A GENERAL MEASURING ARGUMENT FOR FINITE PERMUTATION GROUPS

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ABSTRACT. In Chermak and Delgado’s paper “A measuring argument for finite groups”, a certain “measuring lemma” was shown to hold. This lemma has been successfully applied in many recent papers. We generalize this lemma by expanding the discussion from groups acting on groups to groups acting on sets. As applications, we obtain the main results of several earlier papers.

I. INTRODUCTION

The purpose of this paper is to generalize the results of Chermak and Delgado in [1]. In [1] a certain “measuring lemma” was shown to hold (Lemma 1.1 of [1]). This measuring lemma was then successfully applied in [2] in the context of “failure of Thompson factorization”. Since the results of [1] were further applied in [3, 4, 7, 8], and in other papers as well, we thought that it is of interest to generalize this measuring lemma. In fact in the introduction to [1] it was suggested that such a generalization probably exists. Some applications are given in section IV.

The methods of this paper are variations of those in [1]. However, while [1] deals with a finite group G acting on a finite group H , we expand the discussion to the case of a finite group G acting on a finite set Ω . Furthermore, this paper gives a unified approach to previous “measuring arguments”, which yields, as special cases, the results of Goren’s papers [5] and [6], as well as the basic results of [1] and of Lucchini’s paper [9].

We would like to remark that in Goren’s papers [5] and [6], whenever a finite permutation group G on Ω is mentioned, it should be assumed that G has no fixed points on Ω .

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II. BASIC DEFINITIONS AND RESULTS

Let G denote a finite permutation group of a finite set Ω . This situation will be denoted by: $G \in \text{Per}(\Omega)$. The set of all subsets of Ω will be denoted by $\mathcal{P}(\Omega)$ and the set of all subgroups of G will be denoted by $\mathcal{S}(G)$.

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We denote by $\mathcal{L}$ a subset of $\mathcal{P}(\Omega)$, which is a *complete lattice* with respect to inclusion and which is closed with respect to the action of G . The greatest element of $\mathcal{L}$ is 1 and its least element is 0. This situation will be denoted by: $G \in \text{Per}(\Omega, \mathcal{L})$.

If $A, B \in \mathcal{L}$, we shall denote their least upper bound by $A \vee B$ and their greatest lower bound by $A \wedge B$. Finally, $\mathcal{L}^*$ denotes $\mathcal{L} - \{0\}$ and $\mathbb{R}$ denotes the real numbers.

If Ω is arbitrary, then the typical choice for $\mathcal{L}$ is: $\mathcal{L} = \mathcal{P}(\Omega)$, with $1 = \Omega$, $0 = \emptyset$ and for $A, B \in \mathcal{L}$, $A \vee B = A \cup B$ and $A \wedge B = A \cap B$. If $\Omega = G$, $Z(G) = \{1\}$ and G acts on Ω by conjugation, then the typical choice for $\mathcal{L}$ is: $\mathcal{L} = \mathcal{S}(G)$, with $1 = G$, $0 = \{1\}$ and for $A, B \in \mathcal{L}$, $A \vee B = \langle A, B \rangle$ and $A \wedge B = A \cap B$.

We continue with four basic definitions.

Definition 1: Ascending orbit functions. Let $G \in \text{Per}(\Omega, \mathcal{L})$. A function $\omega : \mathcal{L} \rightarrow \mathbb{R}$ is called an *ascending orbit function* on $\mathcal{L}$ if the following conditions hold:

- (1) $\omega(A^g) = \omega(A)$ for all $A \in \mathcal{L}$ and $g \in G$.
- (2) If $A, B \in \mathcal{L}$ with $A \subseteq B$, then $\omega(A) \leq \omega(B)$.
- (3) $\omega(0) = 1$.
- (4) If $A, B \in \mathcal{L}$, then $\omega(A)\omega(B) \leq \omega(A \vee B)\omega(A \wedge B)$.

If $G \in \text{Per}(\Omega, \mathcal{L})$ and there exists an ascending orbit function ω on $\mathcal{L}$, then we shall write $G \in \text{Per}(\Omega, \mathcal{L}, \omega)$.

Definition 2: Descending orbit functions. Let $G \in \text{Per}(\Omega, \mathcal{L})$. A function $\phi : \mathcal{L} \rightarrow \mathbb{R}$ is called a *descending orbit function* on $\mathcal{L}$ if the following conditions hold:

- (1) $\phi(A^g) = \phi(A)$ for all $A \in \mathcal{L}$ and $g \in G$.
- (2) If $A, B \in \mathcal{L}$ with $A \subseteq B$, then $\phi(B) \leq \phi(A)$.
- (3) $\phi(1) = 1$.
- (4) If $A, B \in \mathcal{L}$, then $\phi(A)\phi(B) \leq \phi(A \vee B)\phi(A \wedge B)$.

If $G \in \text{Per}(\Omega, \mathcal{L}, \omega)$ and there exists a descending orbit function ϕ on $\mathcal{L}$, then we shall write $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi)$.

Definition 3: Definition of m , M , M^* and M_* . Let $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi)$. We define $m = m(G, \Omega, \mathcal{L}, \omega, \phi) = \max\{\omega(A)\phi(A) \mid A \in \mathcal{L}^*\}$; $M = M(G, \Omega, \mathcal{L}, \omega, \phi) = \{A \in \mathcal{L}^* \mid \omega(A)\phi(A) = m\}$; M^* = the set of maximal members of M ; M_* = the set of minimal members of M .

Definition 4. Let $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi)$ and let m, M, M^*, M_* be as defined in Definition 3. This situation will be denoted by $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$.

The following three lemmas deal with $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$. Our first observation is:

Lemma 1. (1) *The subsets M , M^* and M_* are closed under the action of G .*
 (2) *If $A \in \mathcal{L}$, then $\omega(A) \geq 1$ and $\phi(A) \geq 1$.*

Proof. The statements follow from the definitions. □

In the following lemma we prove some basic properties of M . Lemma 2 is a generalization of the “measuring lemma” in [1].

Lemma 2. *Let $A, B \in M$ and suppose that either $A \wedge B \neq 0$ or $m \geq \phi(0)$. Then the following statements hold:*

- (1) $A \vee B \in M$.
- (2) $\phi(A)\phi(B) = \phi(A \vee B)\phi(A \wedge B)$.
- (3) $\omega(A)\omega(B) = \omega(A \vee B)\omega(A \wedge B)$.
- (4) If $A \wedge B \neq 0$, then $A \wedge B \in M$.
- (5) If $A \wedge B = 0$, then $m = \phi(0)$.
- (6) If $m > \phi(0)$, then $A \wedge B \neq 0$.

Proof. Since $A \in M$, we have

$$(2-1) \quad \omega(A)\phi(A) \geq \omega(A \vee B)\phi(A \vee B) \geq \frac{\omega(A)\omega(B)}{\omega(A \wedge B)}\phi(A \vee B),$$

which, after reduction, yields

$$(2-2) \quad \frac{\omega(B)}{\omega(A \wedge B)} \leq \frac{\phi(A)}{\phi(A \vee B)} \leq \frac{\phi(A \wedge B)}{\phi(B)}.$$

It follows from (2-2) that

$$(2-3) \quad \omega(B)\phi(B) \leq \omega(A \wedge B)\phi(A \wedge B).$$

Suppose that $A \wedge B \neq 0$. Then $A \wedge B \in \mathcal{L}^*$ and since $B \in M$, equality must hold in (2-3). This implies that $A \wedge B \in M$ and equalities hold throughout the expressions (2-1) and (2-2). In particular, the equalities in (2-1) imply that $A \vee B \in M$ (as $A \in M$) and $\omega(A)\omega(B) = \omega(A \vee B)\omega(A \wedge B)$ holds. The equalities in (2-2) imply that $\phi(A)\phi(B) = \phi(A \vee B)\phi(A \wedge B)$ holds. We have shown that (1), (2), (3) and (4) hold in this case, as claimed.

Suppose, now, that $A \wedge B = 0$. In this case it follows from our assumptions that $\omega(B)\phi(B) = m \geq \phi(0)$ and (2-3) implies $\omega(B)\phi(B) \leq \phi(0)$. Hence $m = \phi(0)$ and consequently equalities hold throughout the expressions (2-1) and (2-2). In particular, $A \vee B \in M$, $\phi(A)\phi(B) = \phi(A \vee B)\phi(A \wedge B)$ and $\omega(A)\omega(B) = \omega(A \vee B)\omega(A \wedge B)$ hold. We have shown that (1), (2), (3) and (5) hold in this case, as claimed.

Clearly (6) follows from (5). The proof of Lemma 2 is complete. $\square$

Lemmas 1 and 2 yield the following results about the subsets M^* and M_* of M .

Lemma 3. *If $A, B \in M^* \cup M_*$, then one of the following four cases holds:*

$$A = B, \quad A \wedge B = 0, \quad A \subset B \quad \text{or} \quad B \subset A.$$

In particular, if $g \in G$, then $A^g \in M^ \cup M_*$ and either $A = A^g$ or $A \wedge A^g = 0$.*

Proof. Let $A, B \in M^* \cup M_*$ and suppose that $A \wedge B \neq 0$. Then, by Lemma 2, $A \wedge B \in M$ and $A \vee B \in M$. But $A \wedge B \subseteq A \subseteq A \vee B$ and $A \in M^* \cup M_*$; hence either $A = A \wedge B$ or $A = A \vee B$. Thus either $A \subseteq B$ or $B \subseteq A$, as claimed.

If $g \in G$, then $A^g \in M^* \cup M_*$ by Lemma 1(1). Since $|A^g| = |A|$, it follows by the opening remark that either $A = A^g$ or $A \wedge A^g = 0$. $\square$

We define now: $C_G(\emptyset) = G$ and if A is a non-empty subset of Ω , then $C_G(A)$ denotes the pointwise centralizer of A in G .

The next proposition turns out to be very useful for our purposes.

Proposition 4. *If $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$, then the following statements hold:*

- (1) *If $m \geq \phi(0)$, then $M^* = \{U\}$, where $U =_{\text{def}} \bigcup_{A \in M} A$. Moreover, $U^g = U$ for all $g \in G$ and $C_G(U) \trianglelefteq G$.*

- (2) If $m > \phi(0)$, then $M_* = \{I\}$, where $I =_{\text{def}} \bigcap_{A \in M} A$. Moreover, $I^g = I$ for all $g \in G$ and $C_G(I) \trianglelefteq G$.

Proof. (1) If $m \geq \phi(0)$, then it follows by Lemma 2(1) that $U \in M$. Consequently $M^* = \{U\}$ and if $g \in G$, then by Lemma 1(1) $U^g \in M^*$ and hence $U^g = U$ for all $g \in G$. It follows that $(C_G(U))^g = C_G(U^g) = C_G(U)$ for all $g \in G$, which implies that $C_G(U) \trianglelefteq G$. The proof of (1) is complete.

(2) If $m > \phi(0)$, then by Lemma 2(6) $A \wedge B \neq 0$ for all $A, B \in M$ and consequently, by Lemma 2(4), $I \in M$. Hence $M_* = \{I\}$ and by Lemma 1(1) $I^g \in M_*$ for all $g \in G$. It follows that $I^g = I$ and $(C_G(I))^g = C_G(I^g) = C_G(I)$ for all $g \in G$, which implies that $C_G(I) \trianglelefteq G$. The proof of (2) is complete. $\square$

If G is a transitive permutation group on Ω , then the following proposition holds.

Proposition 5. Suppose that $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$ is transitive on Ω and $m \geq \phi(0)$. Then $\Omega = 1 \in M$, $M^* = \{1\}$ and $m = \omega(1) = \omega(\Omega)$.

Proof. Since $m \geq \phi(0)$, it follows by Proposition 4(1) that $M^* = \{U\}$ and $U^g = U$ for each $g \in G$. Since $U \in M$, $U \neq 0$ and hence $U \neq \emptyset$. By the transitivity of G on Ω it follows that $U = \Omega \in M$ and hence $1 = \Omega$. Thus $M^* = \{1\}$ and $m = \omega(1)\phi(1) = \omega(1) = \omega(\Omega)$. $\square$

We conclude this section with an important application of Lemmas 1, 2 and 3.

Theorem 6. Suppose that $Z(G) = \{1\} < G$ and $G \in \text{Per}(\Omega, \mathcal{L})$, where $\Omega = G$ on which G acts by conjugation and $\mathcal{L} = \mathcal{S}(G)$. Define two functions ω and ϕ from $\mathcal{S}(G)$ to $\mathbb{R}$ as follows:

$$\omega(A) = |A| \quad \text{and} \quad \phi(A) = |C_G(A)| \quad \text{for each } A \in \mathcal{S}(G).$$

Then the following statements hold.

- (1) $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$ for the appropriate m and $m \geq |G| = \phi(0)$.
- (2) If $A, B \in M$, then $AB = BA \in M$ and $C_G(A \cap B) = C_G(A)C_G(B)$.
- (3) If $A \in M^* \cup M_*$, then $A \trianglelefteq G$.
- (4) $M^* = \{U\}$, where $U = \bigcup_{A \in M} A$ and $m \leq |G||Z(U)|$.
- (5) If $A, B \in M_*$ and $A \neq B$, then $[A, B] = \{1\}$.
- (6) If G has no non-trivial abelian normal subgroups, then $m = |G|$.

Proof. (1) We must first prove that ω is an ascending orbit function on $\mathcal{L}$ and ϕ is a descending orbit function on $\mathcal{L}$. Since $\mathcal{L} = \mathcal{S}(G)$, we have $0 = \{1\}$, $1 = G$ and for each $A, B \in \mathcal{S}(G)$, $A \vee B = \langle A, B \rangle$ and $A \wedge B = A \cap B$. Hence $\omega(0) = |\{1\}| = 1$, $\phi(0) = |C_G(\{1\})| = |G|$ and $\phi(1) = |C_G(G)| = |Z(G)| = 1$. Furthermore, for each $A, B \in \mathcal{S}(G)$ we have:

$$\omega(A)\omega(B) = |A||B| = |AB||A \cap B| \leq |\langle A, B \rangle||A \cap B| = \omega(A \vee B)\omega(A \wedge B)$$

and

$$\begin{aligned} \phi(A)\phi(B) &= |C_G(A)||C_G(B)| = |C_G(A)C_G(B)||C_G(A) \cap C_G(B)| \\ &\leq |C_G(A \cap B)||C_G(\langle A, B \rangle)| = \phi(A \wedge B)\phi(A \vee B). \end{aligned}$$

It follows easily from these observations that ω is an ascending orbit function on $\mathcal{L}$ and ϕ is a descending orbit function on $\mathcal{L}$. Hence $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$ for the appropriate m . Since $G > \{1\}$, we have $m \geq \omega(G)\phi(G) = |G||C_G(G)| = |G| = \phi(0)$.

(2) Let $A, B \in M$. Since $m \geq \phi(0)$, it follows by Lemma 2 that

$$\phi(A)\phi(B) = \phi(\langle A, B \rangle)\phi(A \cap B) \quad \text{and} \quad \omega(A)\omega(B) = \omega(\langle A, B \rangle)\omega(A \cap B) .$$

The first equality implies that

$$\begin{aligned} |C_G(A)||C_G(B)| &= |C_G(\langle A, B \rangle)||C_G(A \cap B)| = |C_G(A) \cap C_G(B)||C_G(A \cap B)| \\ &= (|C_G(A)||C_G(B)|/|C_G(A)C_G(B)|)|C_G(A \cap B)| \end{aligned}$$

from which it follows that $C_G(A \cap B) = C_G(A)C_G(B)$. The second equality implies that $|A||B| = |\langle A, B \rangle||A \cap B| \geq |AB||A \cap B| = |A||B|$, from which it follows that $AB = BA$ and $AB \in M$ by Lemma 2(1).

(3) Let $A \in M^* \cup M_*$ and suppose, by way of contradiction, that $A \not\leq G$. Then there exists $x \in G - N_G(A)$ and by Lemma 3, $A^x \in M^* \cup M_*$ and $A \cap A^x = \{1\}$. Hence, by (2),

$$G = C_G(\{1\}) = C_G(A \cap A^x) = C_G(A)C_G(A^x) = C_G(A)C_G(A)^x .$$

Thus $G = C_G(A)$ and $A \leq G$, a contradiction.

(4) Since $m \geq \phi(0)$, it follows by Proposition 4(1) that $M^* = \{U\}$ and hence

$$m = \omega(U)\phi(U) = |U||C_G(U)| = |UC_G(U)||U \cap C_G(U)| \leq |G||Z(U)| .$$

(5) Suppose that $A, B \in M_*$ and $A \neq B$. If $A \cap B \neq \{1\}$, then by Lemma 2(4) $A \cap B \in M$, a contradiction. Hence $A \cap B = \{1\}$ and by (3) $A, B \leq G$. Thus $[A, B] = \{1\}$, as claimed.

(6) By (3), $U \leq G$ and hence $Z(U) \leq G$. If G has no non-trivial abelian normal subgroups, then $|Z(U)| = 1$ and as $|G| \leq m \leq |G||Z(U)|$ (by (1) and (4)), it follows that $m = |G|$. The proof is complete. $\square$

III. DOMINATED PRODUCTS

In this section we define and investigate finite permutation groups with “dominated products”. From these investigations the main results of this paper will emerge.

Definition 5. Let $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$. We shall say that G has *dominated products* on $\mathcal{L}$ if

$$m = \omega(1) .$$

First we remark

Proposition 7. Let $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$. Then G has dominated products on $\mathcal{L}$ if and only if $1 \in M$.

Proof. Since $\phi(1) = 1$, if $m = \omega(1)$, then $1 \in M$. Conversely, if $1 \in M$, then $m = \omega(1)$. $\square$

The following result follows immediately from Proposition 5.

Theorem 8. Let $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$ be transitive on Ω and suppose that $m \geq \phi(0)$. Then G has dominated products on $\mathcal{L}$.

For continuation, we need the notion of subgroups class functions.

Definition 6: Subgroups class functions. A function $f : \mathcal{S}(G) \rightarrow \mathbb{R}$ is called a *subgroups class function* of G if the following conditions hold:

- (1) $f(U) = f(U^g)$ for all $U \in \mathcal{S}(G)$ and $g \in G$.

- (2) If $U, V \in \mathcal{S}(G)$ and $U \leq V$, then $f(U) \leq f(V)$.
- (3) $f(\{1\}) = 1$.
- (4) If $U, V \in \mathcal{S}(G)$, then $f(U)f(V) \leq f(\langle U, V \rangle)f(U \cap V)$.

Lemma 9. *Let $G \in \text{Per}(\Omega, \mathcal{L})$ and suppose that $1 = \Omega$ and $C_G(A \vee B) \geq C_G(A) \cap C_G(B)$ for each $A, B \in \mathcal{L}$. Moreover, let f be a subgroups class function of G and define $\phi(A) = f(C_G(A))$ for each $A \in \mathcal{L}$. Then ϕ is a descending orbit function on $\mathcal{L}$.*

Proof. We prove each item of Definition 2 separately.

(1) If $A \in \mathcal{L}$ and $g \in G$, then $A^g \in \mathcal{L}$ and $C_G(A^g) = C_G(A)^g$. Hence $\phi(A^g) = f(C_G(A^g)) = f(C_G(A)^g) = f(C_G(A)) = \phi(A)$.

(2) If $A, B \in \mathcal{L}$ and $A \subseteq B$, then $C_G(B) \leq C_G(A)$ and hence $\phi(B) = f(C_G(B)) \leq f(C_G(A)) = \phi(A)$.

(3) Since $1 = \Omega$ and G is a permutation group on Ω , we get $\phi(1) = f(C_G(\Omega)) = f(\{1\}) = 1$.

(4) If $A, B \in \mathcal{L}$, then $\langle C_G(A), C_G(B) \rangle \leq C_G(A \wedge B)$ and by our assumptions $C_G(A \vee B) \geq C_G(A) \cap C_G(B)$. Thus:

$$\begin{aligned} \phi(A \vee B) &= f(C_G(A \vee B)) \geq f(C_G(A) \cap C_G(B)) \geq \frac{f(C_G(A))f(C_G(B))}{f(\langle C_G(A), C_G(B) \rangle)} \\ &\geq \frac{\phi(A)\phi(B)}{f(C_G(A \wedge B))} = \frac{\phi(A)\phi(B)}{\phi(A \wedge B)}. \end{aligned}$$

□

We conclude this section with a proof of the following theorem, which is a generalization of Theorem 5 in [6].

Theorem 10. *Let $G \in \text{Per}(\Omega, \mathcal{L}, \omega)$ with $\mathcal{L} = \mathcal{P}(\Omega)$ and suppose that G is either transitive on Ω or it is a non-abelian simple group, acting on Ω without fixed points.*

Moreover, let f be a subgroups class function of G and let $\phi(A) = f(C_G(A))$ for each $A \in \mathcal{L}$. Then the following statements hold:

- (1) ϕ is a descending orbit function on $\mathcal{L}$ and $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$ for the appropriate m . Moreover, $\phi(0) = \phi(\emptyset) = f(G)$.
- (2) Either $m < f(G)$ or $m = \omega(1) = \omega(\Omega)$. In particular,

$$m \leq \max(\omega(\Omega), f(G)).$$

- (3) If $m \geq f(G)$, then G has dominated products on $\mathcal{L}$.

In particular, this theorem holds if $f(T) = |T|$ for each $T \in \mathcal{S}(G)$, in which case $\phi(0) = \phi(\emptyset) = f(G) = |G|$.

Proof. (1) Since $\mathcal{L} = \mathcal{P}(\Omega)$, we have $0 = \emptyset$ and $1 = \Omega$. Moreover, for each $A, B \in \mathcal{L}$ we have $A \vee B = A \cup B$, $A \wedge B = A \cap B$ and $C_G(A \vee B) = C_G(A \cup B) = C_G(A) \cap C_G(B)$. Thus, by Lemma 9, ϕ is a descending orbit function on $\mathcal{L}$ and $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$ for the appropriate m . Furthermore, since $C_G(\emptyset) = G$, we have $\phi(0) = \phi(\emptyset) = f(C_G(\emptyset)) = f(G)$.

(2) Suppose that $m \geq f(G) = \phi(0)$. Then, by Proposition 4(1), $M^* = \{U\}$, $U^g = U$ for all $g \in G$ and $C_G(U) \trianglelefteq G$. Since $U \in M$, $U \neq \emptyset$. In the transitive case, $m = \omega(\Omega)$ by Proposition 5. In the simple case, since G acts on Ω without fixed points, $C_G(U) = G$ is impossible. But $C_G(U) \trianglelefteq G$, so we must have $C_G(U) = \{1\}$.

Thus $\omega(\Omega) \leq m = \omega(U)\phi(U) = \omega(U)f(\{1\}) = \omega(U)$. But $U \subseteq \Omega$, so $\omega(U) \leq \omega(\Omega)$, and it follows again that $m = \omega(\Omega) = \omega(1)$, as required.

(3) If $m \geq f(G)$, then by (2), $m = \omega(1)$. Thus G has dominated products on $\mathcal{L}$.

If $f : \mathcal{S}(G) \rightarrow \mathbb{R}$ is defined by $f(T) = |T|$ for each $T \in \mathcal{S}(G)$, then for each $U, V \in \mathcal{S}(G)$ we have $f(U)f(V) = |U||V| = |UV||U \cap V| \leq |\langle U, V \rangle||U \cap V| = f(\langle U, V \rangle)f(U \cap V)$. It is easy to check that the function f also satisfies the other conditions imposed on subgroups class functions of G . So the theorem holds if $f(T) = |T|$ for each $T \in \mathcal{S}(G)$, in which case $\phi(0) = \phi(\emptyset) = f(G) = |G|$. $\square$

IV. APPLICATIONS

As applications, we shall obtain the main results of the papers [1], [5], [6] and [9]. We shall use here the notation of the previous sections.

The following theorem is Theorem 9 of [5].

Theorem 11. *Let $G \in \text{Per}(\Omega)$ and suppose that G is either transitive on Ω or it is a simple group without fixed points on Ω . Then*

$$[G : C_G(A)] \geq |G|^{\frac{|A|}{|\Omega|}}$$

for each $A \in \mathcal{P}(\Omega)$.

Proof. View G as $G \in \text{Per}(\Omega, \mathcal{L})$, where $\mathcal{L} = \mathcal{P}(\Omega)$ and let $\omega(A) = |G|^{\frac{|A|}{|\Omega|}}$ for each $A \in \mathcal{P}(\Omega)$. If $A, B \in \mathcal{L}$, then $|A| + |B| = |A \cup B| + |A \cap B|$ and hence $\omega(A)\omega(B) = \omega(A \cup B)\omega(A \cap B)$. It is easy to check that ω also satisfies the other conditions of an ascending orbit function on $\mathcal{L}$ and hence $G \in \text{Per}(\Omega, \mathcal{L}, \omega)$.

Let $f(T) = |T|$ for all $T \in \mathcal{S}(G)$ and let $\phi(A) = f(C_G(A)) = |C_G(A)|$ for each $A \in \mathcal{P}(\Omega)$. As shown at the end of the proof of Theorem 10, f is a subgroups class function of G and we can apply the results of Theorem 10. Since $f(G) = |G|$ and $\omega(\Omega) = |G|^{\frac{|\Omega|}{|\Omega|}} = |G|$, it follows by Theorem 10 that if $A \in \mathcal{P}^*(\Omega)$, then

$$\omega(A)\phi(A) = |G|^{\frac{|A|}{|\Omega|}}|C_G(A)| \leq m \leq \max(\omega(\Omega), f(G)) = |G|,$$

which implies the required inequality. Since $C_G(\emptyset) = G$, that inequality also holds for $A = \emptyset$ and the proof of the theorem is complete. $\square$

Our next application is Theorem 6 of [6].

Theorem 12. *Let $G \in \text{Per}(\Omega)$ and suppose that G is either transitive on Ω or it is a simple group without fixed points on Ω . Suppose, further, that T is a non-empty normal subset of G satisfying $\{1\} \not\subseteq T$ and A is a non-empty G -invariant subset of Ω . Then the following inequality holds for each $D \in \mathcal{P}(\Omega)$:*

$$\frac{|A \cap D|}{|A|} + \frac{|T \cap C_G(D)|}{|T|} \leq 1.$$

Proof. View G as $G \in \text{Per}(\Omega, \mathcal{L})$, where $\mathcal{L} = \mathcal{P}(\Omega)$. Define a function $\omega : \mathcal{L} \rightarrow \mathbb{R}$ by $\omega(B) = |G|^{\frac{|A \cap B|}{|A|}}$ for each $B \in \mathcal{P}(\Omega)$. Let $\alpha = |G|^{\frac{1}{|A|}}$ and let $B, C \in \mathcal{P}(\Omega)$. Then

$$\begin{aligned} \omega(B)\omega(C) &= \alpha^{|A \cap B| + |A \cap C|} = \alpha^{|(A \cap B) \cup (A \cap C)| + |(A \cap B) \cap (A \cap C)|} \\ &= \alpha^{|A \cap (B \cup C)| + |A \cap (B \cap C)|} = \omega(B \cup C)\omega(B \cap C) = \omega(B \vee C)\omega(B \wedge C). \end{aligned}$$

Moreover, since A is a G -invariant subset of Ω , for each $g \in G$ we have: $|A \cup B^g| = |(A \cup B)^g| = |A \cup B|$, which implies that $\omega(B^g) = \omega(B)$. It is easy to check that ω

also satisfies the other conditions required from an ascending orbit function on $\mathcal{L}$. Hence $G \in \text{Per}(\Omega, \mathcal{L}, \omega)$.

Define another function $f : \mathcal{S}(G) \rightarrow \mathbb{R}$ by $f(S) = |G|^{\frac{|T \cap S|}{|T|}}$ for each $S \in \mathcal{S}(G)$ and let $\beta = |G|^{\frac{1}{|T|}}$. We claim that f is a subgroups class function of G . As above, for each $S, R \in \mathcal{S}(G)$ we have

$$f(S)f(R) = \beta^{|T \cap (S \cup R)| + |T \cap (S \cap R)|} \leq \beta^{|T \cap \langle S, R \rangle|} \beta^{|T \cap (S \cap R)|} = f(\langle S, R \rangle)f(S \cap R)$$

and since T is a normal subset of G , we have $f(S^g) = f(S)$ for each $g \in G$. Moreover, as $\{1\} \not\subseteq T$, it follows that $f(\{1\}) = 1$. It is easy to check that f also satisfies the other conditions required from a subgroups class function of G .

Define, finally, the function $\phi : \mathcal{L} \rightarrow \mathbb{R}$ by $\phi(A) = f(C_G(A))$ for each $A \in \mathcal{L}$. Then, by Theorem 10, ϕ is a descending orbit function on $\mathcal{L}$ and $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$ for the appropriate m . Notice that $\omega(\Omega) = |G| = f(G)$. Thus, if $D \in \mathcal{L}^*$, then it follows by Theorem 10 that

$$\begin{aligned} (|G|^{\frac{|A \cap D|}{|A|}})(|G|^{\frac{|T \cap C_G(D)|}{|T|}}) &= \omega(D)f(C_G(D)) \\ &= \omega(D)\phi(D) \leq m \leq \max(\omega(\Omega), f(G)) = |G|, \end{aligned}$$

which implies the required inequality. If $D = \emptyset$, then $|A \cap D| = 0$ and the required inequality holds in that case too. $\square$

For additional results of this type, see [6].

Next we prove Theorem 2.1 of [1].

Theorem 13. *Let G be a non-abelian simple group. Then $|A||C_G(A)| < |G|$ for every non-trivial proper subgroup A of G . In particular, $|A|^2 < |G|$ for every abelian subgroup A of G .*

Proof. By Theorem 6, $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$, where $\Omega = G$ on which G acts by conjugation, $\mathcal{L} = \mathcal{S}(G)$, $\omega(A) = |A|$ and $\phi(A) = |C_G(A)|$ for every $A \in \mathcal{S}(G)$. If $A \in M^* \cup M_*$, then, by Theorem 6(3), $A \trianglelefteq G$ and since $A \in M$, $A \neq \{1\}$. By the simplicity of G , we must have $A = G$ and hence $M = \{G\}$. It follows that $m = |G||C_G(G)| = |G|$ and $|A||C_G(A)| < m = |G|$ for every non-trivial proper subgroup A of G . If A is a non-trivial abelian subgroup of G , then $A < G$ and $|A|^2 \leq |A||C_G(A)| < |G|$. Clearly $|A|^2 < |G|$ also for $A = \{1\}$. The proof is complete. $\square$

Finally we prove Lemma 1.1 of [9].

Theorem 14. *Suppose that G has no non-trivial abelian normal subgroups. Then:*

- (1) $|A||C_G(A)| \leq |G|$ for every subgroup A of G ;
- (2) $|A||C_G(A)| < |G|$ for every non-trivial solvable subgroup A of G ;
- (3) $|A|^2 < |G|$ for every abelian subgroup A of G .

Proof. (1) By Theorem 6, $G \in \text{Per}(\Omega, \mathcal{L}, \omega, \phi, m)$, where $\Omega = G$ on which G acts by conjugation, $\mathcal{L} = \mathcal{S}(G)$, $\omega(A) = |A|$ and $\phi(A) = |C_G(A)|$ for each $A \in \mathcal{S}(G)$. Moreover, by Theorem 6(6), $m = |G|$ and hence (1) holds for every non-trivial subgroup A of G . But for $A = \{1\}$ we have $|A||C_G(A)| = |G|$, so (1) holds for all subgroups A of G .

(2) We show next that if A is a non-trivial solvable subgroup of G , then $|A||C_G(A)| < |G|$. Suppose, by way of contradiction, that $|A||C_G(A)| = |G|$. Then $A \in M$ and there exists $K \leq A$ which belongs to M_* . Then K is a non-trivial

solvable subgroup of G and $K \trianglelefteq G$ by Theorem 6(3), in contradiction to our assumptions.

(3) Finally, if A is a non-trivial abelian subgroup of G , then (2) implies that $|A|^2 \leq |A||C_G(A)| < |G|$, as required. If $A = \{1\}$, then the inequality certainly holds. The proof of the theorem is complete. $\square$

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