A SUM-DIVISION ESTIMATE OF REALS

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(Communicated by Ken Ono)

ABSTRACT. Let A be a finite set of positive real numbers. We present a sum-division estimate:

$$|A + A|^2 |A/A| \geq \frac{|A|^4}{4}.$$

1. INTRODUCTION

Let A be a finite set of positive real numbers throughout. The sum-set, product-set and ratio-set of A are defined respectively to be

$$\begin{aligned} A + A &= \{a + b : a, b \in A\}, \\ AA &= \{ab : a, b \in A\}, \\ A/A &= \{a/b : a, b \in A\}. \end{aligned}$$

A famous conjecture of Erdős and Szemerédi [6] asserts that for any $\alpha < 2$, there exists a constant $C_\alpha > 0$ such that

$$\max \{|A + A|, |AA|\} \geq C_\alpha |A|^\alpha.$$

In a series of papers [1, 2, 7, 11, 12, 13], upper bounds on α were found by many authors. One highlight in this direction was a proof by Elekes [2], that α can be taken to be $\frac{5}{4}$. His argument utilized a clever application of the Szemerédi-Trotter theorem on point-line incidences. Recently, using the concept of multiplicative energy and an ingenious geometric observation, Solymosi [14] obtained that if A is not a singleton, then

$$(1) \quad |A + A|^2 |AA| \geq \frac{|A|^4}{4 \lceil \log_2 |A| \rceil},$$

which yields

$$(2) \quad \max \{|A + A|, |AA|\} \geq \frac{|A|^{4/3}}{2 \lceil \log_2 |A| \rceil^{1/3}}.$$

One cannot completely drop the logarithmic term in (2), since if we choose $\tilde{A} = \{1, 2, \dots, n\}$, then [4, 5, 8, 15]

$$(3) \quad |\tilde{A}\tilde{A}| = \frac{n^2}{(\ln n)^{\beta+o(1)}}, \quad \beta = 1 - \frac{1 + \ln 2}{\ln 2} = 0.0860713\dots$$

Received by the editors May 12, 2009, and, in revised form, May 18, 2009.

2000 *Mathematics Subject Classification*. Primary 11B75.

Key words and phrases. Sum-product estimate, sum-division estimate.

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There is a subtle difference between $|\tilde{A}\tilde{A}|$ and $|\tilde{A}/\tilde{A}|$. In fact, Elekes and Ruzsa [3] showed that there exists a universal constant $\gamma > 0$ such that

$$(4) \quad |A + A|^6 |A/A| \geq \gamma |A|^8,$$

which yields

$$|\tilde{A}/\tilde{A}| \geq \frac{\gamma}{64} |\tilde{A}|^2$$

by choosing $A = \tilde{A}$. This leads to a natural question: how to give a joint estimate on $|A + A|$ and $|A/A|$? It is not difficult to use the Szemerédi-Trotter theorem on point-line incidences to show that

$$(5) \quad |A + A| |A/A| \geq C |A|^{5/2}$$

holds for some universal constant $C > 0$. Besides, if we carefully analyze Solymosi's proof of (1), then

$$(6) \quad |A + A|^2 |A/A| \geq \frac{|A|^4}{4 \lceil \log_2 |A| \rceil}.$$

The main purpose of this paper is to drop the term $\lceil \log_2 |A| \rceil$ in (6).

Theorem 1. *Let A be a finite set of positive real numbers. Then*

$$|A + A|^2 |A/A| \geq \frac{|A|^4}{4}.$$

This implies a sum-division estimate

$$\max \{|A + A|, |A/A|\} \geq \frac{|A|^{4/3}}{2}.$$

There is an explanation of Theorem 1 in plane geometry. View $\mathbb{R}^2$ naturally as the complex plane $\mathbb{C}$. Given a finite set A of positive real numbers, denote by $\text{Rad}(A \times A)$ and $\text{Ang}(A \times A)$ respectively the radius-set and the angle-set of $A \times A$. Applying Theorem 1 with $\hat{A} = \{a^2 : a \in A\}$ yields

$$\max \{|\text{Rad}(A \times A)|, |\text{Ang}(A \times A)|\} \geq \frac{|A|^{4/3}}{2}.$$

This shows the angle-set and the radius-set of $A \times A$ cannot be small simultaneously.

2. PROOF OF THE MAIN RESULT

Suppose $|A/A| = y$ and $A/A = \{z_i\}_{i=1}^y$. Suppose z_i has m_i representations in $A \times A$, that is,

$$m_i = \left| \left\{ (a, b) \in A \times A : \frac{a}{b} = z_i \right\} \right| \quad (i = 1, 2, \dots, y).$$

Without loss of generality we may order all m_i 's as follows:

$$(7) \quad m_1 \leq m_2 \leq \dots \leq m_y.$$

Since $|A|^2 = \sum_{i=1}^y m_i$, there exists a unique integer k , $1 \leq k \leq y$, such that

$$\sum_{i=1}^{k-1} m_i < \frac{|A|^2}{2} \leq \sum_{i=1}^k m_i \leq k m_k.$$

Hence

$$(8) \quad |A/A| \geq k \geq \frac{|A|^2}{2m_k}$$

and

$$(9) \quad \sum_{i=k}^y m_i = \left(|A|^2 - \sum_{i=1}^{k-1} m_i \right) \geq \frac{|A|^2}{2}.$$

By (7) and Solymosi's geometric observation [14],

$$(10) \quad |A + A|^2 = |(A \times A) + (A \times A)| \geq m_k \sum_{i=k}^y m_i.$$

Multiplying (8), (9) and (10) yields

$$|A + A|^2 |A/A| \geq \frac{|A|^4}{4}.$$

This proves Theorem 1.

Remark. Let $F_n = \{a/q : 1 \leq a \leq q \leq n, (a, q) = 1\}$ be the set of Farey fractions of order n . It is well-known ([10]) that $|F_n| \sim \frac{3}{\pi^2} n^2$ as $n \rightarrow \infty$. Besides, it is not difficult to deduce from (3) (see also [8, 9]) that

$$\max\{|F_n + F_n|, |F_n - F_n|, |F_n F_n|, |F_n/F_n|\} \leq \frac{n^4}{(\ln n)^{\beta+o(1)}} \quad (n \rightarrow \infty).$$

This shows generally that one cannot expect the estimate

$$\max\{|A + A|, |A/A|\} \asymp |A|^2 \quad (|A| \rightarrow \infty).$$

We thank Dimitris Koukoulopoulos for communicating this example to us.

ACKNOWLEDGMENTS

The first author's research was supported by the Mathematical Tianyuan Foundation of China (No. 10826088). The second author's research was partially supported by NSF (CNS 0835834) and Texas Higher Education Coordinating Board (ARP 003615-0039-2007).

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