THE SYMMETRY PRESERVING REMOVAL LEMMA

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ABSTRACT. In this paper we observe that in the hypergraph removal lemma, the edge removal can be done in such a way that the symmetries of the original hypergraph remain preserved. As an application we prove the following generalization of Szemerédi's Theorem on arithmetic progressions. Let A be an Abelian group with subsets $S_1, S_2, \dots, S_t$ such that the number of arithmetic progressions $x, x + d, \dots, x + (t - 1)d$ with $x + (i - 1)d \in S_i$ is $o(|A|^2)$. Then we can shrink each S_i by $o(|A|)$ elements such that the new sets don't have any arithmetic progression of the above type.

1. INTRODUCTION

A directed k -uniform hypergraph H on the vertex set V is a subset of V^k such that there is no repetition in the k coordinates. The elements of H will be called **edges**. A **homomorphism** between two directed k -uniform hypergraphs F and H with vertex sets $V(F)$ and $V(H)$ is a map $f : V(F) \mapsto V(H)$ such that $(f(a_1), f(a_2), \dots, f(a_k))$ is in H whenever $(a_1, a_2, \dots, a_k)$ is in F . The **automorphism group** $\text{Aut}(H)$ is the group of bijective homomorphisms $\pi : V(H) \mapsto V(H)$. The **homomorphism density** $t(F, G)$ of F in G is the probability that a map $f : V(G) \mapsto V(H)$ chosen uniformly at random is a homomorphism.

The so-called hypergraph removal lemma ([1], [3], [5], [7], [10]) (in the directed setting) says that if the density $t(F, G)$ is small, then a small subset of the edges of G can be removed to make it F -free.

Theorem 1 (Removal Lemma). *For every $k \in \mathbb{N}$, $\epsilon > 0$ and k -uniform directed hypergraph F , there is a constant $\delta = \delta(k, \epsilon, F) > 0$ such that for every k -uniform directed hypergraph G with $t(F, G) \leq \delta$ there is a subset $S \subseteq G$ with $|S| \leq \epsilon |V(G)|^k$ such that $t(F, G \setminus S) = 0$.*

Using this deep result, we observe that the edge removal can be done in a way that the symmetries of G remain preserved. In other words the set S in Theorem 1 can be assumed to be a union of orbits of $\text{Aut}(G)$.

Theorem 2 (Symmetry Preserving Removal Lemma). *For every $k \in \mathbb{N}$, $\epsilon > 0$ and k -uniform directed hypergraph F there is a constant $\delta_2 = \delta_2(k, \epsilon, F) > 0$ such that for every k -uniform directed hypergraph G with $t(F, G) \leq \delta_2$, there is a subset $S \subseteq G$ with $|S| \leq \epsilon |V(G)|^k$ such that $t(F, G \setminus S) = 0$ and furthermore $\text{Aut}(G) \subseteq \text{Aut}(G \setminus S)$.*

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Proof. Let $V = V(G)$. Using the original removal lemma it remains to show that if $S \subseteq V^k$ satisfies $t(F, G \setminus S) = 0$, then there is $S' \subseteq V^k$ which is $\text{Aut}(G)$ invariant, $t(F, G \setminus S') = 0$ and $S' \leq |F||S|$. Such an S' is the union of those orbits O of $\text{Aut}(G)$ on V^k for which $|O|/|O \cap S| \leq |F|$. Assume that $f : V(F) \mapsto V$ is a homomorphism from F to $G \setminus S'$. Then for every fixed $e \in F$ and for a random element $\pi \in \text{Aut}(G)$, the probability is that $\pi(f(e)) \notin G \setminus S'$ is less than $1/|F|$ and so there is some $\pi \in \text{Aut}(G)$ with $\pi(f(F)) \subseteq G \setminus S$, which is a contradiction. $\square$

The argument given for the symmetry preserving removal lemma is very general. It applies to various modified versions of the removal lemma. An important such version is the t -partite removal lemma, where t is a fixed natural number. The vertex set of a t -partite k -uniform hypergraph is a t -tuple $\{V_i\}_{i=1}^t$ of finite sets. An edge of a t -partite hypergraph is an element from $\prod_{i=1}^k V_{a_i}$, where $a_1, a_2, \dots, a_k$ are k distinct numbers between 1 and t . Let G_1, G_2 be two t -partite k -uniform hypergraphs with vertex sets $\{V_i\}_{i=1}^t$ and $\{W_i\}_{i=1}^t$. A homomorphism from G_1 to G_2 is a t -tuple of maps $\{\phi_i : V_i \rightarrow W_i\}_{i=1}^t$ such that $(\phi_{a_1}(r_1), \phi_{a_2}(r_2), \dots, \phi_{a_k}(r_k)) \in \prod_{i=1}^k W_{a_i}$ is an edge in G_2 whenever $(r_1, r_2, \dots, r_k) \in \prod_{i=1}^k V_{a_i}$ is an edge in G_1 . An automorphism is a bijective homomorphism from G_1 to G_1 , and the homomorphism density $t(G_1, G_2)$ is the probability that a random t -tuple of maps $\{\phi_i : V_i \rightarrow W_i\}_{i=1}^t$ is a homomorphism.

We give an example of an application of the symmetry preserving removal lemma, and then we generalize it in the next chapter.

Example 1. Let S be a subset of a group T . The Cayley graph $\text{Cy}(T, S) \subseteq T \times T$ is the collection of pairs (a, b) with $ab^{-1} \in S$. The automorphism group of $\text{Cy}(T, S)$ contains T with the action $(a, b) \mapsto (ag, bg)$. Clearly any subset of $T \times T$ invariant under this action of G is a Cayley graph itself. This means that the T -orbits of edges in $\text{Cy}(T, S)$ correspond to elements of S .

We apply the symmetry preserving removal lemma to

$$F = \{(1, 2), (1, 3), (2, 3)\}$$

with $V(F) = \{1, 2, 3\}$ and for $G = \text{Cy}(T, S)$. A homomorphism from F to G is a map $f : \{1, 2, 3\} \mapsto T$ such that $a = f(1)f(2)^{-1}$, $b = f(2)f(3)^{-1}$ and $c = f(1)f(3)^{-1}$ are all in S . Consequently, the number of such homomorphisms is the number

$$|T| |\{(a, b, c) | ab = c, a, b, c \in S\}|.$$

The symmetry preserving removal lemma yields that if $ab = c$ has $o(|T|^2)$ solutions in S , then one can remove $o(|T|)$ elements from S such that in the new set there is no solution to $ab = c$. This was first proved by Ben Green [2] for Abelian groups and generalized to groups by Král, Serra and Vena [4].

2. CAYLEY HYPERGRAPHS

In this section we describe a potential way of generalizing Cayley graphs to the hypergraph setting and then discuss the symmetry preserving removal lemma on such graphs.

Definition 2.1. Let $G_1, G_2, \dots, G_t$ be finite groups and let H be a subgroup of $\prod_{i=1}^t G_i$. The group H acts on each G_i by $(h_1, h_2, \dots, h_t)g = h_i g$, where $(h_1, h_2, \dots, h_t) \in H$ and $g \in G_i$. A t -partite k -uniform hypergraph T on the vertex

set $\{G_i\}_{i=1}^t$ is called a **Cayley hypergraph** if its automorphism group contains H with the previous action.

This definition is very general, so we will start by analyzing a special case. Assume that all the groups $G_1, G_2, \dots, G_t$ are isomorphic to an Abelian group A . Furthermore, to get something interesting we want to assume that H is not too big and not too small. Let $C = \{C_1, C_2, \dots, C_r\}$ be a collection of k -element subsets of $\{1, 2, \dots, t\}$. Each set C_i defines a projection $p_i : H \mapsto A^k$ to the coordinates in C_i . Assume that the factor group $A^{C_i}/p_i(H) \cong A$ and let $\psi_i : A^{C_i} \rightarrow A$ be a homomorphism with kernel $p_i(H)$. Now we pick subsets $S_i \subseteq A$ for $1 \leq i \leq r$ and we define the graph $H_{k,t}(A, \{S_i\}, C)$ as

$$\bigcup_{i=1}^r \psi_i^{-1}(S_i),$$

where $\psi_i^{-1}(S_i)$ is the union of cosets in A^{C_i} of $p_i(H)$ representing an element in S_i . Note that the way we produced $H_{k,t}(A, \{S_i\}, C)$ guarantees that its automorphism group contains H as a subgroup.

The symmetry preserving removal lemma for t -partite hypergraphs directly implies the following lemma.

Lemma 2.1 (Cayley Hypergraph Removal Lemma). *For every k, t natural numbers, t -partite k -uniform hypergraph F and $\epsilon > 0$, there exists a constant $\delta > 0$ such that if*

$$t(F, H_{k,t}(A, \{S_i\}, C)) \leq \delta,$$

then there are subsets S'_i in A of size at most $\epsilon|A|$ such that $t(F, H_{k,t}(A, \{S_i \setminus S'_i\}, C)) = 0$.

Example 2. This example uses an idea by Solymosi [8], who showed that the Hypergraph Removal Lemma implies Szemerédi's theorem on arithmetic progressions (even in a multi-dimensional setting). Let t be a natural number, $k = t - 1$, and A be an Abelian group. We define H to be the subgroup in A^t of the elements $(a_1, a_2, \dots, a_t)$ with $\sum_{i=1}^t a_i = 0$ and $\sum_{i=1}^t (i - 1)a_i = 0$. Now set

$$C = \{\{1, 2, \dots, i - 1, i + i, \dots, t\}\}_{i=1}^t$$

and

$$\psi_i(a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_t) = \sum_{j=1}^t (j - i)a_j.$$

The functions ψ_i are computed in such a way that $\ker(\psi_i)$ is the projection of H to the coordinates in C_i .

Let F be the complete t -partite $(t - 1)$ -uniform hypergraph on t points. Lemma 2.1 applied to F and the above hypergraph $H_{t-1,t}(A, \{S_i\}, C)$ implies that if the system

$$x_i = \sum_{j=1}^t (j - i)a_j \in S_i$$

has $o(|A|^t)$ solutions, then we can delete $o(|A|)$ elements from each S_i such that the previous system has no solution. It is clear that $x_1, x_2, \dots, x_t$ form a t -term arithmetic progression and in fact that any such progression with $x_i \in S_i$ gives rise to $|A|^{t-2}$ solutions of the previous system. Using this we obtain the following:

Theorem 3 (Diagonal Szemerédi Theorem). *For every $\epsilon > 0$ and natural number t there exists $\delta = \delta(\epsilon, t) > 0$ such that if A is an Abelian group, $S_1, S_2, \dots, S_t$ are subsets in A and there are at most $\delta|A|^2$ pairs $x, d \in A$ with $x + (i-1)d \in S_i$ ($1 \leq i \leq t$), then we can shrink each S_i by at most $\epsilon|A|$ elements such that the new sets don't have such a configuration.*

This theorem implies Szemerédi's theorem [9] in the following way. Assume that S is a subset of A not containing any nontrivial k -term arithmetic progression. Let us apply Theorem 3 to $S = S_i$, $1 \leq i \leq t$. We have that if $x + (i-1)d \in S_i$ for every $1 \leq i \leq t$, then d is 0. This means that the condition of Theorem 3 holds with $\delta = |S|/|A|^2 \leq 1/|A|$. Now for every $\epsilon > 0$, if $|A| \geq 1/\delta(\epsilon, t)$ we obtain that every S_i contains a subset Q_i of size at most $\epsilon|A|$ such that the system $\{S_i \setminus Q_i\}_{i=1}^t$ does not contain any “diagonal arithmetic progression”. In particular, $S \setminus \bigcup_{i=1}^t Q_i$ has to be empty, and so $|S| \leq t\epsilon|A|$.

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