LIMITING FRACTIONAL AND LORENTZ SPACE ESTIMATES OF DIFFERENTIAL FORMS

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ABSTRACT. We obtain estimates in Besov, Triebel–Lizorkin and Lorentz spaces of differential forms on $\mathbf{R}^n$ in terms of their L^1 norm.

1. INTRODUCTION

The classical Hodge theory states that if $u \in C_c^\infty(\mathbf{R}^N; \bigwedge^\ell \mathbf{R}^n)$ and if $1 < p < \infty$, one has

$$(1) \quad \|Du\|_{L^p} \leq C(\|du\|_{L^p} + \|\delta u\|_{L^p}).$$

Here Du is the derivative of u , du is its exterior differential of u , δu is its exterior codifferential, and $\bigwedge^\ell \mathbf{R}^n$ is the ℓ th exterior product of $\mathbf{R}^n$. This estimate is known to fail when $p = 1$ or $p = \infty$.

When $p = 1$, J. Bourgain and H. Brezis [2, 3] and L. Lanzani and E. Stein [5] obtained for $2 \leq \ell \leq n - 2$ the estimate

$$\|u\|_{L^{n/(n-1)}} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}),$$

which would be the consequence by the Sobolev embedding of (1) with $p = 1$. When $\ell = 1$ or $\ell = n - 1$ one has to assume that du or δu vanishes.

I. Mitrea and M. Mitrea [6] have in a recent work extended these estimates to homogeneous Besov spaces. Using interpolation theory, they could replace the norm $\|u\|_{L^{n/(n-1)}}$ by $\|u\|_{\dot{B}_{p,q}^s}$ with $\frac{1}{p} - \frac{s}{n} = 1 - \frac{1}{n}$ and $q = \frac{2}{1-s}$. The goal of the present paper is to improve the assumption on q by relying on previous results and methods.

The first result is the estimate for the Besov spaces $\dot{B}_{p,q}^s(\mathbf{R}^n)$. We follow H. Triebel [10] for the definitions of the function spaces.

Theorem 1. *For every $s \in (0, 1)$, $p > 1$ and $q > 1$, if*

$$(2) \quad \frac{1}{p} - \frac{s}{n} = 1 - \frac{1}{n},$$

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then there exists $C > 0$ such that for every $u \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$ with, moreover, $\delta u = 0$ if $\ell = 1$ and $du = 0$ if $\ell = n - 1$, one has

$$\|u\|_{\dot{B}_{p,q}^s} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}).$$

In particular, since $\|u\|_{\dot{W}^{s,p}} = \|u\|_{\dot{B}_{p,p}^s}$, one has the estimate

$$(3) \quad \|u\|_{\dot{W}^{s,p}} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}).$$

In Theorem 1, we assume that $q > 1$. If it held for some $q \in (0, 1]$, then the embedding of $F_{1,2}^1(\mathbf{R}^n) \subset \dot{B}_{n/(n-1),q}^0(\mathbf{R}^n)$ would hold. This can only be the case when $q \geq 1$. Therefore, the only possible improvement of Theorem 1 would be the limiting case $q = 1$.

Open Problem 1. Does Theorem 1 hold for $q = 1$?

The estimate of Theorem 1 follows from the corresponding estimate for homogeneous Triebel–Lizorkin spaces $\dot{F}_{p,q}^s(\mathbf{R}^n)$.

Theorem 2. For every $s \in (0, 1)$, $p > 1$ and $q > 0$, if (2) holds, then there exists $C > 0$ such that for every $u \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$ with, moreover, $\delta u = 0$ if $\ell = 1$ and $du = 0$ if $\ell = n - 1$, one has

$$\|u\|_{\dot{F}_{p,q}^s} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}).$$

Note that here there is no restriction on $q > 0$. Finally, the latter estimate has an interesting consequence for Lorentz spaces.

Theorem 3. For every $q > 1$, there exists $C > 0$ such that for every $u \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$ with, moreover, $\delta u = 0$ if $\ell = 1$ and $du = 0$ if $\ell = n - 1$, one has

$$\|u\|_{L^{\frac{n}{n-1},q}} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}).$$

In Theorem 3 the case $q = 1$ and $\ell = 0$ is equivalent to the embedding of $W^{1,1}(\mathbf{R}^n)$ in $L^{\frac{n}{n-1},1}(\mathbf{R}^n)$, which was obtained by J. Peetre [8] (see also [15]). This raises the following question:

Open Problem 2. Does Theorem 3 hold for $q = 1$ and $\ell \geq 1$?

The proofs of the theorems rely on the techniques developed by the author [11, 12] and on classical embeddings and regularity theory in fractional spaces.

2. THE MAIN TOOL

Our main tool is a generalization of an estimate for divergence-free L^1 vector fields by the author [12].

Proposition 2.1. For every $s \in (0, 1)$, $p > 1$ and $q > 0$ with $sp = n$, there exists $C > 0$ such that for every $f \in (C_c^\infty \cap L^1)(\mathbf{R}^n; \bigwedge^{n-1} \mathbf{R}^n)$ and $\varphi \in C_c^\infty(\mathbf{R}^n; \bigwedge^1 \mathbf{R}^n)$, if $df = 0$,

$$\int_{\mathbf{R}^n} f \wedge \varphi \leq C \|f\|_{L^1} \|\varphi\|_{\dot{F}_{p,q}^s}.$$

Here and in the sequel, $\wedge$ denotes the exterior product of forms. The proof of this proposition follows the method introduced by the author [4, 11, 12, 14] and followed subsequently by L. Lanzani and E. Stein [5] and by I. Mitrea and M. Mitrea [6]. The extension to the case $q = p$ is suggested in a previous work of the author [12, Remark 5] (see also [2, Remark 1], [3, Remark 11], [14, Remark 2], and [4]); the proposition can be deduced therefrom by following a remark in a subsequent paper [13, Remark 4.2].

Proof. Write $\varphi = \varphi^1 dx_1 + \cdots + \varphi^n dx_n$ and $f = f_1 dx_2 \wedge \cdots \wedge dx_n - f_2 dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n + \cdots + (-1)^n f_n dx_1 \wedge \cdots \wedge dx_{n-1}$. Without loss of generality, we shall estimate

$$\int_{\mathbf{R}^n} f_1 \varphi^1.$$

Fix $t \in \mathbf{R}$, and consider the function $\psi : \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ defined by $\psi(y) = \varphi^1(t, y)$. Choose $\rho \in C_c^\infty(\mathbf{R}^n)$ such that $\int_{\mathbf{R}^{n-1}} \rho = 1$, and set for $\varepsilon > 0$ and $y \in \mathbf{R}^{n-1}$ $\rho_\varepsilon(y) = \frac{1}{\varepsilon^{n-1}} \rho(\frac{y}{\varepsilon})$. For every $\alpha \in (0, 1)$, there is a constant $C > 0$ that only depends on ρ and α such that

$$\|\nabla \rho_\varepsilon * \psi\|_{L^\infty} \leq C \varepsilon^{\alpha-1} |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}$$

and

$$\|\psi - \rho_\varepsilon * \psi\|_{L^\infty} \leq C \varepsilon^\alpha |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})},$$

where $|\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}$ is the $C^{0,\alpha}$ seminorm of ψ , i.e.,

$$|\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})} = \sup_{y, z \in \mathbf{R}^{n-1}} \frac{|\psi(z) - \psi(y)|}{|z - y|^\alpha}.$$

One has on the one hand

$$\int_{\mathbf{R}^{n-1}} f_1(t, \cdot) (\psi - \rho_\varepsilon * \psi) \leq C \|f_1(t, \cdot)\|_{L^1(\mathbf{R}^{n-1})} \varepsilon^\alpha |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}.$$

On the other hand, by integration by parts, and since $\sum_{i=1}^n \partial_i f_i = 0$,

$$\begin{aligned} \int_{\mathbf{R}^{n-1}} f_1(t, \cdot) \rho_\varepsilon * \psi &= \sum_{i=2}^n (-1)^i \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^+} f_i(t, y) \partial_i (\rho_\varepsilon * \psi)(y) dt dy \\ &\leq C \|f_1(t, \cdot)\|_{L^1(\mathbf{R}^{n-1})} \varepsilon^{\alpha-1} |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}. \end{aligned}$$

Taking $\varepsilon = \|f\|_{L^1(\mathbf{R}^n)} / \|f(t, \cdot)\|_{L^1(\mathbf{R}^{n-1})}$, one obtains

$$(4) \quad \int_{\mathbf{R}^{n-1}} f_1 \psi \leq C \|f\|_{L^1(\mathbf{R}^n)}^\alpha \|f_1(t, \cdot)\|_{L^1(\mathbf{R}^{n-1})}^{1-\alpha} |\psi|_{C^{0,\alpha}(\mathbf{R}^{n-1})}.$$

Now, by the embedding theorem for Triebel–Lizorkin spaces, one has the estimate

$$|\psi|_{C^{0,\alpha}} \leq C \|\psi\|_{\dot{F}_{p,q}^s(\mathbf{R}^{n-1})}$$

with $\alpha = \frac{1}{p}$; hence from (4) we deduce the inequality

$$\int_{\mathbf{R}^{n-1}} f_1 \psi \leq C \|f\|_{L^1}^{\frac{1}{p}} \|f_1(\cdot, t)\|_{L^1}^{1-\frac{1}{p}} \|\psi\|_{\dot{F}_{p,q}^s(\mathbf{R}^{n-1})}.$$

Now, recalling that as a direct consequence of the Fubini property of Triebel–Lizorkin spaces ([10, Theorem 2.5.13], [1, Théorème 2], [9, Theorem 2.3.4/2])

$$\left(\int_{\mathbf{R}} \|\varphi(t, \cdot)\|_{\dot{F}_{p,q}^s(\mathbf{R}^{n-1})}^p dt \right) \leq C \|\varphi\|_{\dot{F}_{p,q}^s(\mathbf{R}^n)}^p,$$

one concludes, using Hölder's inequality, that

$$\int_{\mathbf{R}^n} f_1 u^1 \leq C \|f\|_{L^1}^{\frac{1}{p}} \int_{\mathbf{R}} (\|f_1(\cdot, t)\|_{L^1}^{1-\frac{1}{p}} \|\varphi(t, \cdot)\|_{\dot{F}_{p,q}^s(\mathbf{R}^{n-1})}) dt \leq C' \|f\|_{L^1} \|\varphi\|_{\dot{F}_{p,q}^s(\mathbf{R}^n)}.$$

□

Proposition 2.2. *For every $s \in (0, 1)$, $p > 1$ with $\frac{1}{p} + \frac{s}{n} = 1$, $q > 1$ and $1 \leq \ell \leq n-1$, there exists $C > 0$ such that for every $f \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$ with $df = 0$, one has*

$$\|f\|_{\dot{F}_{p,q}^{-s}} \leq C \|f\|_{L^1}.$$

Proof. The proposition will be proved by downward induction. The proposition is true for $\ell = n-1$ by Proposition 2.1. Assume now that it holds for $\ell+1$, and let $f \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$. Since $d(f \wedge dx_i) = 0$, Proposition 2.1 is applicable and

$$\|f\|_{\dot{F}_{\frac{n}{n-s},q}^{-s}} \leq \sum_{i=1}^n \|f \wedge dx_i\|_{\dot{F}_{\frac{n}{n-s},q}^{-s}} \leq C \sum_{i=1}^n \|f\|_{L^1} = Cn \|f\|_{L^1}. \quad \square$$

A useful corollary of the previous proposition is

Corollary 2.3. *For every $s \in (0, 1)$, $p > 1$ with $\frac{1}{p} + \frac{s}{n} = 1$, $q > 1$ and $1 \leq \ell \leq n-1$, there exists $C > 0$ such that for every $f \in C_c^\infty(\mathbf{R}^n; \bigwedge^\ell \mathbf{R}^n)$ with $df = 0$, one has*

$$\|f\|_{\dot{B}_{q,p}^{-s,p}} \leq C \|f\|_{L^1}.$$

Proof. This follows from classical embeddings between Besov and Triebel–Lizorkin spaces; see the proof of Theorem 1 below. □

3. PROOFS OF THE MAIN RESULTS

We begin by proving Theorem 2.

Proof of Theorem 2. To fix ideas, assume that $2 \leq \ell \leq n-1$. Recall that one has

$$u = d(K * (\delta u)) + \delta(K * (du)),$$

where the Newton kernel is defined for $n \geq 3$ by $K(x) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}|x|^{n-2}}$ and for $n = 2$ by $K(x) = -\frac{1}{2\pi} \log|x|$. By the classical elliptic estimates for Triebel–Lizorkin spaces,

$$\|K * (\delta u)\|_{\dot{F}_{p,q}^{s+1}} \leq C \|\delta u\|_{\dot{F}_{p,q}^{s-1}} \quad \text{and} \quad \|K * (du)\|_{\dot{F}_{p,q}^{s+1}} \leq C \|\delta u\|_{\dot{F}_{p,q}^{s-1}}.$$

Now, since $d(du) = 0$, Proposition 2.2 is applicable and yields

$$\|K * (du)\|_{\dot{F}_{p,q}^{s+1}} \leq C \|\delta u\|_{L^1}.$$

Since $\delta(\delta u) = 0$, one can, by the Hodge duality between d and δ , treat $\|K * (du)\|_{\dot{F}_{p,q}^{s+1}}$ similarly. □

We can now deduce Theorem 1 from Theorem 2.

Proof of Theorem 1. First assume that $q \geq p$. Then one has

$$\|u\|_{\dot{B}_{p,q}^s} \leq C \|u\|_{\dot{F}_{p,q}^s},$$

and Theorem 1 follows from Theorem 2. Otherwise, if $q < p$, then by the embedding theorems of Besov spaces,

$$\|u\|_{\dot{B}_{p,q}^s} \leq C \|u\|_{\dot{F}_{q,q}^r},$$

with $r = s + n(\frac{1}{q} - \frac{1}{p})$, and Theorem 1 also follows from Theorem 2. □

We finish with the proof of Theorem 3. It relies on

Lemma 3.1. *For every $s > 0$, $p > 1$ and $q > 1$ with $sq < n$ and*

$$(5) \quad \frac{1}{p} = \frac{1}{q} - \frac{s}{n},$$

there exists $C > 0$ such that for every $u \in C_c^\infty(\mathbf{R}^n)$,

$$\|u\|_{L^{p,q}} \leq C \|u\|_{\dot{F}_{q,2}^s}.$$

Proof. One has

$$u = I_s * ((-\Delta)^{\frac{s}{2}} u),$$

where the Riesz kernel I_s is defined for $x \in \mathbf{R}^n$ by

$$I_s(x) = \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}} 2^s \Gamma(\frac{s}{2}) |x|^{n-s}}.$$

One then has, by the Sobolev inequality for Riesz potentials in Lorentz spaces of R. O’Neil [7] (see also e.g. [15, Theorem 2.10.2]),

$$\|u\|_{L^{r,p}} \leq C \|(-\Delta)^{\frac{s}{2}} u\|_{L^p}.$$

One concludes by noting that $\|(-\Delta)^{\frac{s}{2}} u\|_{L^p}$ and $\|u\|_{\dot{F}_{p,2}^s}$ are equivalent norms [10, Theorem 2.3.8 and section 5.2.3]. $\square$

Proof of Theorem 3. Choose $s > 0$ so that (5) holds with $p = \frac{n}{n-1}$. Since $\frac{1}{q} - \frac{s}{n} = 1 - \frac{1}{n}$, one can combine Theorem 2 and Lemma 3.1 to obtain the conclusion. $\square$

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