in \mathbb{N}} K_i(B) \rightarrow \prod_{n \in \mathbb{N}} K_i(B) / \bigoplus_{n \in \mathbb{N}} K_i(B)$ for $i = 0, 1$. Suppose that $\xi = \Pi_i(\{x_n\})$ and $\xi \in H_1(K_0(A), \prod_{n \in \mathbb{N}} K_i(B) / \bigoplus_{n \in \mathbb{N}} K_i(B))$. Then $\{x_n\} \in H_1^{ap}(K_0(A), K_i(B))$.

If $K_i(B) = H_1(K_0(A), K_i(B))$, then $H_1^{ap}(K_0(A), K_i(B)) = \prod_{n \in \mathbb{N}} K_i(B)$.

Recall that, for a unitary u in a unital C^* -algebra A ,

$$\text{cel}(u) = \inf \left\{ \sum_{k=1}^n \|h_k\| : u = \prod_{k=1}^n \exp(ih_k), n \in \mathbb{N}, h_k \in A_{s,a} \right\}$$

and $\text{cel}(A) = \sup_{u \in U_0(A)} \text{cel}(u)$ (see [18]).

Let $r : \mathbb{N} \rightarrow \mathbb{N}$ be a map. The unital C^* -algebra A is said to have K_1 - r -cancellation if $u \oplus 1_{M_{r(n)}}$ and $v \oplus 1_{M_{r(n)}}$ are in the same path connected component of $U(M_{n+r(n)})$ for any $u, v \in M_n(A)$ with $[u] = [v]$ in $K_1(A)$.

Proposition 4.2. *Let A be a unital separable amenable C^* -algebra, let C be a unital C^* -algebra and let $G \subset U(B)$ be a normal subgroup. Suppose that $\text{cel}(B) \leq L$ for some $L \geq \pi$ and B has K_1 - r -cancellation (for some $r : \mathbb{N} \rightarrow \mathbb{N}$). Let $h, \phi : A \rightarrow B$ be two unital monomorphisms. Suppose that there exists a sequence of unitaries $u_n \in U(B)$ such that*

$$\lim_{n \rightarrow \infty} \text{Ad } u_n \circ h(a) = \phi(a) \text{ for all } a \in A.$$

Suppose also that h and ϕ are G -strongly approximately unitarily equivalent. Then there exists a subsequence $\{n(k)\}$ such that $[u_{n(k)}] \in H_1^{ap}(K_0(A), K_1(B)) + \prod \bar{G}$, where $\bar{G}$ is the image of G in $K_1(B)$.

If $K_0(A)$ is finitely generated, then one can require that

$$[u_{n(k)}] \in H_1(K_0(A), K_1(B)) + \bar{G}.$$

Proof. By Lemma 3.2, there exists a sequence of unitaries $v_k \in B$ such that $[v_k] = [u_{n(k)}]$ in $U(B)/G$ for a subsequence $\{n(k)\}$ and

$$\lim_{k \rightarrow \infty} \|v_k h_2(a) - h_2(a) v_k\| = 0 \text{ for all } a \in A.$$

Define a map $\Phi : A \otimes C(S^1) \rightarrow l^\infty(B)$ by $\Phi(a \otimes f) = \{h_2(a)f(v_k)\}$ for $f \in C(S^1)$. Let $\Pi : l^\infty(B) \rightarrow l^\infty(B)/c_0(B)$ be the quotient map. Then $\Psi = \Pi \circ \Phi : A \otimes C(S^1) \rightarrow l^\infty(B)/c_0(B)$ is a homomorphism. Since $\text{cel}(B) \leq L$ and B has K_1 - r -cancellation, it follows from Proposition 2.1(3) of [6] that

$$K_1(l^\infty(B)) \subset \prod K_1(B) \quad \text{and} \quad K_1(l^\infty(B)/c_0(B)) \subset \prod K_1(B) / \oplus K_1(B).$$

Let $\xi = \Pi_{*1}([\{v_k\}])$. Then one obtains a homomorphism $\gamma : K_0(A) \rightarrow \prod K_1(B) / \oplus K_1(B)$ induced by Ψ which maps $[1_A]$ to ξ . Let $\{G_n\}$ be an increasing sequence of finitely generated subgroups with $[1_A] \in G_n$ and $\bigcup_{n=1}^\infty G_n = K_0(A)$. It follows from 8.3 of [12] that there is a homomorphism $f_n : G_n \rightarrow \prod K_1(B)$ such that $\Pi \circ \{f_n(g)\} = \gamma(g)$ for all $g \in G_n$. Let $p_n : \prod K_1(B) \rightarrow K_1(B)$ be the projection on the n coordinate. For each n there is $m(n)$ such that $p_k \circ f_n([1_A]) = [v_k]$ if $k \geq m(n)$. We may assume that $m(n+1) > m(n)$. Define $\phi_n = p_{m(n)} \circ f_n$. Then $\phi_n : G_n \rightarrow K_1(B)$ is a homomorphism and $\phi_n([1_A]) = [v_{m(n)}]$. Thus $\{[v_{m(n)}]\} \in H_1^{ap}(K_0(A), K_1(B))$. It follows that $\{[u_{n(m(k))}]\} \in H_1^{ap}(K_0(A), K_1(B)) + \prod \bar{G}$. If $K_0(A)$ is finitely generated, we may assume that $G_n = K_0(A)$ for all n . So the last part of the statement also follows. $\square$

Theorem 4.3. *Let A be a unital separable C^* -algebra in $\mathcal{N}$. Let $B = C \otimes \mathcal{K}$, where C is unital such that $M(B)/B$ has property (P). Suppose that $K_0(B) = H_1(K_0(A), K_0(B))$. Then two unital full essential extensions of A by B are strongly approximately unitarily equivalent if and only if they are approximate unitarily equivalent.*

Proof. Since $\text{cel}(M(B)/B) \leq \text{cel}(M(B))$, by 5.1.15 of [11], $\text{cel}(M(B)/B) \leq 6\pi$. By Lemma 3.6, $M(B)/B$ also has K_1 - r -cancellation (with $r : \mathbb{N} \rightarrow \mathbb{N}$ as the identity map). Suppose that $\tau_1 : A \rightarrow M(B)/B$ is a unital full essential extension. Let $\{u_n\} \subset M(B)/B$ be a sequence of unitaries such that

$$\tau_2(a) = \lim_{n \rightarrow \infty} \text{Ad } u_n \circ \tau_1(a) \quad \text{for all } a \in A.$$

For each n , since $[u_n] \in H_1(K_0(A), K_0(B))$, as in the proof of 3.7, one obtains a unitary $w_n \in U(M(B)/B)$ such that $[w_n] = [u_n]$ and $w_n \tau_1(a) = \tau_1(a) w_n$. Now let $v_n = w_n^* u_n$. Note $v_n \in U_0(M(B)/B)$ (by Lemma 3.6). Then

$$\tau_2(a) = \lim_{n \rightarrow \infty} \text{Ad } v_n \circ \tau_1(a) \quad \text{for all } a \in A.$$

So τ_1 and τ_2 are strongly approximately unitarily equivalent. $\square$

Theorem 4.4. *Let A be a unital separable C^* -algebra in $\mathcal{N}$. Let $B = C \otimes \mathcal{K}$, where C is unital such that $M(B)/B$ has property (P). Suppose that $\tau : A \rightarrow M(B)/B$ is a unital full essential extension. Then there is an injection from $K_0(B)/H_1(K_0(A), K_0(B))$ to the set of strong approximate unitary equivalence classes of unital full essential extensions $\sigma : A \rightarrow M(B)/B$ for which $[\sigma] = [\tau]$ in $KL^1(A, B)$.*

Proof. Let $x_1, x_2 \in K_0(B)/H_1(K_0(A), K_0(B))$. Suppose that $u_i \in U(M(B)/B)$ such that $[u_i] = x_i$ in $K_0(B)/H_1(K_0(A), K_0(B))$, $i = 1, 2$. Let $\tau_i = \text{Ad } u_i \circ \tau$, $i = 1, 2$. Suppose that there is a sequence of unitaries $w_n \in U_0(M(B)/B)$ such that $\lim_{n \rightarrow \infty} \text{Ad } w_n \circ \tau_1(a) = \tau_2(a)$ for all $a \in A$. It follows that $\lim_{n \rightarrow \infty} \text{Ad } u_2^* \circ \text{Ad } w_n \circ \tau_1(a) = \tau(a)$ for all $a \in A$. It follows from Proposition 4.2 that there is

a subsequence $\{k(n)\}$ such that $\{[u_2^* w_{k(n)} u_1]\} \in H_1^{ap}(K_0(A), K_0(B))$. This implies that $[\overline{u_1}] = [\overline{u_2}]$ or $x_1 = x_2$. $\square$

Lemma 4.5. *Let A be a separable unital C^* -algebra in $\mathcal{N}$ and let B be a unital purely infinite simple C^* -algebra. Let $\tau : A \rightarrow q_\infty(B) = l^\infty(B)/c_0(B)$ be a unital monomorphism and $\zeta \in H_1(K_0(A), K_1(q_\infty(B)))$. Then there is full monomorphism $\tau_1 : A \otimes C(S^1) \rightarrow q_\infty(B)$ such that*

$$[\tau_1|_A] = [\tau] \text{ in } KL(A, q_\infty(B)) \quad \text{and} \quad \Gamma([\tau_1])([1_A \otimes \iota]) = \zeta \text{ in } K_1(q_\infty(B)),$$

where $\iota(z) = z$ for $z \in S^1$.

Proof. It follows from 8.5 and 7.7 of [15] that there is a (group) isomorphism from the approximately unitary equivalence classes of full monomorphisms from A to $q_\infty(B)$ and $KL(A, q_\infty(B))$. Exactly the same proof as that of Lemma 3.5 proves the theorem. One may also use the identification $KK^1(A, Sq_\infty(B)) = KK(A, q_\infty(B))$ and apply Lemma 3.5. $\square$

Theorem 4.6. *Let B be a σ -unital simple C^* -algebra with continuous scale and let A be a unital separable amenable C^* -algebra in $\mathcal{N}$.*

(i) *Let $\tau : A \rightarrow M(B)/B$ be a unital essential extension and let $\{u_n\} \in U(M(B)/B)$ such that $[u_n] \in H_1^{ap}(K_0(A), K_1(M(B)/B))$ and*

$$\sigma(a) = \lim_{n \rightarrow \infty} \text{Ad } u_n \circ \tau(a) \text{ for all } a \in A,$$

where $\sigma : A \rightarrow M(B)/B$ is another essential extension. Then σ and τ are strongly unitarily equivalent.

(ii) *If*

$$\prod_{n \in \mathbb{N}} K_1(M(B)/B) = H_1^{ap}(K_0(A), K_1(M(B)/B))$$

(in particular, if $K_1(M(B)/B) = H_1(K_0(A), K_1(M(B)/B))$), then two unital essential extensions $\tau_1, \tau_2 : A \rightarrow M(B)/B$ are strongly approximately unitarily equivalent if and only if $[\tau_1] = [\tau_2]$ in $KL(A, M(B)/B)$.

(iii) *If $\tau : A \rightarrow M(B)/B$ is a unital essential extension, then there is an injection from $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B))$ to the set of strong approximate unitary equivalence classes of unital essential extensions σ for which $[\sigma] = [\tau]$ in $KL(A, M(B)/B)$.*

(iv) *In (iii), if furthermore, $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B))$ is finite, then there is a bijection from $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B))$ onto the set of strong approximate unitary equivalence classes of unital essential extensions σ for which $[\sigma] = [\tau]$ in $KL(A, M(B)/B)$.*

(v) *If neither τ nor σ is unital, then τ and σ are strongly approximately unitarily equivalent if and only if $[\sigma] = [\tau]$ in $KL(A, M(B)/B)$.*

Proof. Put $Q = M(B)/B$. Then Q is purely infinite and simple (see Remark 2.6). In particular, Q has K_1 - r -cancellation (for $r(n) = n$) and $\text{cel}(M(B)/B) \leq 2\pi + d$ (for any $d > 0$) (see [16]). To see (i), let $\xi = \pi_{*1}([\{u_n\}])$, where $\pi : l^\infty(Q) \rightarrow q_\infty(Q)$. By passing to a subsequence, without loss of generality, we may assume that $\xi \in H_1(K_0(A), K_1(q_\infty(Q)))$. Let $\Phi : A \rightarrow l^\infty(Q)$ be defined by $\Phi(a) = \{\sigma(a), \tau(a), \dots\}$ for all $a \in A$. Define $\bar{\tau} = \pi \circ \Phi$. It follows from Lemma 4.5 that there exists a full monomorphism $\bar{\tau}_1 : A \otimes C(S^1) \rightarrow q_\infty(Q)$ such that $[(\bar{\tau}_1)|_A] = [\Phi]$

in $KL(A, q_\infty(Q))$ and $[\bar{\tau}_1(1 \otimes \iota)] = \xi$. It follows from 7.7 of [14] that Φ is approximately unitarily equivalent to $(\bar{\tau}_1)|_A$. As in the proof of Theorem 3.7 we may assume that $(\bar{\tau}_1)|_A$ is unital. There is a sequence of unitaries $w_n \in q_\infty(Q)$ such that

$$\lim_{n \rightarrow \infty} \text{Ad } w_n \circ \bar{\tau}_1(a \otimes 1) = \Phi(a) \text{ for all } a \in A.$$

Put $z = \bar{\tau}_1(1 \otimes \iota)$. Then

$$\lim_{n \rightarrow \infty} \|w_n^* z w_n \Phi(a) - \Phi(a) w_n^* z w_n\| = 0 \text{ for all } a \in A.$$

It is well known that there are unitaries $w_n(k), z(k) \in Q$ such that $\pi(\{w_n(k)\}) = w_n$ and $\pi(\{z(k)\}) = z$. Thus there is a subsequence $\{n(k)\}$ such that

$$\lim_{n \rightarrow \infty} \|w_{n(k)}(k)^* z(k) w_{n(k)}(k) \sigma(a) - \sigma(a) w_{n(k)}(k)^* z(k) w_{n(k)}(k)\| = 0 \text{ for all } a \in A.$$

Let $v_k = w_{n(k)}(k)^* z(k) w_{n(k)}(k)$. Then $[v_k] = [z(k)]$ in $K_1(Q)$. By the definition of ξ , one checks that $[v_k] = [u_{n(k)}]$ for all sufficiently large k . Note that $v_k^* u_{n(k)} \in U_0(Q)$, since Q is purely infinite and simple. We have

$$\lim_{n \rightarrow \infty} \text{Ad } u_{n(k)} v_k^* \circ \tau(a) = \sigma(a) \text{ for all } a \in A.$$

This proves (i).

Note that (ii) follows from (i) and Theorem 2.5 immediately, and (iii) follows from the same proof as for Theorem 4.4.

To see (iv), we note that, by (ii), we only need to show that the injection is actually surjective in this case. Suppose that $\{u_n\}$ is a sequence of unitaries such that

$$\sigma(a) = \lim_{n \rightarrow \infty} \text{Ad } u_n \circ \tau(a) \text{ for all } a \in A.$$

Let $\phi : K_1(M(B)/B) \rightarrow K_1(M(B)/B)/(\bar{G} + H_1(K_0(A), K_1(M(B)/B)))$ be the quotient map. By the assumption infinitely many $\phi([u_n])$ are the same. Suppose that $\{k(n)\}$ is a subsequence of $\mathbb{N}$ such that $\phi([u_{k(n)}]) = x$ for $n = 1, 2, \dots$ and for some $x \in K_1(M(B)/B)/(\bar{G} + H_1(K_0(A), K_1(M(B)/B)))$. Choose $u \in U(M(B)/B)$ such that $\phi([u]) = x$. Then

$$\sigma(a) = \lim_{n \rightarrow \infty} \text{Ad } u_{k(n)} u^* \circ \text{Ad } u \circ \tau(a) \text{ for all } a \in A.$$

Since $[u_{k(n)} u^*] \in H_1(K_0(A), K_1(M(B)/B))$, by (i), σ is strongly unitarily equivalent to $\text{Ad } u \circ \tau$. This proves (iv).

For (v), by Theorem 2.5, we only need to show the “if” part. Since $M(B)/B$ is a purely infinite simple C^* -algebra (see [13]), there is a unitary $v_1 \in U(M(B)/B)$ such that $v_1^* \tau_2(1) v_1 = \tau_1(1)$. Put $e = 1 - \tau_1(1)$. Again, since $M(B)/B$ is purely infinite and simple, there is a unitary $v'_2 \in e(M(B)/B)e$ such that $[v'_2] = [v_1^*]$. Put $v_2 = v'_2 + (1 - e)$. Then $v_2^* v_1^* \tau_2(1) v_1 v_2 = \tau_1(1)$. Note that $v_1 v_2 \in U_0(M(B)/B)$. Thus we may assume that $\tau_1(1) = \tau_2(1)$. Let $u_n \in M(B)/B$ be a sequence of unitaries such that

$$\lim_{n \rightarrow \infty} \text{Ad } u_n \circ \tau_1(a) = \tau_2(a) \text{ for all } a \in A.$$

Since $M(B)/B$ is purely infinite and simple, we obtain unitaries $w_n \in e(M(B)/B)e$ such that $[w_n] = [u_n]$. Put $z_n = w_n^* + (1 - e)$. Then $z_n u_n \in U_0(M(B)/B)$. One verifies that

$$\lim_{n \rightarrow \infty} \text{Ad } z_n \circ \tau_1(a) = \tau_2(a) \text{ for all } a \in A.$$

□

Remark 4.7. From Theorem 4.6 we also know that for each $[\tau]$ in $KL(A, M(B)/B)$, there are at most $\prod_{n \in \mathbb{N}} K_1(M(B)/B)/H_1^{ap}(K_0(A), K_1(M(B)/B))$ many different strong approximate unitary equivalence classes of unital essential extensions σ for which $[\sigma] = [\tau]$. However, given a sequence of elements $\{x_n\}$ in $K_1(M(B)/B)$, we do not know if there is a sequence of unitaries $\{u_n\}$ such that $[u_n] = x_n$ and $\text{Ad } u_n \circ \tau(a)$ converges for any $a \in A$. This prevents us from determining exactly how many different strong approximate unitary equivalence classes of unital essential extensions σ with $[\sigma] = [\tau]$ there are. In the case that $B = C \otimes \mathcal{K}$, one can make another estimate. There are at most $|K_0(B)/H_1(K_0(A), K_0(B))| \cdot |\text{ext}_{\mathbb{Z}}(K_*(A), K_*(B))|$ many strong approximate unitary equivalence classes of unital essential extensions σ with $[\sigma] = [\tau]$. However, even if $[\sigma] \neq [\tau]$ in $KK^1(A, B)$ but $[\sigma] = [\tau]$ in $KL^1(A, B)$, they could still be strongly approximately unitarily equivalent. For example, this happens when $K_0(B) = H_1(K_0(A), K_0(B))$, since in this case, by Theorem 4.3, all approximately unitarily equivalent unital full essential extensions are strongly approximately unitarily equivalent. Full extensions in different KK -classes could be strongly approximately unitarily equivalent even in the case where $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B)) \neq \{0\}$. For example, suppose that $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B))$ is finite but $\text{ext}_{\mathbb{Z}}(K_1(A), K_0(M(B)/B))$ is infinite. In this case there are only finitely many strong approximate unitary equivalence classes of unital essential extensions which give the same element in $KL^1(A, M(B)/B)$. Therefore there are infinitely many unital essential extensions in different $KK(A, M(B)/B)$ (but in the same $KL(A, M(B)/B)$) that are strongly approximately unitarily equivalent.

On the other hand, the above results show that there are unital full essential extensions which give the same element in $KK(A, M(B)/B)$ (or in $KK^1(A, B)$) but which may not be strongly approximately unitarily equivalent as long as $K_1(M(B)/B)/H_1(K_0(A), K_1(M(B)/B))$ is not trivial. In short, $KK(A, M(B)/B)/KL(A, M(B)/B)$ cannot be used to distinguish strong approximate unitary equivalence from approximate unitary equivalence. It is $H_1(K_0(A), K_1(M(B)/B))$ (or the approximate version of it) that detects the difference.

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