COMPLETE BOUNDED HOLOMORPHIC CURVES IMMERSED IN $\mathbb{C}^2$ WITH ARBITRARY GENUS

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ABSTRACT. Recently, a complete holomorphic immersion of the unit disk $\mathbb{D}$ into $\mathbb{C}^2$ whose image is bounded was constructed by the authors. In this paper, we shall prove the existence of complete holomorphic null immersions of Riemann surfaces with arbitrary genus and finite topology whose image is bounded in $\mathbb{C}^2$.

As an analogue to the above construction, we also give a new method to construct complete bounded minimal immersions (resp. weakly complete maximal surfaces) with arbitrary genus and finite topology in Euclidean 3-space (resp. Lorentz-Minkowski 3-spacetime).

1. INTRODUCTION

In [MUY], the authors constructed a complete bounded minimal immersion of the unit disk $\mathbb{D}$ into $\mathbb{R}^3$ whose conjugate is also bounded. As applications of our results, we show in this paper that the technique developed by F. J. López in [L] can be suitably modified to give the following three things:

- The first examples of complete bounded complex submanifolds with arbitrary genus immersed in $\mathbb{C}^2$.
- A new and simple method to construct complete, bounded minimal surfaces with arbitrary genus in the Euclidean 3-space.
- A method to construct weakly complete, bounded maximal surfaces with arbitrary genus in the Lorentz-Minkowski 3-spacetime.

Actually, a complete and bounded conformal minimal immersion of $\mathbb{D}$ into $\mathbb{R}^3$ whose conjugate is also bounded is the real part of a complete and bounded null holomorphic immersion

$$X_0 : \mathbb{D} \longrightarrow \mathbb{C}^3.$$

As was shown in [MUY], if we consider the projection

$$(1.1) \quad \pi : \mathbb{C}^3 \ni (z_1, z_2, z_3) \longmapsto (z_1, z_2) \in \mathbb{C}^2,$$

the map $\pi \circ X_0 : \mathbb{D} \rightarrow \mathbb{C}^2$ gives a complete bounded complex submanifold immersed in $\mathbb{C}^2$. By a perturbation of X_0 , considering a similar deformation as in [L], we shall prove the following:

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Theorem A. *Let M be a closed Riemann surface. For an arbitrary positive integer N , there exist points $Q_0, Q_1, \dots, Q_e$ ($e \geq N$) on M , pairwise disjoint neighborhoods U_l of Q_l which are homeomorphic to the unit disc $\mathbb{D}$, and a holomorphic immersion*

$$X : M \setminus \bigcup_{l=0}^e \overline{U_l} \longrightarrow \mathbb{C}^2$$

which is complete and bounded.

If we do not specify the choice of M with the fixed arbitrary genus, we can choose M so that the number of ends is at most 3 (see [L, Lemma 3]).

Among other things, López in [L] gave a method to construct complete minimal surfaces between two parallel planes of arbitrary genus. He constructed these examples as a perturbation of the minimal disk given by Jorge and Xavier in [JX]. However, the global boundedness was not treated in [L], because López' technique is not of use when you apply it directly to Nadirashvili's minimal disk [N]. So, the aim of this paper is to introduce some simple but appropriate changes in López' machinery in order to obtain complete bounded minimal surfaces with nontrivial topology. Also the deformation parameter given in [L] is not sufficient to kill the periods of the holomorphic immersions in $\mathbb{C}^2$, and so we must consider another deformation space and calculate the new periods.

On the other hand, the proof of the above theorem allows us to conclude that

Corollary B. *Let M be a closed Riemann surface. For an arbitrary positive integer N , there exist points $Q_0, \dots, Q_e$ ($e \geq N$) on M , pairwise disjoint neighborhoods U_l of Q_l which are homeomorphic to the unit disc $\mathbb{D}$, and a complete conformal minimal immersion into the Euclidean 3-space $\mathbb{R}^3$ (resp. a weakly complete maximal surface in the sense of [UY3], which may have singular points, whose first fundamental form is conformal on the regular set, in the Lorentz-Minkowski 3-spacetime $\mathbb{L}^3$),*

$$x : M \setminus \bigcup_{l=0}^e \overline{U_l} \longrightarrow \mathbb{R}^3 \quad (\text{resp. } \mathbb{L}^3),$$

whose image is bounded.

Though general methods to produce bounded complete minimal immersions into $\mathbb{R}^3$ with higher genus are already known (cf. [AFM], [FMM], [LMM1] and [LMM2]), this corollary gives a new and short way to construct such examples. On the other hand, for a maximal surface in $\mathbb{L}^3$, this corollary produces the first examples of weakly complete bounded maximal surfaces with arbitrary genus. The *weak completeness* of maximal surfaces, which may admit certain kinds of singularities, was defined in [UY3]. In [A] a weakly complete disk satisfying a certain kind of boundedness was given. After that, a bounded example of genus zero with one end was shown in [MUY].

Finally, we remark that the technique in this paper does not produce bounded null holomorphic curves immersed in $\mathbb{C}^3$ with arbitrary genus because of the difficulty in obtaining the required deformation parameters. If we do succeed in finding an example, the above three examples could be all realized as projections of it. So it is still an open problem to show the existence of complete bounded null holomorphic immersion into $\mathbb{C}^3$ with arbitrary genus.

2. PRELIMINARIES

Let M be a compact Riemann surface of genus κ , and fix a point $Q_0 \in M$. Let

$$(2.1) \quad 1 = d_1 < d_2 < \cdots < d_\kappa \leq 2\kappa - 1$$

be the *Weierstrass gap series* at Q_0 ; that is, there exists a meromorphic function on M which is holomorphic on $M \setminus \{Q_0\}$ such that Q_0 is the pole of order d if and only if $d \notin \{d_1, \dots, d_\kappa\}$. Then there exists a meromorphic function f on M which is holomorphic on $M \setminus \{Q_0\}$, and Q_0 is the pole of order m_0 ($> d_\kappa$). Let $\{Q_1, \dots, Q_e\} \subset M$ be the set of branch points of f . Then the divisor (df) of df is written as

$$(2.2) \quad (df) = \frac{\prod_{l=1}^e Q_l^{m_l}}{Q_0^{m_0+1}} \quad (m_0 > d_\kappa),$$

where m_l ($l = 1, \dots, e$) are positive integers, and the divisor is expressed by the multiplication of these branch points.

Remark 2.1. In the latter construction of surfaces in Theorem A and Corollary B, the ends of surfaces in $\mathbb{C}^2$ or $\mathbb{R}^3$ correspond to the points $Q_0, \dots, Q_e$. We set

$$f_{j+1} := (f_j - c_j)^2, \quad f_0 = f \quad (j = 1, 2, 3, \dots)$$

inductively, where the c_j 's are complex numbers. Then, by replacing f by $f_1, f_2, f_3, \dots$, we can make an example such that e is greater than the given number N .

We write

$$(2.3) \quad M_0 := M \setminus \{Q_0, \dots, Q_e\}.$$

Denoting by $H_{\text{hol}}^1(M)$ and $H_{\text{hol}}^1(M_0)$ the (first) holomorphic de Rham cohomology group of M and M_0 (as the vector space of holomorphic differentials factored by the subspace of exact holomorphic differentials), respectively, we find that

$$\dim H_{\text{hol}}^1(M) = \kappa, \quad \dim H_{\text{hol}}^1(M_0) = n$$

hold, where we set, for the sake of simplicity,

$$(2.4) \quad n := 2\kappa + e.$$

Take a basis $\{\zeta_1, \dots, \zeta_\kappa\}$ of $H_{\text{hol}}^1(M)$.

Lemma 2.2 (cf. III.5.13 in [F]). *One can take a basis of $H_{\text{hol}}^1(M_0)$,*

$$\{\zeta_1, \dots, \zeta_\kappa; \xi_1, \dots, \xi_\kappa; \eta_1, \dots, \eta_e\},$$

where ξ_j is a meromorphic 1-form on M which is holomorphic on $M \setminus \{Q_0\}$, and Q_0 is a pole of order $d_j + 1$, and where η_l is a meromorphic 1-form on M which is holomorphic on $M \setminus \{Q_0, Q_l\}$, and Q_0 and Q_l are poles of order 1.

For simplicity, we set

$$(2.5) \quad \begin{aligned} \zeta_{\kappa+j} &:= \xi_j & (j = 1, \dots, \kappa), \\ \zeta_{2\kappa+l} &:= \eta_l & (l = 1, \dots, e). \end{aligned}$$

Then $\{\zeta_1, \dots, \zeta_n\}$ ($n = 2\kappa + e$) is a basis of $H_{\text{hol}}^1(M_0)$.

Lemma 2.3 ([L, Lemma 1]). *There exists a meromorphic function v on M with the following properties:*

- (1) v is holomorphic on $M_0 = M \setminus \{Q_0, \dots, Q_e\}$,

- (2) Q_l ($l = 1, \dots, e$) is a pole of v whose order is greater than or equal to $m_l + 2$,
and
(3) Q_0 is a pole of v whose order is greater than m_0 .

Proof. For each $l = 1, \dots, e$, let v_l be a meromorphic function on M which is holomorphic on $M \setminus \{Q_l\}$ and so that Q_l is a pole of order $\max\{m_l + 2, 2\kappa\}$. On the other hand, take a meromorphic function v_0 on M which is holomorphic on $M \setminus \{Q_0\}$ and Q_0 is a pole of order $\max\{m_0 + 1, 2\kappa\}$. Then $v = v_0 + v_1 + \dots + v_e$ is the function that proves the lemma. $\square$

Using f as in (2.2) and v as in Lemma 2.3, we define

$$(2.6) \quad \mathcal{G}_\Lambda := \lambda_0 v + \frac{1}{df} \sum_{j=1}^n \lambda_j \zeta_j : M \longrightarrow \mathbb{C} \cup \{\infty\},$$

where

$$\Lambda := (\lambda_0, \lambda_1, \dots, \lambda_n) \in \mathbb{C}^{n+1}.$$

Lemma 2.4 ([L, Section 3]). *The function $\mathcal{G}_\Lambda$ as in (2.6) is a meromorphic function on M such that*

- (1) $\mathcal{G}_\Lambda$ is holomorphic on $M_0 = M \setminus \{Q_0, \dots, Q_e\}$,
(2) if $\Lambda \neq \mathbf{0}$, $\mathcal{G}_\Lambda$ is nonconstant on M , and
(3) if $\lambda_0 \neq 0$, $\mathcal{G}_\Lambda$ has poles at $Q_0, \dots, Q_e$.

Proof. (1) and (2) are obvious. (3) follows from the fact that v has a pole of higher order than $\zeta_1/df, \dots, \zeta_n/df$ at Q_j for each $j = 0, \dots, e$. $\square$

We write

$$|\Lambda| = \sqrt{|\lambda_0|^2 + |\lambda_1|^2 + \dots + |\lambda_n|^2}$$

and consider the unit sphere in the space of Λ :

$$\mathcal{S}_1 := \{\Lambda \in \mathbb{C}^{n+1} ; |\Lambda| = 1\}.$$

The following assertion is a modification of [L, Lemma 2], which is much easier to prove. For our purpose, this weaker assertion is sufficient.

Proposition 2.5. *Let $\Lambda_0 = (a_0, a_1, \dots, a_n)$ be a point in $\mathcal{S}_1$ satisfying $a_0 \neq 0$. Then there exist $\varepsilon(> 0)$ and a neighborhood W of Λ_0 in $\mathcal{S}_1$ such that if $0 < |t| < \varepsilon$ and $\Lambda \in W$, the set $\mathcal{G}_{t\Lambda}^{-1}(\mathbb{D})$ is conformally equivalent to a compact surface of genus κ minus $e+1$ pairwise disjoint discs with analytic regular boundaries. In particular, there are no branch points of $\mathcal{G}_{t\Lambda}$ on the boundary $\partial\mathcal{G}_{t\Lambda}^{-1}(\mathbb{D}) = \mathcal{G}_{t\Lambda}^{-1}(\partial\mathbb{D})$.*

Proof. Since the poles of $\mathcal{G}_{\Lambda_0}$ are exactly $Q_0, \dots, Q_e$ and $\mathcal{G}_{t\Lambda}^{-1}(\mathbb{D}) = \mathcal{G}_\Lambda^{-1}((1/t)\mathbb{D})$, for sufficiently small t , the inverse image $\mathcal{G}_{t\Lambda}^{-1}(\mathbb{D})$ of the unit disk $\mathbb{D}$ by $\mathcal{G}_{t\Lambda}$ is homeomorphic to a compact surface of genus κ minus $e+1$ pairwise disjoint discs with piecewise analytic boundaries. Moreover, since the set of branch points of $\mathcal{G}_{\Lambda_0}$ does not have any accumulation points, $\mathcal{G}_{t\Lambda_0}^{-1}(\partial\mathbb{D})$ has no branch points for sufficiently small t , and $\mathcal{G}_{t\Lambda}^{-1}(\partial\mathbb{D})$ consists of real analytic regular curves in M .

Furthermore, since $a_0 \neq 0$, $\mathcal{G}_{\Lambda_0}/\mathcal{G}_\Lambda$ are holomorphic near $Q_0, \dots, Q_e$ for any Λ which is sufficiently close to Λ_0 . Thus $\mathcal{G}_{t\Lambda}^{-1}(\mathbb{D})$ has the same properties as $\mathcal{G}_{t\Lambda_0}^{-1}(\mathbb{D})$. $\square$

Under the situation in Proposition 2.5, let $\{\gamma_1, \dots, \gamma_{2\kappa}\}$ be a family of loops on $\mathcal{G}_\Lambda^{-1}(\mathbb{D})$ which is a homology basis of M . On the other hand, take a loop $\gamma_{2\kappa+l}$ on $\mathcal{G}_\Lambda^{-1}(\mathbb{D})$ for each $l = 1, \dots, e$, surrounding a neighborhood of Q_l (as in Proposition 2.5).

We define the *period matrix* as

$$(2.7) \quad P = (p_{kj}), \quad p_{kj} := \int_{\gamma_k} \zeta_j \quad (1 \leq j, k \leq n),$$

which is a nondegenerate $n \times n$ matrix.

3. PROOF OF THE MAIN THEOREM

In this section, we give a proof of Theorem A in the introduction.

The initial immersion. Let $X_0: \mathbb{D} \rightarrow \mathbb{C}^3$ be a complete holomorphic null immersion whose image is bounded in $\mathbb{C}^3$ (as in Theorem A in [MUY]), where $\mathbb{D} \subset \mathbb{C}$ is the unit disk. We write

$$(3.1) \quad X_0(z) = \int_0^z (\varphi_1(z), \varphi_2(z), \varphi_3(z)) dz,$$

where z is a canonical coordinate on $\mathbb{D} \subset \mathbb{C}$ and φ_j ($j = 1, 2, 3$) are holomorphic functions on $\mathbb{D}$. Since X_0 is null, it holds that

$$(3.2) \quad (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0.$$

Let $(g, \omega := \omega_0 dz)$ be the Weierstrass data of X_0 ; that is,

$$(3.3) \quad \varphi_1 = (1 - g^2)\omega_0, \quad \varphi_2 = i(1 + g^2)\omega_0, \quad \varphi_3 = 2g\omega_0,$$

where $i = \sqrt{-1}$.

Lemma 3.1. *Let $X_0: \mathbb{D} \rightarrow \mathbb{C}^3$ be a null holomorphic immersion as in (3.1) whose image is not contained in any plane. Then there exists a point $z_0 \in \mathbb{D}$ and a complex orthogonal transformation $T: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ in $O(3, \mathbb{C}) = \{a \in GL(3, \mathbb{C}); {}^t a = a^{-1}\}$ (${}^t a$ is the transpose of a) such that, up to replacing X_0 by $T \circ X_0$, the following properties hold:*

- (1) $\varphi_1(z_0) = 0$,
- (2) $\varphi_3(z_0) \neq 0$ and $\varphi'_3(z_0) \neq 0$, where $' = d/dz$,
- (3) $\varphi_2(z_0) = i\varphi_3(z_0)$ and $\varphi'_2(z_0) = -i\varphi'_3(z_0)$.

Moreover, if X_0 is complete and bounded, then so is $T \circ X_0$.

We shall now assume that our initial X_0 satisfies the three properties above and set $z_0 = 0$ by a coordinate change of $\mathbb{D}$.

Proof of Lemma 3.1. Since the image of X_0 is not contained in any plane, at least one of φ_1 , φ_2 and φ_3 , say φ_3 , is not constant. Then we can take $z_0 \in \mathbb{D}$ such that $\varphi_3(z_0) \neq 0$ and $\varphi'_3(z_0) \neq 0$. Moreover, if $\varphi_1 + i\varphi_2$ or $\varphi_1 - i\varphi_2$ vanishes identically, this contradicts the fact that the image of X_0 is not contained in any plane. So we may also assume that

$$\varphi_1(z_0) \neq \pm i\varphi_2(z_0).$$

When $\varphi_1(z_0) \neq 0$, we replace X_0 by $T \circ X_0$, where T is the linear map associated with a complex orthogonal matrix

$$\begin{pmatrix} -c(1+c^2)^{-1/2} & (1+c^2)^{-1/2} & 0 \\ - (1+c^2)^{-1/2} & -c(1+c^2)^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (c = \varphi_2(z_0)/\varphi_1(z_0)).$$

Since T is orthogonal, $T \circ X_0$ is also a null holomorphic immersion. So we get property (1). Then (3.2) implies that $\varphi_2(z_0) = \pm i\varphi_3(z_0)$. Replacing φ_2 by $-\varphi_2$ if necessary, we may assume $\varphi_2(z_0) = i\varphi_3(z_0)$. Differentiating (3.2), we have $\varphi_1(z_0)\varphi'_1(z_0) + \varphi_2(z_0)\varphi'_2(z_0) + \varphi_3(z_0)\varphi'_3(z_0) = 0$. Then

$$-\varphi_3(z_0)\varphi'_3(z_0) = \varphi_1(z_0)\varphi'_1(z_0) + \varphi_2(z_0)\varphi'_2(z_0) = i\varphi_3(z_0)\varphi'_2(z_0)$$

holds. In particular, it holds that $\varphi'_2(z_0) = -i\varphi'_3(z_0)$.

Next we prove the boundedness and completeness of $T \circ X_0$ under the assumption that X_0 is complete and bounded. Since X_0 is bounded and T is continuous, $X_0(\mathbb{D})$ and $T(X_0(\mathbb{D}))$ are both contained in the closed ball $\overline{B_0}(R)$ in $\mathbb{C}^3$ of a certain radius $R > 0$ centered at the origin. We denote by h_0 the canonical metric on $\mathbb{C}^3$ and consider the pull-back metric $h_1 = T^*h_0$ by T . Now we apply Lemma 3.1 in [MUY] for $K := \overline{B_0}(R)$ in $N := \mathbb{C}^3$. Then there exist positive numbers a and b ($0 < a < b$) such that $ah_0 < h_1 < bh_0$ on $\overline{B_0}(R)$. Now, we consider the pullback of the metric h_0 and h_1 by the immersion X_0 . Since

$$(X_0)^*h_1 = (X_0)^*(T^*h_0) = (T \circ X_0)^*h_0 = (X_1)^*h_0,$$

we have that

$$a \cdot (X_0)^*h_0 < (X_1)^*h_0 < b \cdot (X_0)^*h_0$$

on $\mathbb{D}$. Since X_0 is complete, $(X_0)^*h_0$ is a complete Riemannian metric on $\mathbb{D}$. Then the above relation implies that $(X_1)^*h_0$ is also a complete Riemannian metric on $\mathbb{D}$; that is, X_1 is also a complete immersion. $\square$

A family of holomorphic immersions. Let

$$\Lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{C}^{n+1}, \quad \Delta = (\delta_1, \dots, \delta_n) \in \mathbb{C}^n,$$

and take a meromorphic function $\mathcal{G}_\Lambda$ as in (2.6). Define a meromorphic function $\mathcal{F}_\Delta$ as

$$(3.4) \quad \mathcal{F}_\Delta = \frac{1}{df} \sum_{j=1}^n \delta_j \zeta_j : M \longrightarrow \mathbb{C} \cup \{\infty\}$$

and Weierstrass data $(\hat{g}, \hat{\omega})$ on $\mathcal{G}_\Lambda^{-1}(\mathbb{D})$ as

$$(3.5) \quad \hat{g} = \hat{g}_{(\Lambda, \Delta)} := h_\Delta \cdot (g \circ \mathcal{G}_\Lambda), \quad \hat{\omega} = \hat{\omega}_{(\Lambda, \Delta)} := \frac{\omega_0 \circ \mathcal{G}_\Lambda df}{h_\Delta} \quad (h_\Delta := \exp \mathcal{F}_\Delta),$$

where f is the meromorphic function as in (2.2), and define holomorphic 1-forms on $\mathcal{G}_\Lambda^{-1}(\mathbb{D})$ as

$$(3.6) \quad \Psi_1 = (1 - \hat{g}^2)\hat{\omega}, \quad \Psi_2 = i(1 + \hat{g}^2)\hat{\omega}, \quad \Psi_3 = 2\hat{g}\hat{\omega}.$$

The following lemma is a modified version of [L, Theorem 3]. (In fact, our data (3.4) and (3.5) for the surfaces are somewhat different from those in [L].)

Lemma 3.2. *If X_0 as in (3.1) is a complete immersion, the metric*

$$d\hat{s}^2 := (1 + |\hat{g}|^2)^2 |\hat{\omega}|^2$$

determined by $(\hat{g}, \hat{\omega})$ as in (3.5) is a complete Riemannian metric on $\mathcal{G}_\Lambda^{-1}(\mathbb{D})$ for a sufficiently small $(\Lambda, \Delta) \neq (\mathbf{0}, \mathbf{0})$.

Proof. As in the equations (15) and (17) in [L], there exists a positive constant a (< 1) such that

$$a < |h_\Delta| < \frac{1}{a} \quad \text{and} \quad \left| \frac{df}{d\mathcal{G}_\Lambda} \right| > a \quad \text{on } \mathcal{G}_\Lambda^{-1}(\mathbb{D}).$$

Then (setting $z = \mathcal{G}_\Lambda$),

$$\begin{aligned} (1 + |\hat{g}|^2) |\hat{\omega}| &= (1 + |gh_\Delta|^2) \left| \frac{\omega_0}{h_\Delta} \right| \left| \frac{df}{d\mathcal{G}_\Lambda} \right| |dz| \\ &\geq (a^2 + |ag|^2) |a\omega_0| (a|dz|) = a^4 (1 + |g|^2) |\omega|. \end{aligned}$$

Thus we have the conclusion. $\square$

Thus, for each (sufficiently small) $(\Lambda, \Delta) \in \mathbb{C}^{2n+1} \setminus \{\mathbf{0}\}$, there exists a complete null immersion

$$(3.7) \quad X_{(\Lambda, \Delta)} := \int_{z_0}^z (\Psi_1, \Psi_2, \Psi_3) : \widetilde{\mathcal{G}_\Lambda^{-1}(\mathbb{D})} \longrightarrow \mathbb{C}^3,$$

where $\widetilde{\mathcal{G}_\Lambda^{-1}(\mathbb{D})}$ denotes the universal cover of $\mathcal{G}_\Lambda^{-1}(\mathbb{D})$. In fact, the line integral $\int_{z_0}^z \Psi_j$ ($j = 1, 2, 3$) from a base point z_0 depends on the choice of the path but can be considered as a single-valued function on $\widetilde{\mathcal{G}_\Lambda^{-1}(\mathbb{D})}$.

Then we get the following assertion, which can be proven in exactly the same way as Corollary B in [MUY]:

Proposition 3.3. *Let π be the projection as in (1.1). Then $\pi \circ X_{(\Lambda, \Delta)}$ is a complete immersion of $\widetilde{\mathcal{G}_\Lambda^{-1}(\mathbb{D})}$ into $\mathbb{C}^2$.*

The period map. Under the situations above, we define the *period map*

$$(3.8) \quad \text{Per}_1 : \mathbb{C}^{2n+1} \ni (\Lambda, \Delta) \longmapsto \left(\int_{\gamma_1}^t \Psi_1, \dots, \int_{\gamma_n}^t \Psi_1, \int_{\gamma_1} \Psi_2, \dots, \int_{\gamma_n} \Psi_2 \right) \in \mathbb{C}^{2n},$$

where $n = 2\kappa + e$ (see (2.4)), “ $^t(\)$ ” is the transposing operation for matrices, the γ_j ’s are loops as in (2.7), and Ψ_1 and Ψ_2 are as in (3.6). The following assertion is an analogue of [L, Theorem 2]:

Proposition 3.4. *Suppose that X_0 satisfies the three conditions as in Lemma 3.1. Then the $(2n) \times (2n)$ matrix*

$$(3.9) \quad J_1 := \left(\frac{\partial \text{Per}_1}{\partial \lambda_1}, \dots, \frac{\partial \text{Per}_1}{\partial \lambda_n}, \frac{\partial \text{Per}_1}{\partial \delta_1}, \dots, \frac{\partial \text{Per}_1}{\partial \delta_n} \right) \bigg|_{(\Lambda, \Delta) = (\mathbf{0}, \mathbf{0})}$$

is nondegenerate.

Proof. Note that

$$(3.10) \quad \mathcal{G}_\Lambda|_{\Lambda=\mathbf{0}} = \mathcal{F}_\Delta|_{\Delta=\mathbf{0}} = 0, \quad h_\Delta|_{\Delta=\mathbf{0}} = 1.$$

By the definitions, we have

$$\left. \frac{\partial \mathcal{G}_\Lambda}{\partial \lambda_j} \right|_{\Lambda=\mathbf{0}} = \frac{\zeta_j}{df}, \quad \left. \frac{\partial h_\Delta}{\partial \delta_j} \right|_{\Delta=\mathbf{0}} = \left. \frac{\partial \exp \mathcal{F}_\Delta}{\partial \delta_j} \right|_{\Delta=\mathbf{0}} = \frac{\zeta_j}{df}$$

for $j = 1, \dots, n$. Then

$$\begin{aligned} \left. \frac{\partial \hat{g}}{\partial \lambda_j} \right|_{(\Lambda, \Delta)=(\mathbf{0}, \mathbf{0})} &= \left(\left. \frac{dg}{dz} \right|_{z=0} \right) \left(\left. \frac{\partial \mathcal{G}_\Lambda}{\partial \lambda_j} \right|_{\Lambda=\mathbf{0}} \right) = g'(0) \frac{\zeta_j}{df}, \\ \left. \frac{\partial \hat{\omega}}{\partial \lambda_j} \right|_{(\Lambda, \Delta)=(\mathbf{0}, \mathbf{0})} &= \omega'_0(0) \zeta_j, \\ \left. \frac{\partial \hat{g}}{\partial \delta_j} \right|_{(\Lambda, \Delta)=(\mathbf{0}, \mathbf{0})} &= g(0) \frac{\zeta_j}{df}, \quad \left. \frac{\partial \hat{\omega}}{\partial \delta_j} \right|_{(\Lambda, \Delta)=(\mathbf{0}, \mathbf{0})} = -\omega_0(0) \zeta_j, \end{aligned}$$

where $' = d/dz$. Hence we have

$$\begin{aligned} \left. \frac{\partial \Psi_1}{\partial \lambda_j} \right|_{(\Lambda, \Delta)=(\mathbf{0}, \mathbf{0})} &= \left(-2\hat{g} \frac{\partial \hat{g}}{\partial \lambda_j} \hat{\omega} + (1 - \hat{g}^2) \frac{\partial \hat{\omega}}{\partial \lambda_j} \right) \Big|_{(\Lambda, \Delta)=(\mathbf{0}, \mathbf{0})} \\ &= (-2g(0)g'(0)\omega_0(0) + (1 - g(0)^2)\omega'_0(0))\zeta_j \\ &= ((1 - g^2)\omega)' \Big|_{z=0} \zeta_j = \varphi'_1(0)\zeta_j, \\ \left. \frac{\partial \Psi_1}{\partial \delta_j} \right|_{(\Lambda, \Delta)=(\mathbf{0}, \mathbf{0})} &= -(1 + g(0)^2)\omega_0(0) = i\varphi_2(0)\zeta_j, \end{aligned}$$

where the φ_j 's are holomorphic functions on $\mathbb{D}$ as in (3.3). Similarly, we have

$$\frac{\partial \Psi_2}{\partial \lambda_j} = \varphi'_2(0)\zeta_j, \quad \frac{\partial \Psi_3}{\partial \lambda_j} = \varphi'_3(0)\zeta_j, \quad \frac{\partial \Psi_2}{\partial \delta_j} = -i\varphi_1(0)\zeta_j, \quad \frac{\partial \Psi_3}{\partial \delta_j} = 0$$

at $(\Lambda, \Delta) = (\mathbf{0}, \mathbf{0})$. Thus, we have that

$$(3.11) \quad \begin{aligned} \frac{\partial}{\partial \lambda_j} \int_{\gamma_k} \Psi_1 &= \varphi'_1(0) \int_{\gamma_k} \zeta_j, & \frac{\partial}{\partial \delta_j} \int_{\gamma_k} \Psi_1 &= i\varphi_2(0) \int_{\gamma_k} \zeta_j, \\ \frac{\partial}{\partial \lambda_j} \int_{\gamma_k} \Psi_2 &= \varphi'_2(0) \int_{\gamma_k} \zeta_j, & \frac{\partial}{\partial \delta_j} \int_{\gamma_k} \Psi_2 &= -i\varphi_1(0) \int_{\gamma_k} \zeta_j, \\ \frac{\partial}{\partial \lambda_j} \int_{\gamma_k} \Psi_3 &= \varphi'_3(0) \int_{\gamma_k} \zeta_j, & \frac{\partial}{\partial \delta_j} \int_{\gamma_k} \Psi_3 &= 0 \end{aligned}$$

hold at $(\Lambda, \Delta) = (\mathbf{0}, \mathbf{0})$, for $j, k = 1, \dots, n$. Hence the matrix J_1 in (3.9) is written as

$$J_1 = \begin{pmatrix} \varphi'_1(0)P & i\varphi_2(0)P \\ \varphi'_2(0)P & -i\varphi_1(0)P \end{pmatrix} = \begin{pmatrix} \varphi'_1(0)P & i\varphi_2(0)P \\ \varphi'_2(0)P & O \end{pmatrix} \quad (\varphi_2(0), \varphi'_2(0) \neq 0)$$

because of Lemma 3.1, where P is the nondegenerate period matrix as in (2.7). Hence J_1 is nondegenerate. $\square$

The period-killing problem. Since $\text{Per}_1(\mathbf{0}) = \mathbf{0}$, Proposition 3.4 yields that there exists a holomorphic map $c \mapsto (\lambda_1(c), \dots, \lambda_n(c), \delta_1(c), \dots, \delta_n(c))$ such that

$$\text{Per}_1(c, \lambda_1(c), \dots, \lambda_n(c), \delta_1(c), \dots, \delta_n(c)) = 0$$

for sufficiently small c . We set

$$\mathcal{G}_c = \mathcal{G}_{\Lambda(c)}, \quad \text{where } \Lambda(c) := (c, \lambda_1(c), \dots, \lambda_n(c)).$$

Since $\Lambda(0) = \mathbf{0}$, there exists an analytic function $\Lambda^*(c)$ such that $\Lambda(c) = c\Lambda^*(c)$ near $c = 0$. Now we can apply Proposition 2.5 to $\Lambda_0 := \Lambda^*(0)/|\Lambda^*(0)| \in \mathcal{S}_1$. Then, for sufficiently small c , $\mathcal{G}_c^{-1}(\mathbb{D})$ is conformally equivalent to M minus $e+1$ pairwise disjoint discs with analytic regular boundaries, and the map

$$(3.12) \quad \begin{aligned} X_c &:= \pi \circ X_{(\Lambda(c), \Delta(c))}, \\ \Lambda(c) &= (c, \lambda_1(c), \dots, \lambda_n(c)), \\ \Delta(c) &= (\delta_1(c), \dots, \delta_n(c)) \end{aligned}$$

is well-defined on $\mathcal{G}_c^{-1}(\mathbb{D})$. Moreover, by Proposition 3.3, X_c is a complete immersion for any sufficient small c .

Boundedness of X_c . By Proposition 2.5, $d\mathcal{G}_c$ does not vanish on $\partial\mathcal{G}_c^{-1}(\mathbb{D})$ for sufficiently small c . Then if we choose a real number $r \in (0, 1)$ sufficiently close to 1, we have

$$d\mathcal{G}_c \neq 0 \quad \text{on } \mathcal{G}_c^{-1}(\overline{\mathbb{D}} \setminus \mathbb{D}_r),$$

where $\mathbb{D}_r = \{z \in \mathbb{C}; |z| < r\}$. Moreover, $\mathcal{G}_c^{-1}(\overline{\mathbb{D}} \setminus \mathbb{D}_r)$ is exactly a union of $e+1$ closed annular domains surrounding the points $Q_0, Q_1, \dots, Q_e$ in M . To show the boundedness of X_c , it is sufficient to show that the image of each annular domain by X_c is bounded. For the sake of simplicity, we show the boundedness of X_c at Q_0 . We denote by $\overline{\Omega}$ the closed annular domain surrounding the point Q_0 . Then

$$\mathcal{G} := \mathcal{G}_c|_{\overline{\Omega}} : \overline{\Omega} \longrightarrow \overline{\mathbb{D}} \setminus \mathbb{D}_r$$

gives a holomorphic finite covering.

Since $\mathcal{G}_c^{-1}(\overline{\mathbb{D}}_r) \subset \mathcal{G}_c^{-1}(\mathbb{D})$ is compact and $X_c : \mathcal{G}_c^{-1}(\mathbb{D}) \rightarrow \mathbb{C}^2$ is holomorphic, there exists a positive constant K_0 such that

$$(3.13) \quad |X_c| \leq K_0 \quad \text{on } \mathcal{G}_c^{-1}(\overline{\mathbb{D}}_r),$$

where $|X_c| = \sqrt{|X_1|^2 + |X_2|^2}$ ($X_c = (X_1, X_2)$). We denote by Ω the set of interior points of $\overline{\Omega}$ and fix $q \in \Omega$ arbitrarily. Let $\sigma(t)$ ($0 \leq t \leq 1$) be a line segment such that $\sigma(0) = \mathcal{G}(q)$ and $\sigma(1) = r\mathcal{G}(q)/|\mathcal{G}(q)| \in \partial\mathbb{D}_r$; that is,

$$\sigma(t) := (1-t)\mathcal{G}(q) + t \frac{r\mathcal{G}(q)}{|\mathcal{G}(q)|}.$$

Since $\mathcal{G}$ is a covering map, there exists a unique smooth curve $\tilde{\sigma} : [0, 1] \rightarrow \overline{\Omega}$ such that $\tilde{\sigma}(0) = q$ and $\mathcal{G} \circ \tilde{\sigma} = \sigma$. Moreover, there exists a neighborhood U of the line segment $\sigma([0, 1])$ and a holomorphic map $\mathcal{H} : U \rightarrow M$ which gives the (local) inverse of $\mathcal{G}$. By definition, we have $\tilde{\sigma} = \mathcal{H} \circ \sigma$. We set

$$q_1 := \tilde{\sigma}(1) \in \partial\Omega \setminus \mathcal{G}_c^{-1}(\partial\mathbb{D});$$

that is, q_1 lies on the connected component of $\partial\Omega$ which is further from Q_0 (see Figure 1).

FIGURE 1. The segments $\tilde{\sigma}$ and σ

By (3.13), it is sufficient to show that

$$(3.14) \quad |X_c(q_1) - X_c(q)| = \left| \int_{\tilde{\sigma}} (\Psi_1, \Psi_2) \right|$$

is bounded from above by a constant which does not depend on q . Here we have

$$(3.15) \quad \begin{aligned} \left| \int_{\tilde{\sigma}} \Psi_1 \right| &= \left| \int_{\tilde{\sigma}} \left(1 - (h_{\Delta})^2 (g \circ \mathcal{G}_c)^2 \right) \frac{\omega_0 \circ \mathcal{G}_c}{h_{\Delta}} df \right| \\ &= \left| \int_0^1 \left(\left(1 - (g \circ \sigma(t))^2 (h(t))^2 \right) \frac{\omega_0 \circ \sigma(t)}{h(t)} \frac{d(f \circ \mathcal{H}(z))}{dz} \right) \Big|_{z=\sigma(t)} \frac{d\sigma}{dt} dt \right| \\ &= |I_1 + I_2|, \end{aligned}$$

where $h(t) := h_{\Delta} \circ \tilde{\sigma}(t)$ and

$$\begin{aligned} I_1 &= \frac{1}{2} \int_0^1 \left((\varphi_1(z) - i\varphi_2(z)) \Big|_{z=\sigma(t)} \frac{d\sigma}{dt} \right) \left(\frac{1}{h_{\Delta} \circ \mathcal{H}(z)} \frac{df \circ \mathcal{H}(z)}{dz} \Big|_{z=\sigma(t)} \right) dt, \\ I_2 &= \frac{-1}{2} \int_0^1 \left((\varphi_1(z) + i\varphi_2(z)) \Big|_{z=\sigma(t)} \frac{d\sigma}{dt} \right) \left(h_{\Delta} \circ \mathcal{H}(z) \frac{df \circ \mathcal{H}(z)}{dz} \Big|_{z=\sigma(t)} \right) dt. \end{aligned}$$

Here we used the relations $2\omega_0 = \varphi_1 - i\varphi_2$ and $2g^2\omega_0 = -(\varphi_1 + i\varphi_2)$. To estimate I_1 , we shall apply Lemma A.1 in the appendix for

$$a(t) = (\varphi_1(z) - i\varphi_2(z)) \Big|_{z=\sigma(t)} \frac{d\sigma}{dt} \quad \text{and} \quad b(t) = \frac{1}{h_{\Delta} \circ \mathcal{H}(z)} \frac{df \circ \mathcal{H}(z)}{dz} \Big|_{z=\sigma(t)}.$$

Let us check the hypotheses:

$$(A(s) :=) \int_0^s a(t) dt = \int_{\sigma([0,s])} (\varphi_1(z) - i\varphi_2(z)) dz = (X_1(z) - iX_2(z)) \Big|_{z=\sigma(0)}^{z=\sigma(s)},$$

where $X = X_0$ as in (3.1). Since X is bounded, $|A(s)|$ is bounded for all s . On the other hand,

$$b(t) = \tilde{b}(\sigma(t)), \quad \text{where} \quad \tilde{b}(z) = \frac{1}{h_{\Delta} \circ \mathcal{H}(z)} \frac{d(f \circ \mathcal{H}(z))}{dz}.$$

Since $\mathcal{H}(z)$ can be considered as a single-valued holomorphic function on a certain finite covering of $\overline{\mathbb{D}} \setminus \mathbb{D}_r$, both $\tilde{b}(z)$ and $\tilde{b}'(z)$ ($' = d/dz$) are bounded by a constant. Hence by the lemma we have that $|I_1|$ is bounded. Similarly, we can show that $|I_2|$ is bounded, and we can conclude that the integration of Ψ_1 (and similarly of Ψ_2) along $\tilde{\sigma}$ is bounded.

The resulting immersion X_c can have an arbitrary number of ends by setting $f = f_N$ as in Remark 2.1.

4. PROOF OF THE COROLLARY

In this section we shall prove Corollary B in the introduction. Define $\mathcal{G}_\Lambda$, (Ψ_1, Ψ_2, Ψ_3) and the null immersion $X_{(\Lambda, \Delta)}$ as in the previous section, and define real parameters (s_j, t_j) as

$$(4.1) \quad \lambda_0 = s_0 + it_0, \quad \lambda_j = s_j + it_j, \quad \delta_j = s_{n+j} + it_{n+j} \quad (j = 1, \dots, n).$$

Note that for a holomorphic function $F(u)$ in $u = s + it$, one has

$$(4.2) \quad \begin{aligned} \frac{\partial \operatorname{Re} F}{\partial s} &= \operatorname{Re} \frac{dF}{du}, & \frac{\partial \operatorname{Re} F}{\partial t} &= -\operatorname{Im} \frac{dF}{du}, \\ \frac{\partial \operatorname{Im} F}{\partial s} &= \operatorname{Im} \frac{dF}{du}, & \frac{\partial \operatorname{Im} F}{\partial t} &= \operatorname{Re} \frac{dF}{du}. \end{aligned}$$

Minimal surfaces in $\mathbb{R}^3$. First, we treat the case of minimal surfaces in $\mathbb{R}^3$. Let

$$x = x_{(\Lambda, \Delta)} := \operatorname{Re} X_{(\Lambda, \Delta)} = \operatorname{Re} \int (\Psi_1, \Psi_2, \Psi_3) : \widetilde{\mathcal{G}_\Lambda^{-1}(\mathbb{D})} \longrightarrow \mathbb{R}^3,$$

where $\widetilde{\mathcal{G}_\Lambda^{-1}(\mathbb{D})}$ is the universal cover of $\mathcal{G}_\Lambda^{-1}(\mathbb{D})$. Then x is a conformal minimal immersion, and the induced metric is complete because of Lemma 3.2. If $x_{(\Lambda, \Delta)}$ were well-defined on $\mathcal{G}_\Lambda^{-1}(\mathbb{D})$ for sufficiently small (Λ, Δ) , boundedness of the image of $x_{(\Lambda, \Delta)}$ could be proved in a similar way as in the previous section. Hence it is sufficient to solve the period-killing problem to show the corollary.

We define the period map

$$(4.3) \quad \begin{aligned} \operatorname{Per}_2 : \mathbb{R}^{4n+2} \ni (s_0, \dots, s_{2n}; t_0, \dots, t_{2n}) \\ \longmapsto \left(\left(\operatorname{Re} \int_{\gamma_k} \Psi_1 \right)_{k=1, \dots, n}, \left(\operatorname{Re} \int_{\gamma_k} \Psi_2 \right)_{k=1, \dots, n}, \left(\operatorname{Re} \int_{\gamma_k} \Psi_3 \right)_{k=1, \dots, n} \right) \in \mathbb{R}^{3n}, \end{aligned}$$

where (s_j, t_j) are real parameters as in (4.1). Consider the $(3n) \times (4n)$ matrix

$$(4.4) \quad J_2 := \left(\frac{\partial \operatorname{Per}_2}{\partial s_1}, \dots, \frac{\partial \operatorname{Per}_2}{\partial s_{2n}}, \frac{\partial \operatorname{Per}_2}{\partial t_1}, \dots, \frac{\partial \operatorname{Per}_2}{\partial t_{2n}} \right) \Big|_{(\Lambda, \Delta) = (\mathbf{0}, \mathbf{0})}.$$

To solve the period-killing problem, it is sufficient to show that the rank of J_2 is $3n$.

By (3.11) and (4.2) we have

$$J_2 = \begin{pmatrix} \operatorname{Re}(\varphi'_1(0)P) & \operatorname{Re}(i\varphi'_2(0)P) & -\operatorname{Im}(\varphi'_1(0)P) & -\operatorname{Im}(i\varphi'_2(0)P) \\ \operatorname{Re}(\varphi'_2(0)P) & -\operatorname{Re}(i\varphi'_1(0)P) & -\operatorname{Im}(\varphi'_2(0)P) & \operatorname{Im}(i\varphi'_1(0)P) \\ \operatorname{Re}(\varphi'_3(0)P) & O & -\operatorname{Im}(\varphi'_3(0)P) & O \end{pmatrix},$$

where the $n \times n$ matrix P is the period matrix as in (2.7). Here, we remark that the real vectors are linearly independent over $\mathbb{R}$ if and only if they are linearly

independent over $\mathbb{C}$. Since we may assume that X_0 satisfies the three conditions as in Lemma 3.1, we have that

$$\begin{aligned} \text{rank } J_2 &= \text{rank} \begin{pmatrix} \varphi'_1(0)P & i\varphi_2(0)P & \overline{\varphi'_1(0)P} & \overline{i\varphi_2(0)P} \\ \varphi'_2(0)P & -i\varphi_1(0)P & \overline{\varphi'_2(0)P} & -\overline{i\varphi_1(0)P} \\ \varphi'_3(0)P & O & \overline{\varphi'_3(0)P} & O \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} \varphi'_1(0)P & i\varphi_2(0)P & \overline{\varphi'_1(0)P} & \overline{i\varphi_2(0)P} \\ -i\varphi'_3(0)P & O & i\overline{\varphi'_3(0)P} & O \\ \varphi'_3(0)P & O & \overline{\varphi'_3(0)P} & O \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} O & P & * & * \\ P & O & * & * \\ O & O & 2\overline{P} & * \end{pmatrix} = 3n, \end{aligned}$$

since P is nondegenerate. Then we can solve the period problem as we did in the previous section.

Maximal surfaces in the Lorentz-Minkowski spacetime. We denote by $\mathbb{L}^3$ the Lorentz-Minkowski 3-spacetime, that is, $(\mathbb{R}^3; (x_0, x_1, x_2))$ endowed with the indefinite metric $-(dx_0)^2 + (dx_1)^2 + (dx_2)^2$. Under the same settings as above, we set

$$y = y_{(\Lambda, \Delta)} := \text{Re} \int (i\Psi_1, \Psi_2, \Psi_3) : \widetilde{\mathcal{G}_\Lambda^{-1}(\mathbb{D})} \longrightarrow \mathbb{L}^3.$$

Then y gives a maximal surface (a mean curvature zero surface), which could possibly have singular points. In particular, since the holomorphic lift

$$(4.5) \quad \int (i\Psi_1, \Psi_2, \Psi_3) : \widetilde{\mathcal{G}_\Lambda^{-1}(\mathbb{D})} \longrightarrow \mathbb{C}^3$$

is an immersion, y is a *maxface* in the sense of [UY3]. Moreover, the induced metric by (4.5) is complete because of Lemma 3.2. Hence y is a *weakly complete maxface* in the sense of [UY3].

To show the $\mathbb{L}^3$ case of Corollary B, we consider the period map

$$(4.6) \quad \begin{aligned} \text{Per}_3 : \mathbb{R}^{4n+2} \ni (s_0, \dots, s_{2n}; t_0, \dots, t_{2n}) \\ \longmapsto \left(\left(\text{Im} \int_{\gamma_k} \Psi_1 \right)_{k=1, \dots, n}, \left(\text{Re} \int_{\gamma_j} \Psi_2 \right)_{j=1, \dots, n}, \left(\text{Re} \int_{\gamma_j} \Psi_3 \right)_{j=1, \dots, n} \right) \in \mathbb{R}^{3n}. \end{aligned}$$

Then

$$(4.7) \quad J_3 := \left(\frac{\partial \text{Per}_3}{\partial s_1}, \dots, \frac{\partial \text{Per}_3}{\partial s_{2n}}, \frac{\partial \text{Per}_3}{\partial t_1}, \dots, \frac{\partial \text{Per}_3}{\partial t_{2n}} \right) \Big|_{(\Lambda, \Delta) = (\mathbf{0}, \mathbf{0})}$$

has the expression

$$J_3 = \begin{pmatrix} \text{Im}(\varphi'_1(0)P) & \text{Im}(i\varphi_2(0)P) & \text{Re}(\varphi'_1(0)P) & \text{Re}(i\varphi_2(0)P) \\ \text{Re}(\varphi'_2(0)P) & -\text{Re}(i\varphi_1(0)P) & -\text{Im}(\varphi'_2(0)P) & \text{Im}(i\varphi_1(0)P) \\ \text{Re}(\varphi'_3(0)P) & O & -\text{Im}(\varphi'_3(0)P) & O \end{pmatrix},$$

and then it can be easily checked that J_3 is of rank $3n$ as in the case of $\mathbb{R}^3$. Hence we conclude as in the previous cases.

APPENDIX A. A LEMMA TO SHOW BOUNDEDNESS

In this appendix, we show the following lemma:

Lemma A.1. *Let a and $b: I \rightarrow \mathbb{C}$ be smooth functions, where $I = [0, L]$ is an interval in $\mathbb{R}$. Suppose that there exist constants C_1 , C_2 and C_3 such that*

$$\left| \int_0^s a(t) dt \right| < C_1, \quad |b(s)| < C_2, \quad \text{and} \quad |b'(s)| < C_3$$

for all $s \in I$, where $' = d/ds$. Then there exists a constant $C = C(C_1, C_2, C_3, L)$ such that

$$\left| \int_I a(t)b(t) dt \right| < C.$$

Proof. Let

$$A(s) = \int_0^s a(t) dt.$$

Then

$$\begin{aligned} \left| \int_I a(t)b(t) dt \right| &= \left| \int_I A'(t)b(t) dt \right| = \left| A(t)b(t) \Big|_{t=0}^{t=L} - \int_I A(t)b'(t) dt \right| \\ &= \left| A(L)b(L) - \int_I A(t)b'(t) dt \right| \leq |A(L)b(L)| + \left| \int_I A(t)b'(t) dt \right| \\ &\leq C_1 C_2 + L C_1 C_3. \end{aligned} \quad \square$$

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