DISCREPANCY FOR RANDOMIZED RIEMANN SUMS

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ABSTRACT. Given a finite sequence $U_N = \{u_1, \dots, u_N\}$ of points contained in the d -dimensional unit torus, we consider the L^2 discrepancy between the integral of a given function and the Riemann sums with respect to translations of U_N . We show that with positive probability, the L^2 discrepancy of other sequences close to U_N in a certain sense preserves the order of decay of the discrepancy of U_N . We also study the role of the regularity of the given function.

1. INTRODUCTION

Let $N \in \mathbb{N}$ be a given large number, let $U_N = \{u_1, \dots, u_N\}$ be a distribution of N points in the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$, treated as the torus $\mathbb{T}^d$, and let f be a real function on $\mathbb{T}^d$. Suppose that for suitable choices of U_N and f , the Riemann sums

$$\frac{1}{N} \sum_{j=1}^N f(u_j - x)$$

are, after an L^2 average on the variable $x \in \mathbb{T}^d$, good approximations of the integral

$$\int_{\mathbb{T}^d} f(s) \, ds.$$

What corresponding statement can we make concerning those sequences *close* to the sequence U_N ? Do such sequences mostly share the same good behavior?

2. A RANDOMIZATION ARGUMENT

In order to start discussing these questions, we introduce the following randomization of U_N ; see [3, 6] and also [8, 9]. Let $d\mu$ denote a probability measure on $\mathbb{T}^d$. For every $j = 1, \dots, N$, let $d\mu_j$ denote the measure obtained after translating $d\mu$ by u_j . More precisely, for any integrable function g on $\mathbb{T}^d$, we have

$$\int_{\mathbb{T}^d} g(t) \, d\mu_j = \int_{\mathbb{T}^d} g(t - u_j) \, d\mu.$$

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Let dt denote the Lebesgue measure on $\mathbb{T}^d$. For every sequence $V_N = \{v_1, \dots, v_N\}$ in $\mathbb{T}^d$ and every function $f \in L^2(\mathbb{T}^d, dt)$, we introduce, for every $t \in \mathbb{T}^d$, the discrepancy

$$D(t, V_N) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{j=1}^N f(v_j - t) - \int_{\mathbb{T}^d} f(s) ds.$$

Observe that $D(\cdot, V_N)$ is a periodic function with Fourier series

$$\sum_{0 \neq k \in \mathbb{Z}^d} \left(\frac{1}{N} \sum_{j=1}^N e^{-2\pi i k \cdot v_j} \right) \overline{\widehat{f}(k)} e^{2\pi i k \cdot t},$$

and the Parseval identity yields

$$D^2(V_N) \stackrel{\text{def}}{=} \|D(\cdot, V_N)\|_{L^2(\mathbb{T}^d, dt)}^2 = \sum_{0 \neq k \in \mathbb{Z}^d} \left| \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot v_j} \right|^2 |\widehat{f}(k)|^2.$$

We now average $D(V_N)$ in $L^2(\mathbb{T}^d, d\mu_j)$ for every $j = 1, \dots, N$ and consider

$$\mathfrak{D}_{d\mu}(U_N) \stackrel{\text{def}}{=} \left(\int_{\mathbb{T}^d} \dots \int_{\mathbb{T}^d} D^2(V_N) d\mu_1(v_1) \dots d\mu_N(v_N) \right)^{1/2}.$$

In this paper we study the relation between $\mathfrak{D}_{d\mu}(U_N)$ and $D(U_N)$. In the case $N = M^d$, where $M \in \mathbb{N}$, and

$$(1) \quad U_N = \frac{1}{M} \mathbb{Z}^d \cap \left[-\frac{1}{2}, \frac{1}{2} \right)^d,$$

the above quantities were studied in relation to the sharpness of a result of Beck [1] and of Montgomery [10] on irregularities of distribution; see Remark 3 below. In [6] two of the authors compared the quantities $D(U_N)$ and $\mathfrak{D}_{d\mu}(U_N)$ in the case (1) and when f is the characteristic function of a ball. Here we study the problem in our more general setting, and we are mainly interested in whether the inequality

$$(2) \quad \mathfrak{D}_{d\mu}(U_N) \leq c D(U_N)$$

holds.

Throughout this paper, the letters $c, C, \dots$ will denote positive constants, possibly depending on f but independent of N , and which may change from one step to the next. On the other hand, different letters $B, \kappa, \dots$ will denote constants which will not change throughout the paper.

3. AN EXPLICIT FORMULA

We first use a slight modification of an argument in [6] to obtain an explicit formula for $\mathfrak{D}_{d\mu}(U_N)$. We have

$$\begin{aligned}
 (3) \quad \mathfrak{D}_{d\mu}^2(U_N) &= \int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} \sum_{0 \neq k \in \mathbb{Z}^d} \left| \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot v_j} \right|^2 |\widehat{f}(k)|^2 d\mu_1(v_1) \cdots d\mu_N(v_N) \\
 &= \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 \left(\frac{1}{N} + \frac{1}{N^2} \sum_{\substack{j, \ell=1 \\ j \neq \ell}}^N \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} e^{2\pi i k \cdot v_j} e^{-2\pi i k \cdot v_\ell} d\mu_j(v_j) d\mu_\ell(v_\ell) \right) \\
 &= \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 \\
 &\quad \times \left(\frac{1}{N} + \frac{1}{N^2} \sum_{\substack{j, \ell=1 \\ j \neq \ell}}^N e^{2\pi i k \cdot (u_\ell - u_j)} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} e^{2\pi i k \cdot v_j} e^{-2\pi i k \cdot v_\ell} d\mu(v_j) d\mu(v_\ell) \right) \\
 &= \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 \left(\frac{1}{N} + |\widehat{\mu}(k)|^2 \left(\left| \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot u_j} \right|^2 - \frac{1}{N} \right) \right) \\
 &= \frac{1}{N} \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu}(k)|^2) + \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 |\widehat{\mu}(k)|^2 \left| \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot u_j} \right|^2 \\
 &= \frac{1}{N} \left(\|f\|_{L^2(\mathbb{T}^d, dt)}^2 - \|f * d\mu\|_{L^2(\mathbb{T}^d, dt)}^2 \right) + \|D(\cdot, U_N) * d\mu\|_{L^2(\mathbb{T}^d, dt)}^2.
 \end{aligned}$$

There are two natural extremal measures. The first one is $d\mu = \delta_0$, the Dirac measure centred at 0. In this case, we have

$$\mathfrak{D}_{\delta_0}(U_N) = D(U_N).$$

On the other hand, when $d\mu = dt$, we have

$$\mathfrak{D}_{dt}^2(U_N) = \frac{1}{N} \left(\|f\|_{L^2(\mathbb{T}^d, dt)}^2 - \left| \int_{\mathbb{T}^d} f(t) dt \right|^2 \right),$$

the classical Monte-Carlo error.

Note that if $ND^2(U_N) \geq c$, then $\mathfrak{D}_{dt}(U_N) \leq c_1 D(U_N)$, and (2) follows easily.

Another very peculiar case is when $D(U_N) = 0$. We observe that in general this does not imply $\mathfrak{D}_{d\mu}(U_N) = 0$, so that (2) does not hold. Indeed, let U_N be given by (1). Then

$$(4) \quad \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot u_j} = \begin{cases} 1 & \text{if } k \in M\mathbb{Z}^d, \\ 0 & \text{otherwise.} \end{cases}$$

Now choose $f(t) = \exp(2\pi i k_0 \cdot t)$ for some $k_0 \in \mathbb{Z}^d \setminus M\mathbb{Z}^d$. Then $D(U_N) = 0$. On the other hand, it follows from (3) that

$$\mathfrak{D}_{d\mu}^2(U_N) = \frac{1}{N}(1 - |\widehat{\mu}(k_0)|^2) \neq 0$$

whenever $|\widehat{\mu}(k_0)| \neq 1$, which is easily fulfilled, particularly by several measures with small support around the origin.

Hence, throughout the paper, we will be interested only in the case when

$$0 < D(U_N) < N^{-1/2}.$$

Let $0 < \varepsilon_N \leq 1$. For every probability measure $d\mu$ supported on the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$, let $d\mu^{(N)}$ denote the probability measure defined by

$$(5) \quad \int_{\mathbb{R}^d} g(\xi) d\mu^{(N)}(\xi) = \int_{\mathbb{R}^d} g(\varepsilon_N \xi) d\mu(\xi).$$

Then $d\mu^{(N)}$ is supported on the subcube $[-\frac{1}{2}\varepsilon_N, \frac{1}{2}\varepsilon_N]^d$ and can be regarded as a measure on $\mathbb{T}^d$.

4. MAIN RESULT

We first state our main result.

Theorem 1. *Let $f \in L^2(\mathbb{T}^d, dt)$ and let $U_N = \{u_1, \dots, u_N\}$ be a distribution of N points in the cube $[-\frac{1}{2}, \frac{1}{2}]^d$. Assume that $0 < D(U_N) < N^{-1/2}$. Let $d\mu$ be a non-Dirac probability measure on $\mathbb{T}^d$, let $d\mu^{(N)}$ be defined by (5) with $0 < \varepsilon_N \leq 1$, and let*

$$\eta_N = \begin{cases} \varepsilon_N^{2\alpha} & \text{if } \alpha < 1, \\ \varepsilon_N^2 \log(1 + \varepsilon_N^{-1}) & \text{if } \alpha = 1, \\ \varepsilon_N^2 & \text{if } \alpha > 1. \end{cases}$$

(i) *If for some $\alpha > 0$ and for every $\rho > 1$ we have*

$$(6) \quad \sum_{\rho \leq |k| < 2\rho} |\widehat{f}(k)|^2 \leq c \rho^{-2\alpha},$$

then

$$(7) \quad \mathfrak{D}_{d\mu^{(N)}}^2(U_N) \leq c \eta_N N^{-1} + D^2(U_N).$$

(ii) *If there exists an open cone¹ $\Omega \subseteq \mathbb{R}^d$ such that for every subcone $\Gamma \subseteq \Omega$,*

$$(8) \quad \liminf_{\rho \rightarrow +\infty} \rho^{2\alpha} \sum_{\substack{k \in \Gamma \\ \rho \leq |k| < 2\rho}} |\widehat{f}(k)|^2 > 0,$$

then there exist positive constants $\Delta \leq 1$ and c such that if $\varepsilon_N \leq \Delta$, then

$$\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \geq c \eta_N N^{-1}.$$

The following corollary shows that, in some sense, good sequences are never alone. Indeed we give conditions on ε_N that will ensure that $\mathfrak{D}_{d\mu^{(N)}}(U_N)$ and $D(U_N)$ are comparable.

Corollary 2. *Let f , U_N and $d\mu$ be as given in Theorem 1.*

¹In this paper every cone starts from the origin.

(i) Let f be as given in part (i) of Theorem 1 and let

$$(9) \quad \varepsilon_N \leq \begin{cases} (N^{1/2}D(U_N))^{1/\alpha} & \text{if } \alpha < 1, \\ \beta_N & \text{if } \alpha = 1, \\ N^{1/2}D(U_N) & \text{if } \alpha > 1, \end{cases}$$

where β_N satisfies $\beta_N^2 \log(1 + \beta_N^{-1}) = ND^2(U_N)$. Then

$$\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \leq cD^2(U_N).$$

(ii) Let f and Δ be as given in part (ii) of Theorem 1 and let $\kappa > 0$. Then there exists $c > 0$ such that whenever

$$(10) \quad \Delta \geq \varepsilon_N \geq \begin{cases} \kappa(N^{1/2}D(U_N))^{1/\alpha} & \text{if } \alpha < 1, \\ \kappa\beta_N & \text{if } \alpha = 1, \\ \kappa N^{1/2}D(U_N) & \text{if } \alpha > 1, \end{cases}$$

we have

$$\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \geq cD^2(U_N).$$

Remark 3. Consider the particular case when $f = \chi_A$, the characteristic function of a convex body $A \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$. Then (6) holds with $\alpha = \frac{1}{2}$. Let $\varepsilon_N = \Delta ND^2(U_N)$. Then

$$\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \leq cD^2(U_N).$$

If furthermore the boundary of A is smooth and has positive Gaussian curvature, then (8) holds with $\alpha = \frac{1}{2}$; see, for instance, [7]. We then have

$$\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \geq cD^2(U_N).$$

We recall that if A is rotated and contracted, then a result of Beck [1] and of Montgomery [10] says that

$$\int_{SO(d)} \int_0^1 \int_{\mathbb{T}^d} \left| \frac{1}{N} \sum_{j=1}^N \chi_{\sigma(rA)}(u_j - t) - r^d |A| \right|^2 dt dr d\sigma \geq cN^{-1-1/d}$$

for every choice of the point set distribution U_N ; see also [2, 4, 5]. We also recall that this is not true if the contraction is omitted; see [12, Theorem 3.1].

5. DECAY OF THE FOURIER COEFFICIENTS

The assumption (6) concerns the decay of the Fourier coefficients of f . This behavior can be naturally related to the smoothness of the function f as follows. Let $f \in L^2(\mathbb{T}^d)$, define $\Delta_h f(x) = f(x+h) - f(x)$ and, for every integer $\ell \geq 1$, write $\Delta_h^\ell f = \Delta_h \Delta_h^{\ell-1} f$. Let $\alpha > 0$. We say that f belongs to the Nikol'skiĭ space $H_2^\alpha(\mathbb{T}^d)$ if there exists $c > 0$ such that

$$\left(\int_{\mathbb{T}^d} |\Delta_h^\ell f(x)|^2 dx \right)^{1/2} \leq c|h|^\alpha$$

for some $\ell \geq 1$; see [11, Section 4.3.3].

Proposition 4. *Let $f \in H_2^\alpha(\mathbb{T}^d)$. Then (6) holds.*

Proof. Since $\widehat{\Delta_h f}(k) = (e^{2\pi i k \cdot h} - 1)\widehat{f}(k)$, we have $\widehat{\Delta_h^\ell f}(k) = (e^{2\pi i k \cdot h} - 1)^\ell \widehat{f}(k)$. Let $h = (1/10\rho, 0, \dots, 0)$ and $\Gamma = \{k \in \mathbb{Z}^d : k_1^2 \geq k_2^2 + \dots + k_d^2\}$. Observe that when $k \in \Gamma$ and $\rho \leq |k| \leq 2\rho$, we have $|e^{2\pi i k \cdot h} - 1| \geq c$. Therefore

$$\begin{aligned} \sum_{\substack{k \in \Gamma \\ \rho \leq |k| < 2\rho}} |\widehat{f}(k)|^2 &\leq c \sum_{\substack{k \in \Gamma \\ \rho \leq |k| < 2\rho}} |(e^{2\pi i k \cdot h} - 1)^\ell|^2 |\widehat{f}(k)|^2 \leq c \sum_{k \in \mathbb{Z}^d} |\widehat{\Delta_h^\ell f}(k)|^2 \\ &= c \int_{\mathbb{T}^d} |\Delta_h^\ell f(x)|^2 dx \leq c |h|^{2\alpha} = c \rho^{-2\alpha}. \end{aligned}$$

Note here that h is tailored on Γ . Since we can cover $\mathbb{Z}^d$ with a finite number of cones, the proposition follows from the above argument applied to different choices of h . $\square$

The proof of Theorem 1 requires a technical lemma.

Lemma 5. *Let $d\nu$ be a probability measure supported on $[-\frac{1}{2}, \frac{1}{2}]^d$. Then either*

- (i) *$d\nu$ is the Dirac measure δ_{t_0} at a point $t_0 \in [-\frac{1}{2}, \frac{1}{2}]^d$, or*
- (ii) *$1 - |\widehat{\nu}(\xi)|^2 = O(|\xi|^2)$ as $\xi \rightarrow 0$, and any open cone in $\mathbb{R}^d$ contains an open subcone Γ such that $1 - |\widehat{\nu}(\xi)|^2 \geq c|\xi|^2$ for small $\xi \in \Gamma$.*

Proof. Since $d\nu$ is compactly supported, its Fourier transform $\widehat{\nu}$ is smooth and has Taylor expansion

$$\widehat{\nu}(\xi) = 1 + \nabla \widehat{\nu}(0) \cdot \xi + \frac{1}{2} H_{\widehat{\nu}}(0) \xi \cdot \xi + o(|\xi|^2),$$

and so

$$1 - |\widehat{\nu}(\xi)|^2 = 1 - \widehat{\nu}(\xi)\widehat{\nu}(-\xi) = (\nabla \widehat{\nu}(0) \cdot \xi)^2 - H_{\widehat{\nu}}(0) \xi \cdot \xi + o(|\xi|^2) = O(|\xi|^2).$$

Let $F(\xi) = (\nabla \widehat{\nu}(0) \cdot \xi)^2 - H_{\widehat{\nu}}(0) \xi \cdot \xi$, and assume that F does not vanish identically. Let $\Sigma_{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$. Since F is a polynomial, it cannot vanish on an open set, and therefore $\{\xi \in \Sigma_{d-1} : F(\xi) = 0\}$ has empty interior in Σ_{d-1} . Since F is homogeneous and continuous, it follows that for every open cone in $\mathbb{R}^d$, we can find an open subcone Γ such that $|F(\xi)| \geq c|\xi|^2$ for $\xi \in \Gamma$. Therefore $1 - |\widehat{\nu}(\xi)|^2 \geq c|\xi|^2$ for small $\xi \in \Gamma$.

Assume now that $F \equiv 0$. Observe that

$$\frac{\partial \widehat{\nu}}{\partial \xi_j}(0) = -2\pi i \int_{\mathbb{T}^d} x_j d\nu(x)$$

and

$$\frac{\partial^2 \widehat{\nu}}{\partial \xi_j \partial \xi_\ell}(0) = -4\pi^2 \int_{\mathbb{T}^d} x_j x_\ell d\nu(x).$$

Then

$$\nabla \widehat{\nu}(0) \cdot \xi = -2\pi i \int_{\mathbb{T}^d} (x \cdot \xi) d\nu(x)$$

and

$$H_{\widehat{\nu}}(0) \xi \cdot \xi = -4\pi^2 \sum_{i,j} \int_{\mathbb{T}^d} \xi_j \xi_\ell x_j x_\ell d\nu(x) = -4\pi^2 \int_{\mathbb{T}^d} (\xi \cdot x)^2 d\nu(x).$$

Hence

$$\begin{aligned} 0 &= (\nabla \widehat{\nu}(0) \cdot \xi)^2 - H_{\widehat{\nu}}(0) \xi \cdot \xi = -4\pi^2 \left(\int_{\mathbb{T}^d} (x \cdot \xi) d\nu(x) \right)^2 + 4\pi^2 \int_{\mathbb{T}^d} (\xi \cdot x)^2 d\nu(x) \\ &= 4\pi^2 \int_{\mathbb{T}^d} \left(x \cdot \xi - \int_{\mathbb{T}^d} (t \cdot \xi) d\nu(t) \right)^2 d\nu(x). \end{aligned}$$

Let

$$t_0 = \int_{\mathbb{T}^d} t d\nu(t).$$

Since $d\nu(x)$ is positive, it follows that for every fixed ξ , we have

$$\nu(\{x : x \cdot \xi - \xi \cdot t_0 \neq 0\}) = 0.$$

Since ξ is arbitrary, we conclude that $\nu(\{x : x - t_0 \neq 0\}) = 0$, so that $d\nu$ is supported at t_0 . Since $d\nu$ is a probability measure, we have $d\nu = \delta_{t_0}$. $\square$

6. PROOF OF THEOREM 1

By Lemma 5, we have

$$1 - |\widehat{\mu^{(N)}}(k)|^2 = 1 - |\widehat{\mu}(\varepsilon_N k)|^2 = O(\varepsilon_N^2 |k|^2).$$

As $d\mu$ is a probability measure, we have

$$0 \leq 1 - |\widehat{\mu^{(N)}}(k)|^2 \leq \min\{1, c\varepsilon_N^2 |k|^2\}.$$

By (6), we have

$$\begin{aligned} (11) \quad \sum_{k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu^{(N)}}(k)|^2) &\leq \sum_{k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 \min\{1, c\varepsilon_N^2 |k|^2\} \\ &\leq \sum_{j=0}^{+\infty} \min\{1, c\varepsilon_N^2 2^{2j}\} \sum_{2^j \leq |k| < 2^{j+1}} |\widehat{f}(k)|^2 \leq c \sum_{j=0}^{+\infty} \min\{1, \varepsilon_N^2 2^{2j}\} 2^{-2j\alpha} \\ &\leq c\varepsilon_N^2 \sum_{2^j < \varepsilon_N^{-1}} 2^{(2-2\alpha)j} + c \sum_{2^j > \varepsilon_N^{-1}} 2^{-2j\alpha}. \end{aligned}$$

There are three cases. If $\alpha < 1$, we have

$$\sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu^{(N)}}(k)|^2) \leq c\varepsilon_N^{2\alpha}.$$

If $\alpha = 1$, we have

$$\sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu^{(N)}}(k)|^2) \leq c\varepsilon_N^2 \log(1 + \varepsilon_N^{-1}).$$

If $\alpha > 1$, we have

$$\sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu^{(N)}}(k)|^2) \leq c\varepsilon_N^2.$$

Since $d\mu$ is a probability measure, we have

$$(12) \quad \|D(\cdot, U_N) * d\mu\|_{L^2(\mathbb{T}^d, dt)} \leq D(U_N).$$

In view of (11) and (12), the inequality (7) follows from (3).

Let us now prove (ii). By Lemma 5 there exists a subcone $\Gamma \subset \Omega$ such that $1 - |\widehat{\mu}(\xi)|^2 \geq m_1|\xi|^2$ for $|\xi| \leq m_2$, $\xi \in \Gamma$. By (8) there exist m_3 and m_4 such that for $\rho \geq m_3$ we have

$$\sum_{\substack{k \in \Gamma \\ \rho \leq |k| < 2\rho}} |\widehat{f}(k)|^2 \geq m_4 \rho^{-2\alpha}.$$

Thus, for $\varepsilon_N < \min\{m_2/4m_3, 1\}$, we have

$$\begin{aligned} \mathfrak{D}_{d\mu^{(N)}}^2(U_N) &\geq \frac{1}{N} \left(\|f\|_{L^2(\mathbb{T}^d, dt)}^2 - \|f * d\mu^{(N)}\|_{L^2(\mathbb{T}^d, dt)}^2 \right) \\ &= \frac{1}{N} \sum_{0 \neq k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 - |\widehat{\mu}(\varepsilon_N k)|^2) \\ &\geq \frac{1}{N} \sum_{m_3 \leq 2^j \leq \frac{1}{2} m_2 \varepsilon_N^{-1}} \sum_{\substack{k \in \Gamma \\ 2^j \leq |k| < 2^{j+1}}} |\widehat{f}(k)|^2 (1 - |\widehat{\mu}(\varepsilon_N k)|^2) \\ &\geq \frac{\varepsilon_N^2}{N} m_1 m_4 \sum_{m_3 \leq 2^j \leq \frac{1}{2} m_2 \varepsilon_N^{-1}} 2^{-2j\alpha} 2^{2j} \geq c \eta_N N^{-1}. \end{aligned}$$

This completes the proof of Theorem 1.

Remark 6. The estimates from below for $\mathfrak{D}_{d\mu^{(N)}}^2(U_N)$ contained in Theorem 1 and Corollary 2 depend on suitable estimates for the first term

$$\frac{1}{N} \left(\|f\|_{L^2(\mathbb{T}^d, dt)}^2 - \|f * d\mu^{(N)}\|_{L^2(\mathbb{T}^d, dt)}^2 \right)$$

in (3). We observe that in our setting the second term may vanish even in rather natural examples. Indeed, let

$$f(x) = \sum_{k \neq 0} \frac{1}{|k|^\gamma} e^{2\pi i k x}$$

for some $\gamma > d/2 + 1$. One can easily check that (8) holds with $\alpha = \gamma - d/2$. Let U_N be as in (1) and μ be the (normalized) Lebesgue measure restricted to $[-\frac{1}{2}, \frac{1}{2}]^d$, so that, taking $\varepsilon_N = 1/M$, we have

$$\widehat{\mu^{(N)}}(k) = N \prod_{j=1}^d \frac{\sin(\pi k_j/M)}{\pi k_j}.$$

By (4) we have

$$D^2(U_N) = \sum_{k \neq 0} |\widehat{f}(Mk)|^2 = \frac{1}{M^{2\gamma}} \sum_{k \neq 0} \frac{1}{|k|^{2\gamma}} = \frac{c_\gamma}{M^{2\gamma}}$$

and

$$\|D(\cdot, U_N) * d\mu^{(N)}\|_{L^2(\mathbb{T}^d, dt)} = \sum_{k \neq 0} |\widehat{f}(Mk)|^2 |\widehat{\mu^{(N)}}(Mk)|^2 = 0.$$

On the other hand observe that, for large N ,

$$\varepsilon_N = \frac{1}{M} \geq c_\gamma N^{\frac{1}{2}} D(U_N) = c_\gamma M^{d/2-\gamma},$$

and therefore we can apply part (ii) of Corollary 2 and obtain the inequality $\mathfrak{D}_{d\mu^{(N)}}^2(U_N) \geq c D^2(U_N)$.

7. CONCLUSION

Let $d\mu^\otimes$ be defined on $(\mathbb{T}^d)^N$ by

$$\int_{(\mathbb{T}^d)^N} \varphi d\mu^\otimes = \int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} \varphi(v_1 - u_1, \dots, v_N - u_N) d\mu^{(N)}(v_1) \cdots d\mu^{(N)}(v_N).$$

We can now state and prove the result introduced in the abstract.

Corollary 7. *Let f , U_N and $d\mu$ be as given in Corollary 2.*

- (i) *Let f and ε_N be as given in part (i) of Corollary 2. Then for every λ satisfying $0 < \lambda < 1$, there exists a constant $c_\lambda > 0$, independent of U_N and such that $d\mu^\otimes(\{V_N : D(V_N) \leq c_\lambda D(U_N)\}) \geq \lambda$.*
- (ii) *Let f , Δ and ε_N be as given in part (ii) of Corollary 2. Then for a suitable constant $c > 0$, we have $d\mu^\otimes(\{V_N : D(V_N) \geq cD(U_N)\}) > 0$.*

Proof. If (9) holds, then Corollary 2 gives

$$\int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} D^2(V_N) d\mu^\otimes(V_N) \leq c D^2(U_N).$$

By the Chebyshev inequality, we have

$$d\mu^\otimes(\{V_N : D(V_N) > c_\lambda D(U_N)\}) \leq \frac{c}{c_\lambda^2},$$

and so

$$d\mu^\otimes(\{V_N : D(V_N) \leq c_\lambda D(U_N)\}) \geq 1 - \frac{c}{c_\lambda^2}.$$

A suitable choice of c_λ completes the proof of part (i). If (10) and (8) hold, then Corollary 2 gives

$$\int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} D^2(V_N) d\mu^\otimes(V_N) \geq c D^2(U_N),$$

which easily implies $d\mu^\otimes(\{V_N : D(V_N) \geq cD(U_N)\}) > 0$. □

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